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LQ-Optimal Actuator Location and Norm Convergence of Riccati Operators

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Abstract— **In many problems governed by partial differential equations, there is freedom in the choice of actuator (and sensor) location. These locations should be chosen to optimize performance objectives. In practice, approximations are used in controller design and thus in selection of the actuator locations. The optimal cost and location of the approximating sequence should converge to the exact optimal cost and location. In this work conditions for this convergence are given in the case of linear quadratic control.**

I. Introduction

The location of actuators in systems governed by partial differential equations can often be chosen. An important application is active control of interior noise, particularly in automobiles and in aircraft. Performance is known to depend strongly on actuator location. For example, in [15] achievable noise reduction in a duct was shown to vary strongly with actuator location. Actuators should therefore be located at positions that optimize performance. The actuator location problem has been considered by many researchers; see for instance [6], [12], [13].

In this paper we are concerned with linear quadratic regulators. The optimal control is calculated via the solution Π to an algebraic Riccati equation. In practice, the equations for the optimal control cannot be solved and the the control is calculated using an approximation Π_n to Π. Criterion for optimality of the calculated optimal locations with reference to the full partial differential equation model need to be obtained. Conditions for strong convergence of the approximations Π_n to Π, with fixed actuator location, are known; see for instance, [1], [9]–[11].

Determining the optimal actuator location for the optimal control introduces an additional layer of numerical calculation. An example in this paper shows that strong convergence of Π_n is not sufficient to obtain correct results: The optimal cost and corresponding actuator locations of the approximating sequence may not converge to the exact cost and location. The sequence of Riccati operators needs to converge uniformly to the exact operator. For general semigroups, conditions for uniform convergence in the Hilbert-Schmidt norm of Π_n to Π are given in [5], but the approximation space needs to lie in $D(A)$. This assumption is not satisfied by many finite-element type approximation schemes.

It will be shown that compactness of the input operator B and cost C is sufficient to ensure uniform convergence of the approximate Riccati operators and furthermore, the optimal cost is continuous with respect to the actuator location. This leads to the main result of this paper; conditions under which the approximating optimal performance converges to the optimal performance, along with a corresponding sequence of actuator locations. The results are illustrated with an example.

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II. Calculation of Linear-Quadratic Control

Consider systems described by

$$\frac{dz}{dt} = Az(t) + Bu(t), \qquad z(0) = z_0 \tag{2.1}$$

on a Hilbert space $\mathcal{H}$ where A with domain $D(A)$ generates a strongly continuous semigroup $S(t)$ on $\mathcal{H}$ and $B \in \mathcal{L}(U, \mathcal{H})$ for some Hilbert space U.

The linear-quadratic (LQ) controller design objective is to find a control $u(t)$ so that the cost functional

$$J(u, z_0) = \int_0^\infty \langle Cz(t), Cz(t)\rangle + \langle u(t), Ru(t)\rangle dt \tag{2.2}$$

is minimized where $R \in \mathcal{L}(U, U)$ is a self-adjoint positive definite operator weighting the control, $C \in \mathcal{L}(\mathcal{H}, Y)$ (with Hilbert space Y) weights the state, and $z(t)$ is determined by (2.1).

Definition 2.1: The system (2.1) with cost (2.2) is optimizable if for every $z_0 \in \mathcal{H}$ there exists $u \in L_2(0, \infty; U)$ such that the cost is finite.

Definition 2.2: The pair (C, A) is detectable if there exists $F \in \mathcal{L}(Y, \mathcal{H})$ such that $A - FC$ generates an exponentially stable semigroup.

Theorem 2.3: [4, Thm 6.2.4, 6.2.7] If (2.1) with cost (2.2) is optimizable and detectable, then the cost has a minimum for every $z_0 \in \mathcal{H}$. Furthermore, there exists a self-adjoint non-negative operator $\Pi \in \mathcal{L}(H, H)$ such that

$$\min_{u \in L_2(0,\infty;U)} J(u, z_0) = \langle z_0, \Pi z_0\rangle.$$

The operator Π is the unique non-negative solution to the operator equation

$$\langle Az_1, \Pi z_2\rangle + \langle \Pi z_1, Az_2\rangle + \langle Cz_1, Cz_2\rangle - \langle B^*\Pi z_1, R^{-1}B^*\Pi z_1\rangle = 0 \tag{2.3}$$

for all $z_1, z_2 \in D(A)$. Defining $K = R^{-1}B^*\Pi$, the corresponding optimal control is $u = -Kz(t)$ and $A - BK$ generates an exponentially stable semigroup.

Definition 2.4: The pair (A, B) is stabilizable if there exists $K \in \mathcal{L}(U, \mathcal{H})$ such that $A - BK$ generates an exponentially stable semigroup.

