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Fast computation of frequency response functions for a class of nonlinear systems

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Abstract— **One of the important and efficient tools in system analysis is the analysis of responses to harmonic excitations. For linear systems the information on such responses is contained in the frequency response functions, which can be computed analytically. For nonlinear systems there may be even no periodic response to a periodic excitation. Even if such a periodic response exists and is unique, its computation is, in general, a computationally expensive task. In this paper we present a fast method for computing periodic responses to periodic excitations for a class of nonlinear systems. The method allows one to efficiently compute the responses for harmonic excitations corresponding to a grid of excitation frequencies and amplitudes. The results are illustrated by application to a flexible beam with one-sided stiffness subject to harmonic excitation.**

I. INTRODUCTION

A common way to analyze the behavior of a linear (closed-loop) dynamical system is to investigate its response to harmonic excitations. For linear systems, the information on responses to harmonic excitations is contained in frequency response functions, which can be computed analytically. These frequency response functions serve as important analysis and design tools allowing one to quantify frequency-dependent steady-state characteristics of forced linear systems. In the context of control, they allow one to quantify the sensitivity of the closed-loop system to measurement noise and external perturbations and its tracking properties at various frequencies. These characteristics are essential for many control applications, see e.g. [1].

There are many problems on controller design and system analysis for nonlinear (control) systems, where it is important to evaluate quantitative characteristics of steady-state responses to harmonic excitations at various frequencies and amplitudes. An engineering field in which this question arises frequently is that of the passive/active vibration suppression in mechanical structures with one-sided flexibilities (an example of such a system is studied in detail in Section VII). These types of systems are commonly encountered in practice, e.g. think of tower cranes, suspension bridges [2], snubbers on solar panels on satellites [3], floating platforms for oil exploration [4], safety stops in car suspensions, etc. Very often these systems are subject to exogenous disturbances that induce undesirable vibrations, which, in turn, may cause damage to the mechanical structure and may lead to inferior system's performance. As a consequence, measures to reduce these vibrations, such as active control, are of great importance. To support a control design aiming at superior disturbance attenuation properties, the closed-loop responses to exogenous disturbances need to be assessed. It is a common practice to evaluate quantitative characteristics of responses to harmonic disturbances, since many disturbances are either nearly harmonic, think of engine-induced periodic vibrations in vehicles [5], or can be approximated as such. It is exactly in the scope of such performance analysis (where responses to a range of excitation amplitudes, excitation frequencies and control gain settings need to be evaluated) that tools for fast computation or evaluation of the periodic responses of a nonlinear system are imperative.

A Lyapunov approach to estimating quantitative characteristics of steady-state responses to oscillatory excitations generated by an exosystem is presented in [6]. For linear differential inclusions these estimates can be found by solving certain linear matrix inequalities. Another approach to quantitative analysis of periodic responses to periodic excitations for a class of nonlinear systems is presented in [7], [8]. In these papers the machinery of integral quadratic constraints is used to identify conditions under which a nonlinear system exhibits a unique periodic response (not necessarily stable) to a periodic excitation. Some quantitative characteristics of such periodic responses are then estimated. For the class of convergent systems [9], [10], periodic responses to periodic excitations are unique and globally asymptotically stable. Moreover, for harmonic excitations of all frequencies and amplitudes, these responses are uniquely characterized by one function, which can be found by solving certain partial differential equation (PDE), [11]. From this function one can determine all the necessary quantitative data on the corresponding periodic responses. Yet, solving this PDE is a nontrivial task by itself. Alternatively, one can simply simulate a convergent system with a periodic excitation from an arbitrary initial condition until its response converges to the globally asymptotically stable periodic response. Despite its simplicity, this approach is very computationally inefficient, especially if one needs to compute the responses for a wide range of excitations and (controller) parameters. To alleviate this computational burden, in certain situations one can approximate the periodic responses using the describing function method [12], which is much faster, but less accurate. Alternatively, one can exploit period solvers, such as the finite difference method, the shooting method or the collocation method, possibly in combination with a path-following technique, to determine these periodic responses [13], [14]. This method has been used in [15]. Drawbacks of such methods are, firstly, the fact that generally relatively accurate initial guesses for the periodic solutions need to be available in order for convergence to occur and, secondly, the fact that these are computationally rather expensive. The latter fact is especially prohibitive when one needs to compute periodic

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solutions for a large set of periodic excitations.

In this paper we present an iterative numerical procedure for fast computation of periodic responses to periodic excitations for Lur'e-type nonlinear systems. The procedure has guaranteed convergence for an arbitrary initial guess of the periodic response. In this procedure, periodic responses are represented in terms of their Fourier coefficients. A periodic response of a nonlinear system contains, in general, an infinite number of harmonics, while only a finite number of them can be stored in the computer memory. For this reason we present an estimate of the error in the output of this iterative procedure which appears due to truncation of the corresponding Fourier series at each iteration step. The efficiency of the method is based on performing linear operations in the frequency domain, where they can be done in a computationally inexpensive way, nonlinear operations in the time domain, and transforming signals between the time and frequency domain using fast Fourier transform algorithms. This method allows us to compute periodic responses to harmonic excitations for a whole range of excitation frequencies and amplitudes (frequency response functions) in a very efficient manner. The fast computational time makes *estimations* of periodic responses obsolete, since for the studied class of systems they can be easily *computed* with any given accuracy. At the same time, in computing periodic responses to periodic excitations this method appears to be more reliable, fast and accurate than the methods mentioned above. To illustrate the efficiency of the method, we compute frequency response functions of a periodically excited nonlinear mechanical system in closed loop with a disturbance attenuation controller.

