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A collocation model for water-hammer dynamics with application to leak detection

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Abstract— **This paper presents a new model for so-called water hammer equations based on a collocation method. This model is shown to fairly represent possible leak effects in a pipeline and thus to be useful in the purpose of leak detection. This is illustrated in simulation by an example of observer-based leak detector relying on this model.**

Keywords: water-hammer equations, collocation method, leak detection, Extended Kalman Filter

I. INTRODUCTION

The problem of leak detection in pipelines is very important so as to avoid undesired losses of the transported liquids and/or subsequent environmental problems. For this reason it has motivated a lot of work even in the control community (see [1]-[7]).

In most of those works, the leak detection is based on a finite dimensional model derived from the so-called water hammer equations known to describe the fluid dynamics in a pipeline, on the basis of finite differences methods (see [8]-[11]). In the present work instead, a so-called orthogonal collocation method is investigated in order to obtain a model which can be used for the leak detection in pipelines. The collocation method is a special case of the so-called weighted-residuals methods, commonly used in computational physics for solving PDE (e.g.[12]). This method is fairly simple, the computational effort is small, and it provides interesting properties for the leak detection as a point-wise dynamic representation of the flow along the pipeline.

The standard 'water-hammer' model for the flow dynamic in a pipeline is first recalled in section II, and from it the proposed collocation model is presented. Its possible use for leak detection via appropriate observer design is then discussed and illustrated in section III. Some conclusions and perspectives finally end the paper in section IV.

II. DYNAMICS PRESENTATION AND COLLOCATION MODEL

The water hammer behavior is the transmission of pressure waves along the pipeline resulting from a change in liquid flow velocity. This phenomenon is described by a set of hyperbolic partial differential equations (PDE) formed by one-dimensional continuity and momentum equations (basis of hydraulic transients), which are used to solve problems of unsteady flow in pipelines [13], [14].

A. The dynamical model

Assuming convective changes in velocity to be negligible, and that the liquid density and pipe cross-sectional area are constant, the momentum and continuity equations governing the dynamics of the fluid in the pipeline can be expressed as

$$\frac{\partial Q(z,t)}{\partial t} + gA\frac{\partial H(z,t)}{\partial z} + \frac{fQ^2(z,t)}{2DA} = 0 \qquad (1)$$

$$\frac{\partial H(z,t)}{\partial t} + \frac{b^2}{gA}\frac{\partial Q(z,t)}{\partial z} = 0 \qquad (2)$$

where H is the pressure head (m), Q the flow rate in the pipeline (m^3/s), b the wave speed in the fluid (m/s), g the gravitational acceleration (m/s^2), A the cross-sectional area of the pipe (m^2), D the diameter of the pipe, f the friction coefficient, t and z the time (s) and space (m) coordinates respectively. Here $z \in [0, L]$ where L is the length of the pipe. Initial conditions as usual correspond to the values of $Q(z,t)$, $H(z,t)$ along the pipe at $t = 0$

The boundary conditions in the flow transient equations can represent the end of the pipe in a tank, a valve, the connection between two pipes or a kind of different element, for example a pump, a leak, by-pass valves, etc. In this work the boundary conditions to be handled are: imposed pressures at each ends of the pipe and possible leaks at different points of pipeline. The imposed pressure heads will be respectively denoted by $u_1(t)$ and $u_2(t)$.

A leak at point z_f of the pipeline with outflow is represented by

$$Q_{z_f}(t) = \lambda\sqrt{H(z_f,t)} \qquad (3)$$

where $\lambda = A_f C_f \geq 0$, A_f is the sectional area of the leak and C_f the discharge coefficient. Equation (3) produces a discontinuity in system (1) and (2), and consequently requires a specific attention in the modeling.

A close-form solutions of these equations is not available. However, several methods have been used to numerically integrate them, such as method of characteristics, finite-difference method, finite element method, and linear element method [15]-[18]. In the present paper, we propose to use orthogonal collocation.

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B. The Orthogonal Collocation Method (OCM)

In order to apply the OCM to the PDE system given by (1) and (2) and obtain an ODE system, the pipe is spatially sectioned into n nodes defined as z_i where the subindex $i = 1,..,n$ represents the node index. The external nodes z_1 and z_n are fixed with the values 0 and L respectively, whereas the interior nodes $z_2, z_3, ..., z_{n-1}$ can take any distance value along the pipe. A boundary condition representing a leak can be added to any interior node.

