Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Discrete-time sliding mode neural observer for continuous time mechanical systems

Juan Resendiz, Wen Yu and Leonid Fridman

Abstract— **This paper proposes a novel discrete-time velocity observer which uses neural network and sliding mode for unknown continuous time mechanical systems. The neural observer in this paper has two stages: first a dead-zone neural observer assures that the observer error is bounded, then super-twisting second-order sliding-mode is used to guarantee the convergence of the estimation errors to a domain. This observer solves the infinite time convergence problem of neural observers with sliding mode compensation, and the chattering phenomenon of sliding mode observer.**

I. INTRODUCTION

State observation problem is one of the most important problems in control theory. Linear observers [14] do not achieve adequate performance for the mechanical systems with Coulomb friction. Model-based nonlinear observers [12] are usually restricted to the cases when the model is exactly known. In the case of the unknown plant parameters, the nonlinear adaptive observer was proposed in [15]. High-gain differentiators [4] are not exact with any fixed finite gain and feature the peaking effect with high gains. To avoid these problems, the sliding mode observers are widely used [5][9]. In order to obtain the finite-time convergence, robustness with respect to uncertainties and the possibility of uncertainty estimation, the sliding mode observers require partial knowledge of the system and the relative degree of the system with respect to the unknown inputs to be one [18].

A new generation of observers based on the second-order sliding-mode algorithms has been recently developed [2][16]. These observers require the proof of a separation principle theorem due to the asymptotic convergence of the estimated values to the real ones. A robust exact differentiator featuring finite time convergence was designed as an application of the super-twisting algorithm [13]. In [8], a super-twisting second-order sliding-mode observer, which reconstruct the velocity of mechanical systems, was proposed. It requires the knowledge of a nominal part of the system and an upper bound for acceleration.

When we have no complete modelling information, a model-free nonlinear observer is required. If the nonlinear system is given in the normal linearized form, the high-gain observers may estimate the derivative of the output [4], but such filters loose their capability in presence of the output immeasurable perturbations because of their high sensibility to such effects. More recently in [19], the 1st order sliding-mode observer, which does not requires of the knowledge of the system, can obtain infinite time convergence. The sliding mode gain should be bigger than the upper bound of the unknown nonlinear function. Neural networks can be considered as an alternative model-free observer because it offers much potential benefit for non-linear modeling [1]. The dynamic neural networks have been also applied to design a Luemberger-like observer [11]. Due to neural modeling error, neural observers are not asymptotically stable. A robust neuro observer with time delay term was designed in [17], it was proved asymptotic stability on average.

Normal combinations of neural networks and sliding mode methods are to apply them at same time, where sliding mode is used to compensate neural modeling error [6]. This type of neural observers with sliding mode compensation cannot assure finite time convergence [21]. In this paper, neural observer and sliding mode compensator are connected serially, it is called two-stage neural observer. The neural network is used to approximate the nonlinear function of the mechanical system. A dead-zone training algorithm is applied. After the observer error enters the dead-zone, the super-twisting second-order sliding-mode is used to guarantee the finite time convergence of the neural observer. This observer solves the infinite time convergence problem of the sliding-mode neural observers, and the chattering phenomenon of pure sliding mode observers.

II. PROBLEM STATEMENT

Generally a second order mechanical system has the form

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+P(\dot{q})+G(q)+d(t,q,\dot{q})=\tau \quad (1)$$

where $q\in\Re^n$ is the state vector, $M(q)$ is the inertia matrix, $C(q,\dot{q})$ is the Coriolis and centrifugal forces matrix, $P(\dot{q})$ is the Coulomb friction, $G(q)$ is the term of gravitational forces, $d(t,q,\dot{q})$ is an uncertainty disturbance and τ is the torque produced by the actuators. Introducing

Juan Resendiz and Wen Yu are with the Departamento de Control Automatico, CINVESTAV-IPN, Av.IPN 2508, México D.F., 07360, México . Leonid Fridman is with División de Ingeniería Eléctrica, Facultad de Ingeniería, Universidad Nacional Autónoma de México, UNAM, 04510, México D.F.