The paper is organized as follows. In Section II we present the notations and preliminaries used in the paper. Section III contains a result on existence and uniqueness of a periodic response to a periodic excitation for a class of Lur'e systems. An iterative algorithm for computing this periodic response is presented in Section IV. Numerical implementation of this algorithm is discussed in Section V. Application of this algorithm to the computation of frequency response functions is discussed in Section VI and illustrated with an example in Section VII. Conclusions are presented in Section VIII.

II. Notations and Mathematical Preliminaries

Throughout the paper we will use the following notations. By $\mathbb{Z}$ we denote the set of integer numbers; $i := \sqrt{-1}$. By $L_2(T)$ we denote the space of piecewise-continuous real-valued T-periodic scalar functions $y(t)$ satisfying $\|y\|_{L_2} < +\infty$, where $\|y\|^2_{L_2} := \frac{1}{T}\int_0^T |y(t)|^2 dt$. By l_2 we denote the space of complex-valued sequences $V = \{V[m]\}_{m\in\mathbb{Z}}$ satisfying $\|V\|_{l_2} < +\infty$, where $\|V\|^2_{l_2} = \sum_{m\in\mathbb{Z}} |V[m]|^2$. Both $L_2(T)$ and l_2 are Banach spaces.

The sequence of Fourier coefficients of $y \in L_2(T)$ is denoted by Y. The elements of the sequence are given by

$$Y[m] = \frac{1}{T}\int_0^T y(t)e^{-i\omega m t}dt, \quad m \in \mathbb{Z},$$

where $\omega := 2\pi/T$. The inverse Fourier transform is given by

$$y(t) = \sum_{m\in\mathbb{Z}} Y[m]e^{im\omega t}.$$

For any $y \in L_2(T)$ and its Fourier coefficients Y the Parseval's equality holds:

$$\|y\|_{L_2} = \|Y\|_{l_2}. \quad (1)$$

For a linear single-input-single-output system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \quad (2)$$

excited by a T-periodic input $u(t)$, $u \in L_2(T)$, if the matrix A has no eigenvalues at $im\omega$, for $\omega := \frac{2\pi}{T}$ and all $m \in \mathbb{Z}$, there exists a unique T-periodic solution $x_u(t)$ with the corresponding output $y_u(t)$ ($y_u \in L_2(T)$). Hence system (2) defines a linear operator $\mathcal{G}_{yu} : L_2(T) \to L_2(T)$ according to $\mathcal{G}_{yu}u(t) = y_u(t)$. In the frequency domain we define the linear operator $\hat{\mathcal{G}}_{yu} : l_2 \to l_2$ that maps the Fourier coefficients U of the function $u(t)$ to the Fourier coefficients Y_U of the function $y_u(t)$, i.e. $\hat{\mathcal{G}}_{yu}U := Y_U$. It is known that

$$\left(\hat{\mathcal{G}}_{yu}U\right)[m] = G_{yu}(im\omega)U[m], \quad m \in \mathbb{Z}, \quad (3)$$

where $G_{yu}(s) := C(sI - A)^{-1}B$ is the transfer function of system (2) from input u to output y. Due to (3) it is straightforward to verify that

$$\|\hat{\mathcal{G}}_{yu}U\|_{l_2} \le \sup_{m\in\mathbb{Z}} |G_{yu}(im\omega)|\|U\|_{l_2}, \quad (4)$$

and, by the Parseval's equality (1), we also conclude that

$$\|\mathcal{G}_{yu}u\|_{L_2} \le \sup_{m\in\mathbb{Z}} |G_{yu}(im\omega)|\|u\|_{L_2}. \quad (5)$$

III. Periodic Responses of Periodically Excited Lur'e-Type Systems

In this section we consider systems of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Dv(t) \quad (6)\\ y &= Cx, \\ u &= \varphi(y), \quad (7) \end{aligned}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$ is the output, $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity and $v(t)$ is a scalar piecewise continuous input. Let $G_{yu}(s) := C(sI - A)^{-1}B$ and $G_{yv}(s) := C(sI - A)^{-1}D$ denote the transfer functions from u to y and from v to y respectively. The next theorem formulates the conditions under which system (6), (7) exhibits a unique T-periodic response to a T-periodic excitation $v(t)$.

Theorem 1

Consider system (6), (7) excited by a T-periodic input $v(t)$. Suppose

A1 *the matrix A has no eigenvalues at $im\omega$, for $\omega := \frac{2\pi}{T}$ and $m \in \mathbb{Z}$,*

A2 *the nonlinearity $\varphi(y)$ satisfies $\varphi(0) = 0$ and is Lipschitz with Lipschitz constant L, i.e.*

$$|\varphi(y_1) - \varphi(y_2)| \le L|y_1 - y_2|, \quad \forall\ y_1, y_2 \in \mathbb{R}, \quad (8)$$

A3 *the transfer function $G_{yu}(s)$ satisfies*

$$\sup_{m\in\mathbb{Z}} |G_{yu}(im\omega)| =: \gamma < \frac{1}{L}. \quad (9)$$

Then system (6), (7) has a unique T-periodic solution $\bar{x}(t)$ with the corresponding periodic output $\bar{y}(t)$ satisfying

$$\|\bar{y}\|_{L_2} \leq \frac{1}{1-\gamma L} \sup_{m\in\mathbb{Z}} |G_{yv}(im\omega)| \|v\|_{L_2}. \tag{10}$$

Proof. Due to condition A1, system (6) defines the linear operators $\mathcal{G}_{yu} : L_2(T) \to L_2(T)$ and $\mathcal{G}_{yv} : L_2(T) \to L_2(T)$ that map T-periodic inputs to T-periodic outputs (see Section II). Due to linearity of system (6), its periodic output $y \in L_2(T)$ corresponding to the inputs $u \in L_2(T)$ and $v \in L_2(T)$ equals

$$y = \mathcal{G}_{yu}u + \mathcal{G}_{yv}v. \tag{11}$$

Notice that since $\mathcal{G}_{yu}$ is a linear operator, $\mathcal{G}_{yu}u_1 - \mathcal{G}_{yu}u_2 = \mathcal{G}_{yu}(u_1 - u_2)$. Applying (5) and (9) to the last equality, we conclude that

$$\|\mathcal{G}_{yu}u_1 - \mathcal{G}_{yu}u_2\|_{L_2} \leq \gamma\|u_1 - u_2\|_{L_2}, \tag{12}$$

for any $u_1, u_2 \in L_2(T)$.