Classically, the following approximation series are used for $Q(z,t)$ and $H(z,t)$:

$$Q(z,t) = \sum_{j=1}^{n} Q_j(t) N_j(z) \quad (4)$$

$$H(z,t) = \sum_{j=1}^{n} H_j(t) N_j(z) \quad (5)$$

for some basis functions $N_j(z)$. For this model we have chosen Lagrange interpolation functions as basis functions (as in [19] for shallow water dynamics for instance), which are given by

$$N_j(z) = \prod_{\substack{i=1 \\ j \neq i}}^{n} \frac{z - z_i}{z_j - z_i} \quad (6)$$

These basis functions have the following property:

$$N_j(z_i) = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ if } j \neq i \end{cases} \quad (7)$$

In the present paper in order to explicitly take possible leaks Q_{fk} at nodes z_{k+1} into account, let us rather consider a modified approximation for $Q(z,t)$ as follows:

$$Q(z,t) = \sum_{j=1}^{n} N_j(z) \left(Q_j(t) + \sum_{k=1}^{j-1} Q_{fk}(t) \right) \quad (8)$$

This indeed means that for any node z_j, Q_j corresponds to the flow at z_j^+, namely the flow $Q(z_j,t)$ reduced by the cumulated leaks occurring at nodes z_2 up z_j.

The derivatives w.r.t. time and space of (5) and (8) are

$$\frac{\partial H(z,t)}{\partial t} = \sum_{j=1}^{n} N_j(z) \frac{dH_j(t)}{dt} \quad (9)$$

$$\frac{\partial Q(z,t)}{\partial t} = \sum_{j=1}^{n} N_j(z) \left(\frac{dQ_j(t)}{dt} + \sum_{k=1}^{j-1} \frac{dQ_{fk}(t)}{dt} \right) \quad (10)$$

$$\frac{\partial H(z,t)}{\partial z} = \sum_{j=1}^{n} \frac{dN_j(z)}{dz} H_j(t) \quad (11)$$

$$\frac{\partial Q(z,t)}{\partial z} = \sum_{j=1}^{n} \frac{dN_j(z)}{dz} \left(Q_j(t) + \sum_{k=1}^{j-1} Q_{fk}(t) \right) \quad (12)$$

Notice that in (10) $\dot{Q}_{fk}$ should be computed according to (3) but in order to simplify the model this derivative will be here neglected (it has actually been checked that this does not significantly affect the response of the model for the leak detection purpose).

Using notations $a_1 = -ga$, $a_2 = -\frac{b^2}{gA}$, $\mu = \frac{f}{2DA}$ the system (1) and (2) with n points of collocation and $n-2$ possible leaks (at the interior nodes) is finally given by:

$$\dot{Q}_i(t) = a_1 \sum_{j=1}^{n} N'_{ij}(z) H_j - \mu \left(Q_i(t) + \sum_{k=1}^{i-1} Q_{fk}(t) \right)^2 \quad (13)$$

$$\dot{H}_i(t) = a_2 \sum_{j=1}^{n} N'_{ij}(z) \left(Q_j(t) + \sum_{k=1}^{j-1} Q_{fk}(t) \right) \quad (14)$$

with $N'_{ij} = \dfrac{\partial N_j(z_i)}{\partial z}$, $H_1 = u_1, H_5 = u_2$ and $Q_{fn-1} = 0$.

C. Simulation Results

Let us present here some simulation results in order to analyze the behavior system (13)-(14) with five nodes, which becomes