It is straightforward to show that the assumption of optimizability in Theorem 2.3 is equivalent to stabilizability.

In practice, the operator equation (2.3) cannot generally be solved and the control is calculated using an approximation. Let $\mathcal{H}_n$ be a finite-dimensional subspace of $\mathcal{H}$ and P_n be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$. The space $\mathcal{H}_n$ is equipped with the norm inherited from $\mathcal{H}$. Consider a sequence of operators $A_n \in \mathcal{L}(\mathcal{H}_n, \mathcal{H}_n)$, $\quad B_n \in \mathcal{L}(U, \mathcal{H}_n)$. This leads to a sequence of approximations

$$\frac{dz}{dt} = A_n z(t) + B_n u(t), \qquad z(0) = z_{n0} = P_n z_0 \quad (2.4)$$

with cost functional

$$J(u, z_0) = \int_0^\infty \langle Cz(t), Cz(t)\rangle + \langle u(t), Ru(t)\rangle dt \quad (2.5)$$

where $C_n = C|_{\mathcal{H}_n}$. If (A_n, B_n) is stabilizable and (A_n, C_n) is detectable, then the cost functional has the minimum cost $\langle P_n z_0, \Pi_n P_n z_0\rangle$ where Π_n is the unique non-negative solution to the algebraic Riccati equation

$$A_n^* \Pi_n + \Pi_n A_n - \Pi_n B_n R^{-1} B_n^* \Pi_n + C_n^* C_n = 0 \quad (2.6)$$

on the finite-dimensional space $\mathcal{H}_n$.

The feedback control $K_n = R^{-1} B_n^* \Pi_n$, is used to control the original system (2.1). Assumptions that guarantee that Π_n converges to Π in some sense are required in order for this approach to be valid.

There have been many papers written describing conditions under which approximations lead to approximating controls that converge to the control for the original infinite-dimensional system, see for instance, [1], [9]–[11]. The following set of assumptions is standard.

(A1) Let $S_n(t)$ indicate the semigroup generated by A_n. For each $z \in \mathcal{H}$, we have

$$(i) \quad \|S_n(t)P_n z - S(t)z\| \to 0,$$

$$(ii) \quad \|S_n^*(t)P_n z - S^*(t)z\| \to 0$$

uniformly in t on bounded intervals.

(A2) (i) For each $u \in U$, $\|B_n u - Bu\| \to 0$, and for each $z \in \mathcal{H}$, $\|B_n^* P_n z - B^* z\| \to 0$,

(ii) For each $z \in \mathcal{H}$, $\|C_n P_n z - Cz\| \to 0$, and for each $y \in Y$, $\|C_n^* y - C^* y\| \to 0$.

(A3) (i) The family of pairs (A_n, B_n) is uniformly exponentially stabilizable, that is, there exists a uniformly bounded sequence of operators $K_n \in \mathcal{L}(\mathcal{H}_n, U)$ such that

$$\left\| e^{(A_n - B_n K_n)t} P^n \, z \right\| \leq M_1 \, e^{-\omega_1 t} \, |z|$$

for some positive constants $M_1 \geq 1$ and ω_1.

(ii) The family of pairs (A_n, C_n) is uniformly exponentially detectable, that is, there exists a uniformly bounded sequence of operators $F_n \in \mathcal{L}(Y, \mathcal{H}_n)$ such that

$$\left\| e^{(A_n - F_n C_n)t} P_n \right\| \leq M_2 \, e^{-\omega_2 t}, \quad t \geq 0,$$

for some positive constants $M_2 \geq 1$ and ω_2.

Common approximation schemes such as modal approximations and finite-elements typically satisfy these assumptions. See, for instance, [1], [11], [14].

Assumption (A1i) is required for convergence of initial conditions. Assumption (A1)(i) is often satisfied by ensuring that the conditions of the Trotter-Kato Theorem hold , see for instance, [16, Chap. 3, Thm. 4.2]. The convergence (A1)(ii) of the adjoint semigroup sequence $S_n^*(t)$ is required for the strong convergence of the approximating Riccati operators Π_n. A counter-example may be found in [2]. Note that assumption (A1) implies that $P_n z \to z$ for all $z \in \mathcal{H}$. If U and Y are finite-dimensional, as is usual, the strong convergence $P_n z \to z$, and definitions of $B_n = P_n B$ and $C_n = C|_{\mathcal{H}_n}$ imply that (A2) is satisfied. Assumption (A3) is standard in the literature on convergence of approximating controls. It is not known to what extent assumption (A3) is necessary.