Next consider the nonlinear operator $\mathcal{F} : L_2(T) \to L_2(T)$ defined as $\mathcal{F}y(t) := \varphi(y(t))$. Since $\varphi(y)$ is Lipschitz (see condition A2), the operator $\mathcal{F}$ is also Lipschitz:

$$\|\mathcal{F}y_1 - \mathcal{F}y_2\|_{L_2} \leq L\|y_1 - y_2\|_{L_2}. \tag{13}$$

From (12) and (13) we conclude that the operator $\mathcal{G}_{yu} \circ \mathcal{F} : L_2(T) \to L_2(T)$ is also Lipschitz. Namely,

$$\begin{aligned} \|\mathcal{G}_{yu} \circ \mathcal{F}y_1 - \mathcal{G}_{yu} \circ \mathcal{F}y_2\|_{L_2} &\leq \gamma\|\mathcal{F}y_1 - \mathcal{F}y_2\|_{L_2} \\ &\leq \gamma L\|y_1 - y_2\|_{L_2}. \end{aligned} \tag{14}$$

Since $\gamma L < 1$ (see A3), $\mathcal{G}_{yu} \circ \mathcal{F}$ is a contraction mapping. Applying the Banach fixed point theorem (see e.g. [16]) we conclude that there exists a unique $\bar{y} \in L_2(T)$ satisfying

$$\bar{y} = \mathcal{G}_{yu} \circ \mathcal{F}\bar{y} + \mathcal{G}_{yv}v. \tag{15}$$

This implies that $\bar{y}(t)$ is the unique T-periodic output of system (6), (7). The corresponding periodic solution $\bar{x}(t)$ is the unique periodic solution of the linear system (6) with the input $u(t) = \varphi(\bar{y}(t))$.

It remains to show that inequality (10) holds. Since $\bar{y}$ satisfies (15), it holds that

$$\|\bar{y}\|_{L_2} \leq \|\mathcal{G}_{yu} \circ \mathcal{F}\bar{y}\|_{L_2} + \|\mathcal{G}_{yv}v\|_{L_2}. \tag{16}$$

Applying inequality (14) for $y_1 = \bar{y}$ and $y_2 = 0$ and expressing $\|\bar{y}\|_{L_2}$, we obtain

$$\|\bar{y}\|_{L_2} \leq \frac{1}{1-\gamma L}\|\mathcal{G}_{yv}v\|_{L_2}. \tag{17}$$

Finally, application of inequality (5) gives (10). □

Although a similar result on existence and uniqueness of periodic solutions can be found in [7], it is the proof of Theorem 1 that will allow us to develop an efficient numerical procedure for computing periodic response.

IV. Iterative computation of periodic responses with convergence and accuracy guarantees

The proof of existence and uniqueness of the periodic response in Theorem 1 is based on the Banach fixed point theorem, which also provides a method for iterative computation of the periodic response $\bar{y}(t)$. It can be found as the limit of the iterative process $y_{k+1} = \mathcal{G}_{yu} \circ \mathcal{F}y_k + \mathcal{G}_{yv}v$ with an arbitrary initial value $y_0 \in L_2(T)$. The convergence of this process is characterized by the inequality

$$\|y_k - \bar{y}\|_{L_2} \leq (\gamma L)^k \|y_0 - \bar{y}\|_{L_2},$$

where $\gamma L < 1$ is the measure of contraction of $\mathcal{G}_{yu} \circ \mathcal{F}$ (see the proof of Theorem 1 and conditions A2, A3 of this theorem). Since $\gamma L < 1$, this iteration converges exponentially. To implement this iterative process, we decompose it into the following equivalent one:

$$\begin{aligned} u_{k+1} &= \mathcal{F}y_k & (18)\\ y_{k+1} &= \mathcal{G}_{yu}u_{k+1} + \mathcal{G}_{yv}v. & (19) \end{aligned}$$

According to the definition of the operator $\mathcal{F}$ (see the proof of Theorem 1), given a function $y_k \in L_2(T)$, $u_{k+1} \in L_2(T)$ can be computed according to

$$u_{k+1}(t) = \varphi(y_k(t)). \tag{20}$$

As follows from the definition of the linear operators $\mathcal{G}_{yu}$ and $\mathcal{G}_{yv}$ (see Section II), $y_{k+1}(t)$ computed at step (19) is the periodic output of the linear time-invariant system (6) with the T-periodic inputs $v(t)$ and $u(t) = u_{k+1}(t)$.

It is possible to implement both steps (18) and (19) in the time domain with any given accuracy. For (18) we can use (20), while the periodic solution of the linear system (6) excited by T-periodic inputs $v(t)$ and $u(t) = u_{k+1}(t)$ can be found using the Cauchy formula for the general solution of a linear system with the boundary condition $x(T) = x(0)$. Yet, it has been noticed that such a method is not computationally efficient. For example, when the periodic solution of the nonlinear system (6), (7) is globally asymptotically stable, one can simply simulate the system with a given input $v(t)$ until its solution converges to the periodic solution. The computational time of this simple simulation-based method is comparable to the computational time of a time-domain implementation of the iterative algorithm (18), (19). For this reason we have opted not to proceed with the time-domain implementation of (18), (19).