$$\begin{aligned}
\dot{Q}_1 &= a_1(N'_{11}H_1 + N'_{12}H_2 + N'_{13}H_3 + N'_{14}H_4 \\
&\quad + N'_{15}H_5) - \mu(Q_1)^2 \\
\dot{Q}_2 &= a_1(N'_{21}H_1 + N'_{22}H_2 + N'_{23}H_3 + N'_{24}H_4 \\
&\quad + N'_{25}H_5) - \mu(Q_2 + Q_{f1})^2 \\
\dot{Q}_3 &= a_1(N'_{31}H_1 + N'_{32}H_2 + N'_{33}H_3 + N'_{34}H_4 \\
&\quad + N'_{35}H_5) - \mu(Q_3 + Q_{f1} + Q_{f2})^2 \\
\dot{Q}_4 &= a_1(N'_{41}H_1 + N'_{42}H_2 + N'_{43}H_3 + N'_{44}H_4 \\
&\quad + N'_{45}H_5) - \mu(Q_4 + Q_{f1} + Q_{f2} + Q_{f3})^2 \\
\dot{Q}_5 &= a_1(N'_{51}H_1 + N'_{52}H_2 + N'_{53}H_3 + N'_{54}H_4 \\
&\quad + N'_{55}H_5) - \mu(Q_5 + Q_{f1} + Q_{f2} + Q_{f3})^2 \\
\dot{H}_2 &= a_2(N'_{21}Q_1 + N'_{22}(Q_2 + Q_{f1}) \\
&\quad + N'_{23}(Q_3 + Q_{f1} + Q_{f2}) \\
&\quad + N'_{24}(Q_4 + Q_{f1} + Q_{f2} + Q_{f3}) \\
&\quad + N'_{25}(Q_5 + Q_{f1} + Q_{f2} + Q_{f3})) \\
\dot{H}_3 &= a_2(N'_{31}Q_1 + N'_{32}(Q_2 + Q_{f1}) \\
&\quad + N'_{33}(Q_3 + Q_{f1} + Q_{f2}) \\
&\quad + N'_{34}(Q_4 + Q_{f1} + Q_{f2} + Q_{f3}) \\
&\quad + N'_{35}(Q_5 + Q_{f1} + Q_{f2} + Q_{f3})) \\
\dot{H}_4 &= a_2(N'_{41}Q_1 + N'_{42}(Q_2 + Q_{f1}) \\
&\quad + N'_{43}(Q_3 + Q_{f1} + Q_{f2}) \\
&\quad + N'_{44}(Q_4 + Q_{f1} + Q_{f2} + Q_{f3}) \\
&\quad + N'_{45}(Q_5 + Q_{f1} + Q_{f2} + Q_{f3}))
\end{aligned} \quad (15)$$

where H_1 and H_5 are considered as inputs, Q_i (where $i = 1, 2, ..., 5$), H_2, H_3 and H_4 as state variables. Moreover here, the Q_{fk}'s will be expressed as in (3). The data for the simulation are taken from [9] and presented in Table I.

TABLE I

EXPERIMENTAL DATA

Constants	Value	Input	Value
g	9.8 m/s^2	H_1	7 m
L	132.56 m	H_5	1 m
b	1250 m/s		
D	0.105 m		
f	0.005 m		

1) A single leak: The leak is located at the node $z_2 = 20$ m and it happens at $t = 100$ s. The flow rate at each one of the nodes can be observed in Fig.1. It can be noticed that before the leak appears, the flow at each node maintains its initial value, but when the leak appears the flow rates Q_2, Q_3, Q_4 and Q_5 decrease and converge to another steady state. This is fairly consistent with the physical interpretation of the effect of a leak at this point.

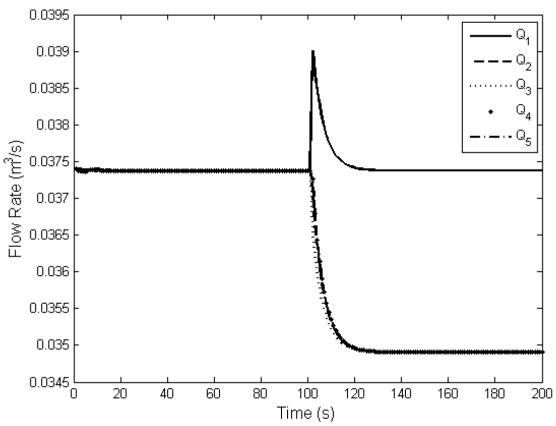

Fig. 1. Response of the system in presence of a single leak

2) Three leaks at different times: The leaks are located at the nodes $z_1 = 20$ m, $z_2 = 70$ m, $z_3 = 125$ m and appears at times $t_1 = 150s$, $t_2 = 250s$, $t_3 = 350s$. The response of the flow rate and pressure head at each node is shown in Fig.2. It can be noticed in this figure that whenever a leak appears a bifurcation of the flow occurs when and where the leak is present. Thus, two flows exist at the node where the leak happens, one flow that represents the previous response of the system and another one that represents the response of the system after the leak. This means that each node contains the information of the state previous to the leak and the information of the lost flow after the leak. On the other hand, the first node contains the information of the system before the first leak appears and the last node has the total information of the overall lost flow caused by all the leaks. This property is of particular interest for a purpose of leak detection.