Theorem 2.5: [1, Thm. 6.9], [10, Thm. 2.1, Cor. 2.2] Assume that (A1)-(A3) are satisfied and that (A, B) is stabilizable and (A, C) is detectable. Then for each n, the finite-dimensional ARE (2.6) has a unique nonnegative solution Π_n with $\sup \|\Pi_n\| < \infty$. There exists constants $M_2 \geq 1$, $\alpha_2 > 0$, independent of n, such that

$$\|e^{(A_n - B_n R^{-1} B_n^* \Pi_n)t}\| \leq M_2 e^{-\alpha_2 t}.$$

Furthermore, for all $z \in \mathcal{H}$,

$$\lim_{n\to\infty} \|\Pi_n P_n z - \Pi z\| = 0$$

and

$$\lim_{n\to\infty} \|K_n P_n z - Kz\| = 0.$$

The above result provides sufficient conditions for strong convergence of the Riccati operators. However, as shall be shown in the next section, convergence of optimal actuator locations obtained using approximations requires uniform convergence of the Riccati operators.

The first point to consider is that since $\mathcal{H}_n$ is finite-dimensional, Π_n has finite rank, and therefore Π must be a compact operator in order for uniform convergence to occur, regardless of the choice of approximation method. The following simple example illustrates that Π is not always a compact operator.

Example 2.6: [4] Consider (2.1) with A, B, C such that $A^* = -A$ and $C = B^*$. Then $\Pi = I$ is a solution to the ARE. This operator is not compact on any infinite-dimensional Hilbert space.

Conditions additional to those required for existence of an optimal control need to be imposed in order to guarantee that Π is compact and hence can be approximated by a finite-rank operator.

Conditions for systems where the semigroup is analytic can be found in [11]. If the input and output spaces are finite-dimensional, and (A, B) is stabilizable, then Π is a Hilbert-Schmidt operator [3, Thm. 4], a special type of compact operator. The following result assumes that B and C are compact in order to show that Π is compact.

Theorem 2.7: Assume that (A, B) is stabilizable , (A, C) is detectable and that B and C are both compact. Then Π is compact.

Proof: The assumptions guarantee that the linear-quadratic control problem (2.2) has a solution Π and that $A - BR^{-1}B^*\Pi$ generates an exponentially stable semigroup, $S_K(t)$. This implies that, for any $z_0 \in \mathcal{H}$,

$$\langle z_0, \Pi z_0\rangle = \int_0^\infty \langle Cz(t), Cz(t)\rangle + \langle u(t), Ru(t)\rangle dt$$

with $z(t) = S_K(t)z_0$, $u(t) = -KS_K(t)z_0$. Rewriting this equation and using the fact that z_0 is arbitrary,

$$\Pi = \int_0^\infty S_K(t)^* \left[C^*C + \Pi^*BR^{-1}B\Pi\right] S_K(t)dt.$$

Compactness of B and C implies that

$$D_T = \int_0^T S_K(t)^* \left[C^*C + \Pi^*BR^{-1}\Pi\right] S_K(t)dt$$

is compact for all $T \geq 0$. Since there is $M \geq 0$, $\alpha > 0$ such that $\|S_K(t)\| \leq Me^{-\alpha t}$, D_T converges uniformly to Π and Π is also compact. □

Example 2.6 fails to satisfy the assumptions of the above theorem since a system with $A^* = -A$ cannot be stabilized by a non-compact feedback [8].

Extension of the proof in [9] to general infinite-dimensional systems leads to the following result on uniform convergence. Apart from compactness of B and C, required to ensure that Π is compact, the only additional assumption is that $\lim_{n\to\infty} \|B_n - P_nB\| = 0$. This is obviously satisfied by the typical case where $B_n = P_nB$. The more general definition of B_n is needed later in this paper.

Theorem 2.8: Assume that (A, B) is stabilizable and (A, C) is detectable, and that B and C are both compact. Let (A_n, B_n, C_n) be a sequence of approximations to (A, B, C) such that assumptions (A1)-(A3) are satisfied and also $\lim_{n\to\infty} \|B_n - P_nB\| = 0$. Then, the minimal non-negative solution Π_n to (2.6) converges uniformly to the non-negative solution Π to (2.3).

Proof: As in the proof of Theorem 2.7, write

$$\Pi = \int_0^\infty S_K(t)^* \left[C^*C + \Pi^*BR^{-1}B\Pi\right] S_K(t)dt$$

and write Π_n similarly. Define $K = R^{-1}B^*\Pi$, $M = C^*C + K^*RK$ and define similarly K_n, M_n. We have

$$\begin{aligned} \|K_nP_n - K\| &\leq \|R^{-1}\|\|B_n^*\Pi_nP_n - B^*\Pi)\| \\ &\leq \|R^{-1}\| \left(\|(B_n^* - B^*P_n^*)\Pi_nP_n\| \ldots \right. \\ &\quad \left. + \|B^*(P_n^*\Pi_nP_n - \Pi)\|\right) \\ &= \|R^{-1}\| \left(\|B_n - P_nB\|\|\Pi_n\| \ldots \right. \\ &\quad \left. + \|(P_n^*\Pi_nP_n - \Pi)B\|\right). \end{aligned}$$

Since Π_n converges strongly to Π, and B is compact, $(P_n^*\Pi_nP_n - \Pi)B$ converges uniformly to zero. Since by assumption $\|B_n - P_nB\|$ converges to zero, K_n converges uniformly to K. Since C is compact, $C_n = CP_n^*$ converges uniformly to C also and we obtain that M_n converges uniformly to M.