Alternatively, one can implement the algorithm (18), (19) in the frequency domain by representing the T-periodic functions $u_k(t)$, $y_k(t)$ and $v(t)$ by their respective Fourier coefficients U_k, Y_k and V, and substituting the operators $\mathcal{G}_{yu}$, $\mathcal{G}_{yv}$ and $\mathcal{F}$ by their frequency domain counterparts $\hat{\mathcal{G}}_{yu}$, $\hat{\mathcal{G}}_{yv}$ and $\hat{\mathcal{F}}$, respectively. Then the algorithm (18), (19) takes the form

$$\begin{aligned} U_{k+1} &= \hat{\mathcal{F}}Y_k & (21)\\ Y_{k+1} &= \hat{\mathcal{G}}_{yu}U_{k+1} + \hat{\mathcal{G}}_{yv}V. & (22) \end{aligned}$$

Using inequalities (12), (13) and (14) and taking into account Parseval's equality (1), one can show that the operator $\hat{\mathcal{G}}_{yu}\circ\hat{\mathcal{F}}$ is a contraction on l_2 and by the Banach fixed point theorem the iterative process (21), (22) will exponentially converge to the unique solution $\bar{Y}$ of the equation

$$\bar{Y} = \hat{\mathcal{G}}_{yu} \circ \hat{\mathcal{F}}\bar{Y} + \hat{\mathcal{G}}_{yv}V. \tag{23}$$

The main advantage of such a frequency domain implementation is the fact that step (19), which is the most computationally demanding in the time-domain implementation is now substituted by the computationally cheap step (22), see (3). The difficulties arising in this case are twofold. First of all, due to the nonlinear operator $\hat{\mathcal{F}}$, the number of nonzero entries in U_{k+1} will, in general, always be infinite despite of the fact that the number of non-zero entries in V (the spectrum of the excitation $v(t)$) may be finite. This would require the storage of an infinite number of Fourier coefficients U_k and Y_k, which is impossible. So, we need to truncate U_{k+1} at each step. Another argument for truncation stems from the difficulty in the implementation of the nonlinear operator $\hat{\mathcal{F}}$ corresponding to the nonlinearity $\varphi(y)$. For a general nonlinearity $\varphi(y)$ it is impossible to find an analytic expression for the implementation of $\hat{\mathcal{F}}$. Therefore it is suggested to firstly transform the Fourier coefficients Y_k to the periodic function $y_k(t)$ in the time domain, compute $u_{k+1}(t) = \varphi_{k+1}(t)$ and then apply the Fourier transform to transform $u_{k+1}(t)$ into U_{k+1}. Numerical implementation of this algorithm can be done very efficiently using Fast Fourier Transform algorithms, but such an operation will always imply a truncation of U_{k+1}. Thus the algorithm becomes

$$U_{k+1} = (\hat{\mathcal{F}}Y_k)_N \tag{24}$$
$$Y_{k+1} = \hat{\mathcal{G}}_{yu}U_{k+1} + \hat{\mathcal{G}}_{yv}V, \tag{25}$$

where $(\cdot)_N$ denotes a truncation operation:

$$(U)_N[m] = \begin{cases} U[m], & \text{for } |m| \leq N \\ 0, & \text{for } |m| > N, \end{cases} \tag{26}$$

and $N > 0$ is a truncation parameter. In general, introduction of truncation in such an iterative algorithm can cause large errors in the limit solution and even prevent the convergence of the algorithm. However, in the next theorem we prove that, in fact, under the conditions of Theorem 1, the iterative sequence (24), (25) will converge for any value of the truncation parameter N. Moreover, we obtain an estimate on the accuracy of the algorithm with truncation.

Theorem 2
Under the conditions of Theorem 1, for any $N > 0$ there is a unique limit $\bar{Y}^N$ for the sequence Y_k, $k = 1, 2, \ldots$, resulting from the iterative process with truncation (24), (25). Moreover

$$\|\bar{Y}^N - \bar{Y}\|_{l_2} \leq \sup_{|m|>N} |G_{yu}(im\omega)| \sup_{m\in\mathbb{Z}} |G_{yv}(im\omega)| \frac{L\|V\|_{l_2}}{(1-\gamma L)^2}. \tag{27}$$

Proof. Notice that, as follows from (3), for any $U \in l_2$ it holds that $\hat{\mathcal{G}}_{yu}(U)_N = (\hat{\mathcal{G}}_{yu})_N U$, where $(\hat{\mathcal{G}}_{yu})_N : l_2 \to l_2$ is a linear operator defined as

$$(\hat{\mathcal{G}}_{yu})_N U[m] = \begin{cases} G_{yu}(im\omega)U[m], & \text{for } |m| \leq N \\ 0, & \text{for } |m| > N. \end{cases} \tag{28}$$

Hence, instead of (24), (25) one can consider the equivalent iterative process

$$\tilde{U}_{k+1} = \hat{\mathcal{F}}Y_k \tag{29}$$
$$Y_{k+1} = (\hat{\mathcal{G}}_{yu})_N \tilde{U}_{k+1} + \hat{\mathcal{G}}_{yv}V, \tag{30}$$

which is of the similar form as (21), (22). So, in order to prove its convergence we only need to show that $(\hat{\mathcal{G}}_{yu})_N \circ \hat{\mathcal{F}}$ is a contraction mapping from l_2 to l_2.