3) Three leaks at the same time: The leaks are located at the nodes $z_1 = 20$ m, $z_2 = 70$ m, $z_3 = 125$ m and they happen at $t = 100s$. It can be noticed in Fig. 3 that when several leaks appear at the same time in different nodes,

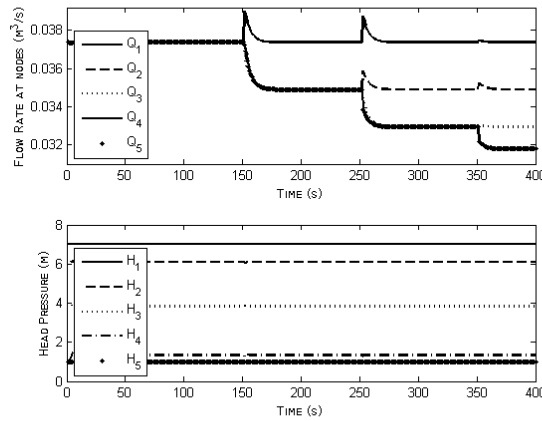

Fig. 2. Response of the system in presence of successive three leaks

the flow is divided in several ramifications at the time they appear. In this case the first node keeps the initial value, the second node contains the information of the lost flow at this node and the initial value, the next nodes contain the response of the lost flow at such node plus the previous lost flows.

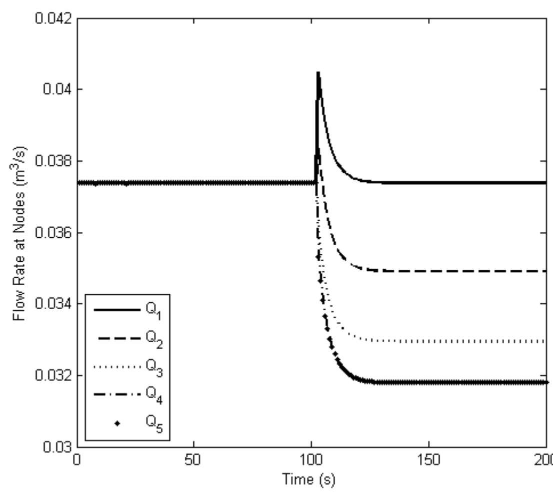

Fig. 3. Response of the system in presence of three simultaneous leaks

4) Comparison with the model by finite differences: The method of finite differences computes the flow by sections, namely if a leak occurs the computation of the overall lost flow is divided in one flow before the leak, which is proportional to the size of the section before the leak and in another one after the leak proportional to the remaining section after the leak. For a leak occurring at node z_i in a pipeline of length L, the proportion of the flow at this node affecting the section before the leak by $\kappa_b = \frac{z_i}{L}$ and after the leak will be given by $\kappa_a = 1 - \frac{z_i}{L}$.

In other words, to compare the OCM model with the model by finite differences, one has to reconstruct from the

OCM model flows some equivalent ones for the sections before and after the leak node as follows (written in variations):

$$\dot{\tilde{Q}}_a = \dot{Q}_1 - (\kappa_a)\,\dot{Q}_n \quad (16)$$

$$\dot{\tilde{Q}}_b = \dot{Q}_1 + (\kappa_b)\,\dot{Q}_n \quad (17)$$

Where $\dot{\tilde{Q}}_a$ and $\dot{\tilde{Q}}_b$ are the modified flows before and after the leak respectively. For the implementation of the finite differences method in this work the pipe is divided into three sections, thus there are three flows corresponding to each section of the pipe (for more implementation details of this method see [9]).