We can write the error in $\Pi_n - \Pi$ as

$$\begin{aligned} \Pi_nP_n - \Pi = &\int_0^\infty S_{Kn}(t)^* \left[M_n - P_nM\right] S_{Kn}(t)P_ndt \ldots \\ + &\int_0^\infty \left[S_{Kn}(t)^*P_n - S_K^*(t)\right] MS_{Kn}(t)P_ndt \ldots \\ + &\int_0^\infty S_K^*(t)M \left[S_{Kn}(t)P_n - S_K(t)\right] dt. \end{aligned}$$

Uniform convergence of M_n to M and uniform exponential stability of S_{Kn} implies uniform convergence to zero of the first term. Compactness of M and strong convergence of S_{Kn} to S_K, uniformly on bounded intervals of time, leads to

$$\lim_{n\to\infty} \|(S_{Kn}(t)^*P_n - S_K^*(t))\, M\| = 0$$

where the convergence is uniform on bounded intervals of time. This, together with uniform exponential stability implies uniform convergence to zero of the second term. Similarly, the third term converges uniformly to zero. Thus, Π_n converges uniformly to Π. □

III. Optimal Actuator Location for Linear-Quadratic Control

Consider now the situation where there are m actuators with locations that can be varied over some compact set $\Omega \subset R^q$. Parametrize the actuator locations by r and indicate the corresponding input operator by $B(r)$. Note that r is a vector of length m with components in Ω so that r varies over a space denoted by Ω^m. For each r we have an optimal control problem (2.2) which we indicate by $J^r(u, z_0)$ with corresponding optimal cost $\langle\Pi(r)z_0, z_0\rangle$. We wish to choose the actuator location in order to minimize the response to the worst choice of initial condition. In other words, choose r in order to minimize

$$\max_{\substack{z_0\in\mathcal{H}\\ \|z_0\|=1}} \min_{u\in L_2(0,\infty;U)} J^r(u, z_o) = \|\Pi(r)\|.$$

The performance for a particular r is $\mu(r) = \|\Pi(r)\|$ and the optimal performance

$$\hat{\mu} = \inf_{r\in\Omega^m} \|\Pi(r)\|.$$

For the sequence of approximating problems $(A_n, B_n(r), C_n)$ define similarly $J_n^r(u, z_0)$, $\mu_n(r)$ and $\hat{\mu}_n$.

Approximations must generally be used to calculate the optimal actuator location. As the following example shows, strong convergence of Π_n to Π is not sufficient to guarantee that $\lim_{n\to\infty} \hat{\mu}_n = \hat{\mu}$.

Example 3.1: Weakly Damped Beam Consider a simply supported Euler-Bernoulli beam and let $w(r, t)$ denote the deflection of the beam from its rigid body motion at time t and position r. The deflection is controlled by applying a force $u(t)$ at a point. The control is a force centered on the point r with width Δ. If we normalize the variables and include viscous damping with parameter $c_d = .1$, we obtain the partial differential equation

$$\frac{\partial^2 w}{\partial t^2} + c_d \frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} = b_r u(t), \qquad t \geq 0, 0 < x < 1,$$

where, letting $\Delta = 0.001$ indicate the width of the actuator and r its location,

$$b_r(x) = \begin{cases} 1/\Delta, & |r - x| < \frac{\Delta}{2} \\ 0, & |r - x| \geq \frac{\Delta}{2} \end{cases}.$$

The boundary conditions are

$$\begin{aligned} w(0,t) = 0, \quad w''(0,t) = 0, \\ w(L,t) = 0, \quad w''(L,t) = 0. \end{aligned} \tag{3.7}$$

Define the state-space $\mathcal{H} = \mathcal{H}_0^2(0,1) \times \mathcal{L}_2(0,1)$ with state $z(t) = (w(\cdot,t), \frac{\partial}{\partial t} w(\cdot,t))$. A state-space formulation of the above partial differential equation problem is

$$\frac{d}{dt} x(t) = Ax(t) + Bu(t),$$

where

$$A = \begin{bmatrix} 0 & I \\ -\frac{d^4}{dr^4} & -c_d I \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ b_r \end{bmatrix},$$

with domain

$$D(A) = \{(\phi, \psi) \in \mathcal{H}_0^2(0,1) \times \mathcal{H}_0^2(0,1) \text{ with } \phi'' \in \mathcal{H}_0^2(0,1)\}.$$

An obvious choice of weight for the state is $C = I$. Since there is only one control, choose control weight $R = 1$.