It is straightforward to verify that

$$\|(\hat{\mathcal{G}}_{yu})_N U\|_{l_2} \leq \sup_{|m|\leq N} |G_{yu}(im\omega)| \|U\|_{l_2}.$$

Taking into account (9), we obtain $\|(\hat{\mathcal{G}}_{yu})_N U\|_{l_2} \leq \gamma \|U\|_{l_2}$. From this and from the linearity of $(\hat{\mathcal{G}}_{yu})_N$ we conclude that for any $U_1, U_2 \in l_2$ it holds that

$$\|(\hat{\mathcal{G}}_{yu})_N U_1 - (\hat{\mathcal{G}}_{yu})_N U_2\|_{l_2} \leq \gamma \|U_1 - U_2\|_{l_2}. \tag{31}$$

Using Parseval's equality (1) and (13) we conclude that

$$\|\hat{\mathcal{F}}Y_1 - \hat{\mathcal{F}}Y_2\|_{l_2} \leq L\|Y_1 - Y_2\|_{l_2}, \tag{32}$$

for any $Y_1, Y_2 \in l_2$. In the same way as in (14), inequalities (31) and (32) imply

$$\|(\hat{\mathcal{G}}_{yu})_N \circ \hat{\mathcal{F}}Y_1 - (\hat{\mathcal{G}}_{yu})_N \circ \hat{\mathcal{F}}Y_2\|_{l_2} \leq \gamma L \|Y_1 - Y_2\|_{l_2}. \tag{33}$$

Since $\gamma L < 1$ (see condition A3 in Theorem 1), the operator $(\hat{\mathcal{G}}_{yu})_N \circ \hat{\mathcal{F}}$ is a contraction. By the Banach fixed point theorem, there exists a unique $\bar{Y}^N \in l_2$ satisfying

$$\bar{Y}^N = (\hat{\mathcal{G}}_{yu})_N \circ \hat{\mathcal{F}}\bar{Y}^N + \hat{\mathcal{G}}_{yv}V, \tag{34}$$

and this solution $\bar{Y}^N$ can be found as a limit of the iterative sequence (29), (30) or, equivalently, of the sequence (24), (25).

It remains to show that (27) holds. From (23) and (34), we conclude that

$$\begin{aligned} \|\bar{Y} - \bar{Y}^N\|_{l_2} &= \|\hat{\mathcal{G}}_{yu} \circ \hat{\mathcal{F}}\bar{Y} - (\hat{\mathcal{G}}_{yu})_N \circ \hat{\mathcal{F}}\bar{Y}^N\|_{l_2} \\ &\leq \|(\hat{\mathcal{G}}_{yu})_N \circ \hat{\mathcal{F}}\bar{Y} - (\hat{\mathcal{G}}_{yu})_N \circ \hat{\mathcal{F}}\bar{Y}^N\|_{l_2} \\ &\quad + \|(\hat{\mathcal{G}}_{yu})_N^{res} \circ \hat{\mathcal{F}}\bar{Y}\|_{l_2}, \end{aligned}$$

where $(\hat{\mathcal{G}}_{yu})_N^{res} := \hat{\mathcal{G}}_{yu} - (\hat{\mathcal{G}}_{yu})_N$. Taking into account (33), we obtain

$$\|\bar{Y} - \bar{Y}^N\|_{l_2} \leq \gamma L \|\bar{Y} - \bar{Y}^N\|_{l_2} + \|(\hat{\mathcal{G}}_{yu})_N^{res} \circ \hat{\mathcal{F}}\bar{Y}\|_{l_2}.$$

Since $\gamma L < 1$, it follows that

$$\|\bar{Y} - \bar{Y}^N\|_{l_2} \leq \frac{1}{1-\gamma L} \|(\hat{\mathcal{G}}_{yu})_N^{res} \circ \hat{\mathcal{F}}\bar{Y}\|_{l_2}. \tag{35}$$

Notice that $(\hat{\mathcal{G}}_{yu})_N^{res}$ is defined as

$$(\hat{\mathcal{G}}_{yu})_N^{res} U[m] = \begin{cases} G_{yu}(im\omega)U[m], & \text{for } |m| > N \\ 0, & \text{for } |m| \leq N. \end{cases} \tag{36}$$

Hence it can be easily verified that

$$\|(\hat{\mathcal{G}}_{yu})_N^{res} U\|_{l_2} \leq \sup_{|m|>N} |G_{yu}(im\omega)| \|U\|_{l_2}. \tag{37}$$

Since $\hat{\mathcal{F}}$ is Lipschitz with the Lipschitz constant L and $\hat{\mathcal{F}}0 = 0$ (this follows from the condition that $\varphi(0) = 0$) we obtain

$$\|\hat{\mathcal{F}}\bar{Y}\|_{l_2} \leq L\|\bar{Y}\|_{l_2}. \tag{38}$$

Uniting (35), (37), (38) and (10) with the Parseval's equality (1) we obtain (27). □

Remark 1

Using the Parseval's equality (1), in the time domain the accuracy estimate (27) takes the form

$$\|\bar{y}^N - \bar{y}\|_{L_2} \leq \sup_{|m|>N} |G_{yu}(im\omega)| \sup_{m\in\mathbb{Z}} |G_{yv}(im\omega)| \frac{L\|v\|_{L_2}}{(1-L\gamma)^2}. \tag{39}$$

From (39) we see that for a given input function $v(t)$ and a given tolerance $\varepsilon > 0$ one can always choose the truncation parameter N such that $\|\bar{y}^N - \bar{y}\|_{L_2} \leq \varepsilon$. Namely, the transfer function $G_{yu}(s)$ is strictly proper and for this reason one can always choose N sufficiently large to minimize $\sup_{|m|>N} |G_{yu}(im\omega)|$ to a desired level.