Fig. 4 shows the flow rate response of the system when a single leak occurs at $t = 100s$ in the positions: a) $z_1 = 29.44$ and b) $z_1 = 117.76$ by both methods. It can be observed that the response of both models agrees (after modifications (16) and (17)). However, it is necessary to emphasize that the response of the OCM model (without modification) has better properties in a framework of leak detection, because the OCM model computes the flow in a punctual way as if there was a flowmeter in each node where the leak happens, that has registered the flow rate before the event and give the flow after at each leak point.

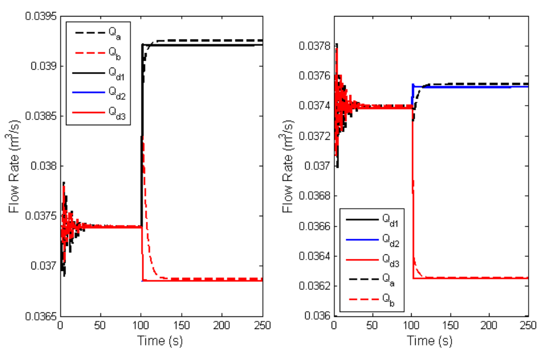

Fig. 4. Comparison between OMC and finite difference model simulation results

III. APPLICATION TO LEAK DETECTION

Let us here illustrate how the model previously presented can be used in a purpose of leak detection. To that end, the idea is to rely on an observer for a extended system including leak position and magnitude in the state variables in a similar way as in [20]. The observer will be designed on the basis of an extended Kalman filter (as in [21]).

Considering a nonlinear system represented by a state representation:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \quad (18)$$

where $x(t) \in \mathbb{R}^q$ is the state, $u(t) \in \mathbb{R}^p$ the input and $y(t) \in \mathbb{R}^m$ the output, an observer (18), can then be designed as follows:

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + K(t)[y(t) - h(\hat{x}(t))] \quad (19)$$

where the state estimate is denoted by $\hat{x}(t)$ and the observer gain $K(t)$ is a time-varying $q \times m$. To calculate this gain the following differential Ricatti equation matrix is considered (as in EKF):

$$\begin{aligned} \dot{\mathbf{P}}(t) &= (\mathbf{A}(t) + \alpha\mathbf{I})\mathbf{P}(t) + \mathbf{P}(t)(\mathbf{A}^{\mathrm{T}}(t) + \alpha\mathbf{I}) \\ &\quad - \mathbf{P}(t)\mathbf{C}^{\mathrm{T}}(t)\mathbf{R}^{-1}\mathbf{C}(t)\mathbf{P}(t) + \mathbf{Q} \end{aligned} \quad (20)$$

with

$$\mathbf{A}(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)), \mathbf{C}(t) = \frac{\partial h}{\partial x}(\hat{x}(t))$$

$$\mathbf{P}(0) = \mathbf{P}(0)^{\mathrm{T}} > \mathbf{0}, \mathbf{Q} = \mathbf{Q}^{\mathrm{T}} \geq \mathbf{0}, \mathbf{R} = \mathbf{R}^{\mathrm{T}} > \mathbf{0}$$

and a positive real number $\alpha > 0$. The observer gain is defined by

$$K(t) = \mathbf{P}(t)\mathbf{C}^{\mathrm{T}}(t)\mathbf{R}^{-1} \quad (21)$$

Using a model with three points of collocation obtained from (13) and (14) for a single leak (where the state and input vectors are given by $x = \begin{bmatrix} Q_1 & Q_2 & Q_3 & H_2 \end{bmatrix}^{\mathrm{T}}$ and $u = \begin{bmatrix} H_1 & H_3 \end{bmatrix}^{\mathrm{T}}$), and assuming that the unknown magnitude (represented by λ) is constant, as well as the unknown location (represented here by the second node z_2), the state model with two additional state variables, $x_5 = z_2$ and $x_6 = \lambda$ has the structure given by (22) below. From this model an observer (19) can be designed in order to detect the leak position and magnitude by direct estimation.

For the simulation, model (15) with five nodes is used as the system to be observed, the leak appears at $t = 100\ s$, the position leak is chosen $z_2 = 44.18\ m$, and the magnitude $\lambda = 1 \times 10^{-3}$. The initial conditions for the observer are given by equilibrium values corresponding to $u_1 = 7\ m$ and $u_2 = 1\ m$, while the initial condition for leak location is chosen $\hat{x}_5(0) = 70\ m$ and the leak magnitude $\hat{x}_6(0) = 0$.