Let $\phi_i(x)$ indicate the eigenfunctions of $\frac{\partial^4 w}{\partial x^4}$ with boundary conditions (3.7). Defining $\mathcal{X}_n$ to be the span of ϕ_i, $i = 1..n$, we choose $\mathcal{H}_n = \mathcal{X}_n \times \mathcal{X}_n$. This type of approximation satisfies all the assumptions of Theorem 2.5 and the sequence of solutions Π_n to the corresponding finite-dimensional ARE's converge strongly to the exact solution Π. However, as shown in Figures 1 and 2 this does not imply convergence of the optimal actuator locations, or of the corresponding actuator locations.

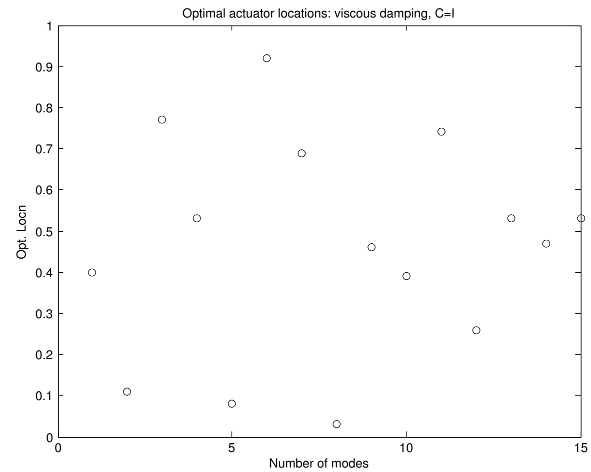

Fig. 1. Optimal actuator location, Viscously damped beam, $C = I$

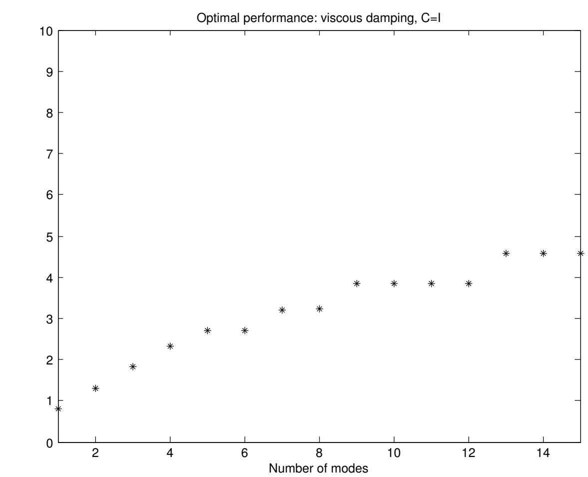

Fig. 2. Performance at optimal location, Viscously damped beam, $C = I$

Conditions under which the optimal cost is continuous with respect to the actuator location will be given. The following theorem is needed. It is a special case of [7, Thm. 5.3] where it is stated with respect to time-varying systems, with possibly varying weights C and R. This theorem applies to the situation where the perturbed system is defined on the same state space, possibly infinite-dimensional, as the original system.

Theorem 3.2: For Hilbert spaces U, Y and $\mathcal{Z}$ with $C \in \mathcal{L}(\mathcal{Z}, Y)$ and positive definite $R \in \mathcal{L}(U,U)$ consider a series of optimal control problems

$$J(u, z_0) = \int_0^\infty \langle Cz(t), Cz(t)\rangle + \langle u(t), Ru(t)\rangle dt \tag{3.8}$$

governed by

$$\dot{z}(t) = F_i z(t) + G_i u(t),$$

where F_i generates a C_0-semigroup $S(t)$ on $\mathcal{Z}$ and $G_i \in \mathcal{L}(U, \mathcal{Z})$. Assume that there is a closed operator F on $\mathcal{Z}$ generating a semigroup $S(t)$ on $\mathcal{Z}$ and $G \in \mathcal{L}(U, \mathcal{Z})$ such that

1) for each $z \in \mathcal{Z}$, we have

$$(i) \|S_i(t)z - S(t)z\| \to 0$$

$$(ii) \quad \|S_i^*(t)z - S^*(t)z\| \to 0,$$

uniformly in t on bounded intervals, and

2) $\|G_i - G\| \to 0$.