Notice that if V has a finite number of non-zero elements (i.e. the excitation $v(t)$ has a finite spectrum), one can always choose N large enough such that $V[m] = 0$ for $|m| > N$. In this case, at each iteration step of the algorithm (24), (25) the sequences U_k and Y_k satisfy $U_k[m] = 0$, $Y_k[m] = 0$ for $|m| > N$. Therefore, at each step we need to store only two complex-valued $2N+1$-dimensional vectors corresponding to non-zero entries of U_k and Y_k. Moreover, if the transfer function $G_{yu}(s)$ has good filtering properties, the number N characterizing the dimension of these vectors can be chosen rather small without significant deterioration of the algorithm accuracy. This is definitely a benefit for numerical implementation of this algorithm since smaller N implies smaller number of operations at each iteration of the algorithm.

V. Numerical implementation of the iterative computation of periodic responses

For a numerical implementation of (24), (25) we need to have the following assumptions and initial data. First of all, it is assumed that V contains a finite number of nonzero entries. Moreover, it is assumed that the truncation parameter N is chosen in accordance with (27) to guarantee a desired accuracy of the algorithm. Moreover, it is assumed that the nonzero entries of V correspond to indexes within $[-N, N]$. In addition to this we will introduce a parameter $\epsilon_{reltol} > 0$ for stopping the iterative process (24), (25) if

$$\frac{\|Y_k - Y_{k-1}\|_{l_2}}{\|Y_{k-1}\|_{l_2}} < \epsilon_{reltol}. \tag{40}$$

Let us also choose a number $M = 2^b$ for some positive integer b and satisfying $M \geq 2N$. This parameter will be used in the direct and inverse Fast Fourier Transforms. In general, M needs to be chosen large enough to guarantee sufficient accuracy of the transformations. To initiate the algorithm, we need an initial guess Y_0 containing Fourier coefficients corresponding to frequencies from $-N\omega$ to $N\omega$. Notice that under these assumptions at each step of the iterative process (24), (25) the sequences Y_k and U_k have non-zero elements only for indexes from $-N$ to N corresponding to the frequencies from $-N\omega$ to $N\omega$. Therefore in this section we consider Y_k and U_k as complex-valued vectors of dimension $2N+1$.

At step $k+1$ we firstly compute U_{k+1} according to (24). To do this, we transform Y_k to the time domain, perform the nonlinear operation $u_{k+1}(t) = \varphi(y_k(t))$, transform $u_{k+1}(t)$ into frequency domain U_{k+1} and then truncate the elements $U_{k+1}[m]$ with indexes satisfying $|m| > N$. This operation can be implemented in the following way. First, define a vector Y_k^{ext} with the entries $Y_k^{ext}[m] = Y_k[m]$ for $|m| \leq N$ and $Y_k^{ext}[m] = 0$ for $N < |m| \leq M$. Then transform this vector of Fourier coefficients to the time domain using the inverse discrete Fourier transform for sampled signals:

$$y_k(t_l) = \frac{1}{T}\sum_{m=-M/2}^{M/2} Y_k^{ext}[m]e^{\frac{2\pi ilm}{M}},\ l = 0, 1, \ldots M-1, \tag{41}$$

nonlinear operation

$$u_{k+1}(t_l) = \varphi(y_k(t_l)), \quad l = 0, 1, \ldots M-1. \tag{42}$$

Then transform the function $u_{k+1}(t)$ into frequency domain using discrete Fourier transform given by the formula

$$U_{k+1}^{ext}[m] = \frac{1}{M}\sum_{l=0}^{M-1} u_{k+1}(t_l)e^{-\frac{2\pi ilm}{M}},\ m = 0, \pm 1, \ldots \pm \frac{M}{2}. \tag{43}$$

Finally, the resulting U_{k+1} is obtained by taking only the elements of U_{k+1}^{ext} with the indexes within $|m| \leq N$, i.e. $U_{k+1}[m] = U_{k+1}^{ext}[m]$, for $|m| \leq N$. Thus we obtain U_{k+1}. Next we compute Y_{k+1} given by (25). As follows from (3),

$$Y_{k+1}[m] = G_{yu}(im\omega)U[m] + G_{yv}(im\omega)V[m],\ \ |m| \leq N. \tag{44}$$

These operations are continued until condition (40) is satisfied. Notice that since Y_k is a $(2N+1)$-dimensional vector, the l_2 norm in (40) becomes simply the second vector norm.

In (43) and (41) we approximate the direct and inverse Fourier transforms for continuous signals by discrete Fourier transform for sampled signals. The inaccuracy introduced by this approximation is not accounted for in the analysis in Section IV, but it can be reduced by increasing the parameter M. Precise analysis of the errors introduced by this approximation is left for future work.

The direct and inverse discrete Fourier transforms (43) and (41) can be computed very efficiently using Fast Fourier Transform (FFT) algorithms, while (44) requires only a relatively small number of summations and multiplications. This makes the algorithm very efficient, as will be illustrated by an example in Section VII.

VI. Computation of frequency response functions

A particular application of the numerical algorithm presented in the previous section is to compute periodic responses to harmonic excitations of the form $v(t) = a\sin\omega t$ for a whole range of amplitudes a and frequencies ω given

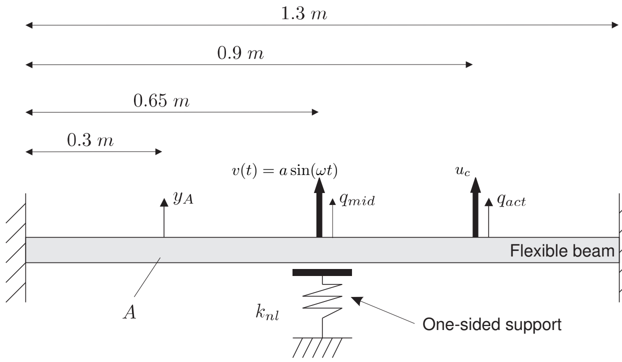

Fig. 1: Perturbed flexible beam system with a one-sided flexible support.

by a certain grid. In this case the efficiency of the algorithm can be enhanced further by choosing the initial guess Y_0 in a smart way. In particular, if one has computed the limit $\bar{Y}^N$ for a particular amplitude and frequency (a, ω) on the grid, this $\bar{Y}^N$ can be used as the initial guess Y_0 for computing the limit solution corresponding to the excitations with the amplitude and frequency given by $(a + \Delta a, \omega)$ or $(a, \omega + \Delta\omega)$, where Δa and $\Delta\omega$ are the increments corresponding to the neighboring points in the grid. Such a choice of the initial guess allows one to significantly reduce the number of iterations in the algorithm presented in the previous section. Such a procedure is often referred to as sequential continuation [17].