The outputs considered to be measured are the flow rates of the pipe's ends, in this case because the model (15) is the observed system, the flows rates are Q_1 and Q_5.

For the observer $\mathbf{Q} = \mathbf{I}$, $\mathbf{R} = 0.0001\mathbf{I}$ and $\mathbf{P}(0) = \mathbf{I}$ while α is chosen so as to tune the observer performances, and $\mathbf{A}$ and $\mathbf{C}$ are de jacobian of (22).

$$\begin{aligned} \dot{x}_1 &= a_1(N'_{11}u_1 + N'_{12}x_6 + N'_{13}u_2 - \mu(x_1)^2) \\ \dot{x}_2 &= a_1(N'_{21}u_1 + N'_{22}x_6 + N'_{23}u_2 - \mu(x_2 + x_6\sqrt{x_4})^2) \\ \dot{x}_3 &= a_1(N'_{31}u_1 + N'_{32}x_6 + N'_{33}u_2 - \mu(x_3 + x_6\sqrt{x_4})^2) \\ \dot{x}_4 &= a_2(N'_{21}x_1 + N'_{22}(x_2 + x_6\sqrt{x_4}) \\ &\quad + N'_{23}(x_3 + x_6\sqrt{x_4})) \\ \dot{x}_5 &= 0 \\ \dot{x}_6 &= 0 \end{aligned} \quad (22)$$

The simulation results are given on Fig.5 and Fig.6. It can be appreciated how both position and magnitude have been detected by the observer. The convergence time of the observer is $t_c \approx 200\ s$. Can be mentioned that in this observer configuration, the estimation results are sensitive to initial values and observer tuning. In the same way the time of convergence depends of the tuning parameters.

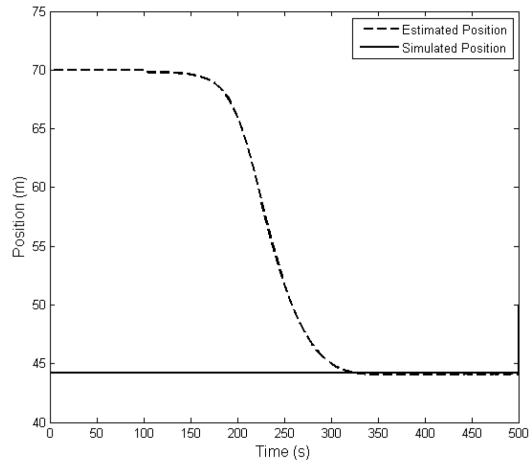

Fig. 5. Estimation of the leak position by the designed observer

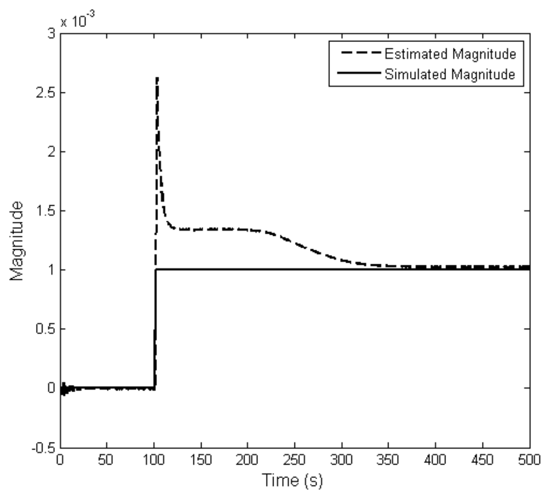

Fig. 6. Estimation of the leak magnitude by the designed observer

IV. CONCLUSIONS AND FUTURE WORKS

An orthogonal collocation approach has been proposed to get an approximation of water hammer equations. It has been shown how this approach allows to include the effect of possible leaks in specific collocation points which makes it of particular interest for leak detection. The proposed model has been compared with a more standard finite difference scheme, showing fairly similar responses. The interest of the proposed model for leak detection has been illustrated via an example of observer based leak detection.

The proposed detection scheme in that respect is mostly for an illustration purpose, and the design of some more sophisticated and accurate observers will be part of future studies, in particular with the purpose of multi-leak detection.

Finally some experimental validation of the model and its use for leak detection will also be further considered.

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