Assume that (F_i, G_i) are stabilizable and (F_i, C) are detectable. Let Π_i be the minimal non-negative solution of the ARE for the i^{th} problem and let S_{Ki} be the semigroup generated by $A_i - G_i R^{-1} G_i^* \Pi_i$. If there exists $M > 0, \beta > 0$ such that

$$\|S_{Ki}(t)\| \leq M e^{-\beta t}, \quad t \geq 0,$$

and M_1 such that

$$\|\Pi_i\| < M_1$$

then there exists a non-negative solution Π to the optimal control problem for (F,G) such that for all $z \in \mathcal{Z}$,

$$\lim_{i\to\infty} \Pi_i z = \Pi z.$$

Letting $S_K(t)$ indicate the semigroup generated by $A - GR^{-1}G^*\Pi$,

$$\|S_K(t)\| \le Me^{-\beta t},$$

and for all $z \in \mathcal{Z}$,

$$\lim_{i\to\infty} S_{Ki}(t)z = S_K(t)z,$$

uniformly on bounded intervals of time.

This general result applies both to perturbations (A_i, B_i) of the original system (A,B) and also to perturbations (A_{ni}, B_{ni}) of an approximating system (A_n, B_n). The following theorem and corollary apply to both the original system and to an approximating system.

Theorem 3.3: Let $B(r) \in \mathcal{L}(U, \mathcal{H})$, $r \in \Omega^m$, be a family of compact input operators such that for any $r_0 \in \Omega^m$,

$$\lim_{r\to r_0} \|B(r) - B(r_0)\| = 0.$$

Assume that $(A, B(r))$ are all stabilizable and that (A,C) is detectable where $C \in \mathcal{L}(\mathcal{H}, Y)$ is a compact operator. Then the corresponding Riccati operators $\Pi(r)$ are continuous functions of r in the operator norm:

$$\lim_{r\to r_0} \|\Pi(r) - \Pi(r_0)\| = 0.$$

Proof: Consider $(A, B(r_0))$ at some arbitrary point $r_0 \in \Omega$. Choose some K so that $A - B(r_0)K$ generates an exponentially stable semigroup with bound $Me^{-\alpha t}$, where $M \ge 1$, $\alpha > 0$. Let δ be such that $A - B(r)K$ generates an exponentially stable semigroup with bound $Me^{-\frac{\alpha}{2}t}$ for all $\|B(r) - B(r_0)\| < \delta$. There is $\epsilon > 0$ such that for all $|r - r_0| < \epsilon$, $\|B(r) - B(r_0)\| < \delta$. We thus have a sequence of uniformly exponentially stabilizable systems $(A, B(r))$. Let $T_r(t)$ indicate the semigroup generated by $A - B(r)K$. For any $z_0 \in \mathcal{H}$,

$$\begin{aligned} \langle \Pi(r)z_0, z_0\rangle &\le J(-Kz(t), z_0) \\ &= \int_0^\infty \|CT_r(t)z_0\|^2 + \|R^{1/2}KT_r(t)z_0\|^2 dt \\ &\le c\|z_0\|^2 \end{aligned}$$

for some constant $c > 0$. This implies that $\|\Pi(r)\| \le c$. This, and Datko's Theorem implies that the semigroups $S_r(t)$ generated by $A - B(r)R^{-1}B(r)^*\Pi(r)$ are bounded by $M_2e^{-\beta t}$ for some $M_2 \ge 1$, $\alpha > 0$. (See the proof of Theorem 2.1 in [10] for details.) Thus, any sequence $(A, B(r_i))$ with $r_i \to r_0$ satisfies the assumptions of Theorem 3.2 and hence for all $z \in \mathcal{H}$

$$\lim_{r\to r_0} \|\Pi(r)z - \Pi(r_0)z\| = 0.$$

Furthermore, letting $S_0(t)$ indicate the semigroup generated by $A - B(r_0)R^{-1}B(r_0)^*\Pi(r_0)$, $\|S_0(t)\| \le M_2e^{-\beta t}$ and $S_r(t)$ converge strongly to $S_0(t)$, uniformly on bounded intervals of time. As in the proof of Theorem 2.8, we can then show that

$$\lim_{r\to r_0} \|\Pi(r) - \Pi(r_0)\| = 0. \quad \square$$

The following result now follows immediately from the compactness of Ω.

Corollary 3.4: There exists an optimal actuator location $\hat{r}$ such that

$$\|\Pi(\hat{r})\| = \inf_{r\in\Omega^m} \|\Pi(r)\| = \hat{\mu},$$

and similarly for each n there exists $\hat{r}_n$ such that

$$\|\Pi_n(\hat{r}_n)\| = \inf_{r\in\Omega^m} \|\Pi_n(r)\| = \hat{\mu}_n.$$

It will now be shown that compactness of C and B leads to convergence of the optimal cost and of a corresponding sequence of optimal actuator locations.

Theorem 3.5: Assume a family of control systems depending on actuator location such that

1) $(A, B(r))$ are stabilizable and (A,C) are detectable,
2) $B(r)$, $r \in \Omega^m$, is compact and such that for any $r_0 \in \Omega$,
$$\lim_{r\to r_0} \|B(r) - B(r_0)\| = 0,$$
3) C is compact.