VII. Example

In this section, we apply the proposed algorithm for the computationally efficient determination of periodic responses to a controlled mechanical system. More specifically, we consider a flexible steel beam with a one-sided flexible support (modelled as a one-sided linear spring), see Figure 1. A harmonic force $v(t) = a\sin(\omega t)$ is applied to the middle of the beam, where ω is the excitation frequency and a is the amplitude of the excitation force. An actuator is mounted on the beam in order to control the beam dynamics by means of an actuator force u_c.

The dynamics of the system is described by a four degree-of-freedom (DOF) model [18], [19]:

$$M_r\ddot{q} + B_r\dot{q} + K_r q + f_{nl}(q) = h_1\, v(t) + h_2\, u_c, \quad (45)$$

where $h_1 = [1 \quad 0 \quad 0 \quad 0]^T$, $h_2 = [0 \quad 1 \quad 0 \quad 0]^T$ and $q = [q_{mid} \quad q_{act} \quad q_{\xi,1} \quad q_{\xi,3}]^T$ are the generalised coordinates. Herein, q_{mid} is the displacement of the middle of the beam and q_{act} is the displacement of the point where the actuator is mounted at the beam, see Figure 1. The variables $q_{\xi,1}$ and $q_{\xi,2}$ reflect the contribution of two eigenmodes of the beam that occur at 21Hz and 55Hz, respectively. M_r, B_r and K_r are the mass, damping and stiffness matrices of the reduced model, respectively. The numerical values of the matrices M_r $[kg]$, K_r $[N/m]$ and B_r $[Ns/m]$ are:

$$M_r = \begin{bmatrix} 3.38062 & 1.2961 & 2.0957 & -0.4958 \\ 1.2961 & 38.6548 & 16.3153 & -14.6109 \\ 2.0957 & 16.3153 & 8.6864 & -6.2413 \\ -0.4958 & -14.6109 & -6.2413 & 6.5893 \end{bmatrix},$$

$$K_r = 10^6 \begin{bmatrix} 2.4151 & 0.0521 & 1.1445 & -0.0199 \\ 0.0521 & 6.3914 & 2.6420 & -2.4342 \\ 1.1445 & 2.6420 & 1.6270 & -1.0107 \\ -0.0199 & -2.4342 & -1.0107 & 1.0542 \end{bmatrix},$$

$$B_r = \begin{bmatrix} 109.3370 & 25.8569 & 61.4792 & -9.8913 \\ 25.8569 & 294.2009 & 128.7864 & -108.5757 \\ 61.4792 & 128.7864 & 85.1265 & -49.2662 \\ -9.8913 & -108.5757 & -49.2662 & 55.5620 \end{bmatrix}.$$

Moreover, in (45) f_{nl} is the restoring force of the one-sided spring: $f_{nl}(q) = k_{nl}\ h_1\ \min(0, h_1^T q) = k_{nl}\, h_1 \min(0, q_{mid})$, where $k_{nl} = 1.6 \cdot 10^5 \frac{N}{m}$ is the stiffness of the one-sided spring.

Here, we apply a state-feedback controller of the form $u_c = -K_1 q^T - K_2\dot{q}$, with $K_1 = \begin{bmatrix}-7524.4 & 4831.3 & -16196.0 & 499.03\end{bmatrix}$ and $K_2 = \begin{bmatrix}26.791 & 54.566 & -236.63 & -0.2323\end{bmatrix}$, to achieve disturbance attenuation. Note that, in general the entire state can not be measured for such mechanical systems and output-feedback design should be considered. Therefore, in [19] observer-based output-feedback control designs are proposed for the system under study here. For the sake of simplicity we limit ourselves to the state-feedback case here.

In state-space form, the model of the beam system can be written as a Lur'e-type system of the form (6), (7) with

$$A = \begin{bmatrix} 0_{4\times 4} & I_{4\times 4} \\ -M_r^{-1}\left(K_r + h_2K_1 + \frac{k_{nl}}{2}h_1h_1^T\right) & -M_r^{-1}\left(B_r + h_2K_2\right) \end{bmatrix}$$
$$D = \begin{bmatrix} 0_{4\times 1} \\ M_r^{-1}h_1 \end{bmatrix}, \quad B = -D,$$
$$C = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \quad (46)$$

$\varphi(y) = -\frac{k_{nl}}{2} \mid y \mid$ and where $x = [q^T \quad \dot{q}^T]^T \in \mathbb{R}^8$ represents the system state.

Now let us apply Theorem 1 to this system. Clearly, the nonlinearity φ as defined above satisfies condition A2 with $L = \frac{k_{nl}}{2} = 8 \times 10^4$ N/m. Condition A1 is satisfied since matrix A is Hurwitz. Moreover, since $\gamma = \sup_{\omega\in\mathbb{R}} |G_{yu}(i\omega)| = 1.127 \times 10^{-5} < \frac{1}{L} = \frac{2}{k_{nl}} = 1.25 \times 10^{-5}$ m/N, condition A3 for the controlled beam system is satisfied for any ω. Consequently, we can conclude, based on Theorem 1, that system (6), (7), (46) has a unique T-periodic response for every excitation $v(t) = a\sin\omega t$, with $T = 2\pi/\omega$.

Let us now apply the algorithm, proposed in Sections V and VI, to efficiently compute these periodic responses of system (6), (7), (46) to harmonic excitations $v(t) = a\sin(\omega t)$ for a grid of excitation amplitudes and excitation frequencies: $a \in \{a_{min}, a_{min} + \Delta a, a_{min} + 2\Delta a, \ldots, a_{max}\}$, $\omega \in \{\omega_{min}, \omega_{min} + \Delta\omega, \omega_{min} + 2\Delta\omega, \ldots, \omega_{max}\}$, with $a_{min} = 1$ N, $\Delta a = 1$ N, $a_{max} = 10$ N, $\omega_{min} = 2\pi$ rad/s, $\Delta\omega = 2\pi$ rad/s, $\omega_{max} = 2\pi 200$ rad/s. The parameter N (the number of harmonics used to characterise the periodic solution) is taken to be $N = 64$, while we use $M = 2N = 128$. The relative tolerance threshold ϵ_{reltol} used in condition (40) is chosen $\epsilon_{reltol} = 1 \times 10^{-6}$.

The results of the application of the algorithm are depicted in Figure 2, which displays the L_2-norm of the computed periodic response $\bar{y}(t)$: $\|\bar{y}(t)\|_{L_2}$ for the grid of excitation amplitudes and frequencies[1]. The upper bound on the L_2-

[1] Due to Parseval's equality, $\|\bar{y}(t)\|_{L_2} = \|\bar{Y}\|_{l_2}$, where $\bar{Y}$ is the sequence of Fourier coefficients of $\bar{y}(t)$. The algorithm provides $\bar{Y}^N$ which is an approximation of $\bar{Y}$ with the guaranteed accuracy bound (27).

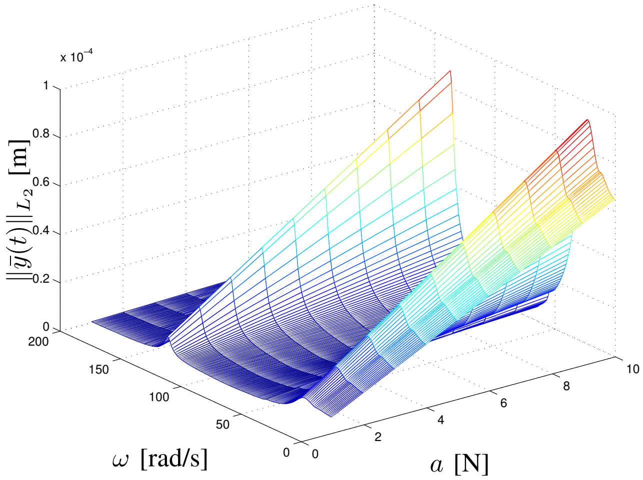

Fig. 2: The norm $\|\bar{y}(t)\|_{L_2}$ of the computed periodic response to $v(t) = a \sin \omega t$ for the grid of excitation amplitudes a and frequencies ω.

norm of difference between the steady-state solutions $\bar{y}$ and $\bar{y}^N$ (the steady-state solution evolving from the iterative procedure including truncation) can readily be computed from (39), which gives $\|\bar{y}^N - \bar{y}\|_{L_2} \leq 9.3 \times 10^{-3}$. Note that to compute this bound we adopted the worst case scenario, which related to the minimal excitation frequency ω_{min} (then $\sup_{|m|>N} |G_{yu}(im\omega)|$ is maximal in (39)) and the maximal amplitude a_{max} (then $\|v\|_{L_2}$ is maximal in (39)). This upperbound decreases monotonically with N; e.g. for $N = 512$ it gives $\|\bar{y}^N - \bar{y}\|_{L_2} \leq 3.7 \times 10^{-5}$. Note that 2000 periodic steady-state solutions corresponding to the harmonic excitations on this (amplitude, frequency)-grid have been computed. For the sake of comparison, these periodic solutions have also been computed using standard numerical simulation, requiring the same level of accuracy. Both approaches have been implemented in MATLAB [20] and have been executed on an Intel Pentium 1.7 GHz processor. The computational time involved in computing the 2000 periodic solutions using the algorithm proposed in this paper (with $N = 64$) is approximately 5 s, whereas the approach using standard numerical simulation involves a computational time of the order of seconds *for a single periodic solution*, which indicates the computational efficiency of the proposed algorithm. It is exactly this computational efficiency that will be able to support the use of this algorithm in the context of optimising the controller gains for the performance in terms of disturbance attenuation.

VIII. Conclusions

In this paper an efficient iterative numerical method for computing periodic responses of nonlinear Lur'e-type systems to periodic excitations is proposed. At each iteration step the algorithm makes use of both frequency and time domain methods with the support of fast Fourier transform for the transition between the domains. This allows us to significantly reduce the computational costs for computing the periodic responses. For the case of harmonic excitations, the proposed numerical algorithm makes it possible to compute the periodic responses for a whole range of amplitudes and frequencies very efficiently. This is achieved by further optimization of the initial guess needed for the iterative algorithm. With this method, quantitative characteristics of the periodic responses corresponding to harmonic excitations of a whole range of amplitudes and frequencies can be established in a fast manner. The efficiency of the proposed method is clearly illustrated by application to a mechanical nonlinear system excited by harmonic excitations.This method supports the analysis of frequency and amplitude dependent steady-state characteristics of the considered class of nonlinear systems. Moreover, it opens a way for tuning controller gains for optimizing the frequency- and amplitude-dependent steady-state performance characteristics of a closed-loop control system.

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