Choose some approximation scheme such that assumptions (A1)-(A3) are satisfied for each $(A, B(r), C)$ with $B_n(r) = P_nB(r)$, $C_n = C|_{\mathcal{H}_n}$. Letting $\hat{r}$ be an optimal actuator location for $(A, B(r), C)$ with optimal cost $\hat{\mu}$ and defining similarly $\hat{r}_n$, $\hat{\mu}_n$, it follows that

$$\hat{\mu} = \lim_{n\to\infty} \hat{\mu}_n,$$

and there exists a subsequence $\{\hat{r}_m\}$ of $\{\hat{r}_n\}$ such that

$$\hat{\mu} = \lim_{m\to\infty} \|\Pi(\hat{r}_m)\|.$$

Proof:

$$\begin{aligned} \hat{\mu}_n &= \inf_{r\in\Omega^m} \|\Pi_n(r)\| \\ &\le \|\Pi_n(\hat{r})\| \\ &\le \|\Pi_n(\hat{r}) - \Pi(\hat{r})\| + \|\Pi(\hat{r})\| \\ &= \|\Pi_n(\hat{r}) - \Pi(\hat{r})\| + \hat{\mu}. \end{aligned}$$

Since $\|\Pi_n(\hat{r}) - \Pi(\hat{r})\| \to 0$ (Thm. 2.8),

$$\limsup \hat{\mu}_n \le \hat{\mu}.$$

It remains only to show that

$$\liminf \hat{\mu}_n \ge \hat{\mu}.$$

To this end, choose a subsequence $\mu_m \to \liminf \hat{\mu}_n$, with corresponding actuator locations r_m. Since the sequence $\{r_m\}$ lies in a compact set it has a convergent subsequence, also denoted $\{r_m\}$, with limit $\underline{r}$. Since $B_m = P_mB$,

$$\|B_m(r_m) - P_mB(\underline{r})\| = \|P_mB(r_m) - P_mB(\underline{r})\|.$$

Compactness of B along with uniform convergence of $B(r_m)$ to $B(\underline{r})$ and convergence of r_m to $\underline{r}$ implies that $\|B_m(r_m) - P_mB(\underline{r})\|$ converges to zero. By assumption

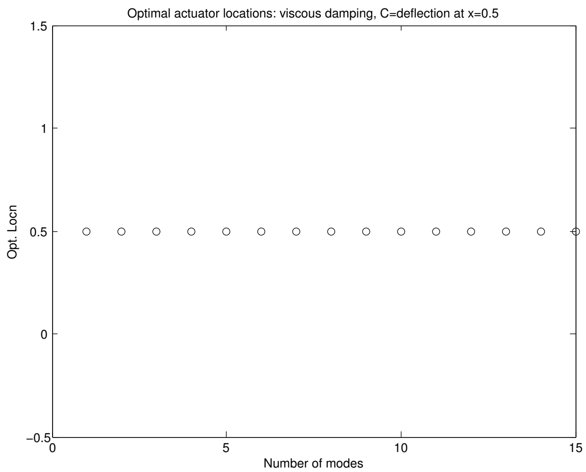

Fig. 3. Optimal actuator location, Viscously damped beam, $C =$ deflection at $x = 0.5$

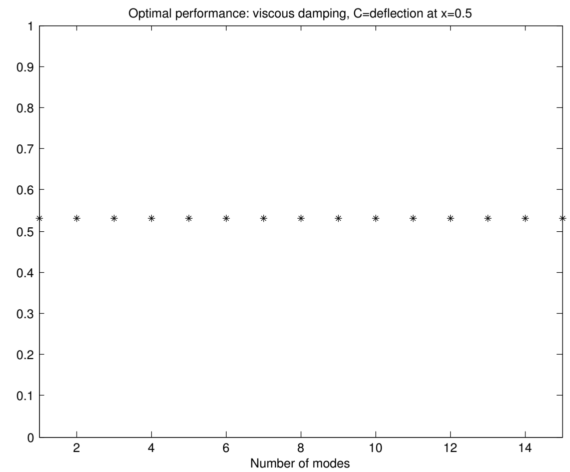

Fig. 4. Performance, Viscously damped beam, $C =$ deflection at $x = 0.5$

(A3), there is a uniformly bounded sequence $K_m(\underline{r}) \in \mathcal{L}(\mathcal{H}, U)$ such that $A_m - B_m(\underline{r})K_m(\underline{r})$ generate semigroups bounded by $Me^{-\omega_1 t}$ for some $M > 0, \omega_1 > 0$. For some $\epsilon < \omega_1/M$, choose N large enough that $\|B_m(r_m) - B_m(\underline{r})\| < \epsilon$ for $m > N$. Then for all $m > N$, $A_m - B_m(r_m)K_m(\underline{r})$ generates an exponentially stable C_0-semigroup with bound $Me^{(-\omega_1 + M\epsilon)t}$. The assumptions of Theorem 2.8 are satisfied by the sequence $(A_m, B_m(r_m), C_m)$ and so $\|\Pi_m(r_m) - \Pi(\underline{r})\| \to 0$. Thus,

$$\begin{aligned} \liminf \hat{\mu}_n &= \lim_{m\to\infty} \mu_m & (3.9)\\ &= \lim_{m\to\infty} \|\Pi_m(r_m)\| \\ &= \|\Pi(\underline{r})\| & (3.10)\\ &\geq \hat{\mu}. \end{aligned}$$

Thus, $\liminf \hat{\mu}_n \geq \hat{\mu}$ and so $\lim \hat{\mu}_n = \hat{\mu}$ as required.

Since $\hat{\mu} = \lim \hat{\mu}_n = \liminf \hat{\mu}_n$, (3.10) implies that

$$\begin{aligned} \hat{\mu} &= \liminf \mu_n \\ &= \|\Pi(\underline{r})\| \\ &= \lim_{m\to\infty} \|\Pi(\hat{r}_m)\|. \end{aligned}$$

where the latter equality follows from continuity of performance with respect to acuator location (Theorem 3.3). Thus, as was to be shown, a sequence of approximating actuator locations yield performance arbitrarily close to optimal. □

Example 3.6: Viscously Damped Beam, cont. Consider the same control system as above, except that now instead of trying to minimize the norm of the entire state, $C = I$, we consider only the position at the midpoint. The weight $Cz = w(0.5)$ where w is the first component of the state z. Both B and C are compact operators on $\mathcal{H}$. Using the same modal approximations as before, we obtain the sequence of optimal actuator locations shown in Figure 3 with corresponding performance shown in Figure 4. As predicted by the theory, the optimal locations and performance converge.

REFERENCES

[1] H.T. Banks and K. Kunisch. The linear regulator problem for parabolic systems. *SIAM J. Control and Optim.*, 22(5):684–698, 1984.

[2] J.A. Burns, K. Ito, and G.Propst. On non-convergence of adjoint semigroups for control systems with delays. *SIAM J. Control and Optim.*, 26(6):1442–1454, 1988.

[3] F. M. Callier, L. Dumortier, and J. Winkin. On the nonnegative self-adjoint solutions of the operator Riccati equation for infinite dimensional systems. *Integral Equations and Operator Theory*, 22:162–195, 1995.

[4] R. F. Curtain and H. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer Verlag, Berlin, 1995.

[5] A. De Santis, A. Germani, and L. Jetto. Approximation of the algebraic Riccati equation in the Hilbert space of Hilbert-Schmidt operators. *SIAM J. Control Optim.*, 31(4):847–874, 1993.

[6] F. Fahroo and M. A. Demetriou. Optimal actuator/sensor location for active noise regulator and tracking control problems. *J. Comp. Appl. Mech.*, 114:137–158, 2000.

[7] J.S. Gibson. The Riccati integral equations for optimal control problems on Hilbert spaces. *SIAM J. Control and Optim.*, 17(4):637–665, 1979.

[8] J.S. Gibson. A note on stabilization of infinite-dimensional linear oscillators by compact linear feedback. *SIAM J. Control and Optim.*, 18(3):311–316, 1980.

[9] J.S. Gibson. Linear-quadratic optimal control of hereditary differential systems: infinite dimensional Riccati equations and numerical approximations. *SIAM J. Control and Optim.*, 21(1):95–139, 1983.

[10] K. Ito. Strong convergence and convergence rates of approximating solutions for algebraic Riccati equations in Hilbert spaces. In W. Schappacher F. Kappel, K. Kunisch, editor, *Distributed Parameter Systems*. Springer-Verlag, 1987.

[11] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, volume I. Cambridge University Press, 2000.

[12] Y. Lou and P. Christofides. Optimal actuator/sensor placement for nonlinear control of the Kuramoto-Sivashinksy equation. *IEEE Trans. Cont. Sys. Tech.*, 11(5):737–745, 2003.

[13] M. Moallem, M.R. Kermani, R.V. Patel, and M. Ostojic. Flexure control of a positioning system using piezoelectric transducers. *IEEE Trans. Cont. Sys. Tech.*, 12(5):757–752, 2004.

[14] K.A. Morris. Design of finite-dimensional controllers for infinite-dimensional systems by approximation. *Jour. of Mathematical Systems, Estimation and Control*, 4(1-30), 1994.

[15] K.A. Morris. Noise reduction achievable by point control. *ASME Journal on Dynamic Systems, Measurement and Control*, 120(2):216–223, 1998.

[16] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Number 44. Springer-Verlag, 1983.