$u = \tau \in \Re^n$, $x_1 = q \in \Re^n$, $x_2 = \dot{q} \in \Re^n$, this system can be rewritten in the state-space form

$$\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(t,x,u) + \xi_1(t,x,u) \\ y = Cx \end{array} \tag{2}$$

where $y \in \Re^m$, $C \in \Re^{m\times 2n}$. In order to assure the system is observable, we let $m = n$. $x = \left[x_1^T, x_2^T\right]^T \in \Re^{2n}$, and

$$f_1(t,x,u) = -M(x_1)^{-1}\left(C(x)x_2 + P(x_2) + G(x_1) + u\right) \tag{3}$$

while the uncertainties are concentrated in the term $\xi(t,x,u) \in \Re^n$. The solutions of the system (4) are understood in Filippov's sense [10]. For our purpose, the following assumptions are required.

Let x_1, x_2, y be measured at discrete times with the time interval δ, and let t_i, t_{i+1}be successive measurement times. Consider a discrete modification of the mechanical system (4) (the Euler scheme)

$$\begin{array}{l} x_1(t_{i+1}) = x_2(t_i) \\ x_2(t_i+1) = f\left[x(t_i), u(t_i)\right] + \xi(t_i,x,u) \\ y = Cx \end{array} \tag{4}$$

A1: The system states are bounded for all time, it is BIBO stable.

A2: f and the uncertainty term $\xi = \xi(t_i,x,u)$ are Lebesgue- measurable and uniformly bounded in any compact region of the state space x_1, x_2, with

$$\|\xi\|^2_{\Lambda_\xi} = \xi^T\Lambda_\xi\xi \le \bar{\xi} < \infty \qquad \Lambda_\xi = \Lambda_\xi^T > 0 \tag{5}$$

The normalizing matrix Λ_ξ introduced ensures the possibility of working with components of different physical nature, this matrix is given a priory , $\bar{\xi}$ represent the power of the corresponding perturbation.

When $f\left[x(t_i), u(t_i)\right]$ is known, the discrete-time sliding mode observer proposed in [8] has the form

$$\begin{array}{l} \hat{x}_1(t_{i+1}) = \hat{x}_2(t_i) + \lambda\left|\tilde{x}_1\right|^{1/2} sign(\tilde{x}_1) \\ \hat{x}_2(t_{i+1}) = f\left[\hat{x}(t_i), u(t_i)\right] + \alpha sign(\tilde{x}_1) \end{array} \tag{6}$$

where $\hat{x}_1$ and $\hat{x}_2$ are the state estimations and $\tilde{x}_1 = x_1(t_i) - \hat{x}_1(t_i)$. The sliding mode gain α should be big enough, such that it can cancel the uncertainty ξ [8].

When $f\left[x(t_i), u(t_i)\right]$ is unknown, the uncertainty ξ will include $f\left[x(t_i), u(t_i)\right]$, the sliding mode observer proposed in [19] is similar as (6) without $f(t,x,u)$. The model-based sliding-mode observer (6) require the discretization form of f defined in (3), it is very difficult. The sliding mode gain α should be much bigger than before and the chattering is big [8]. Nevertheless it is not always possible to have a good knowledge on the system, so the chattering phenomenon is inevitable [18].

The model-free sliding-mode observer does not require the model f. But it cannot arrive finite time convergence, the sliding mode gain α should be bigger than the upper bounds of the uncertainty term ξ_t. This causes big chattering phenomenon. If we have incomplete information about the nominal nonlinear function $f\left[x(t_i), u(t_i)\right]$, and of course ξ_t, the chattering will be increase. It seems to be natural to construct its estimate $\hat{f}(t, \hat{x}_1, \hat{x}_2|W_t)$ depending on a parameter W_t, which can be adjusted online by means of an updating law

$$W(t_{i+1}) = \Phi(t_i, \hat{x}_1, \hat{x}_2, W(t_i), y) \tag{7}$$

In this paper, we will use neural networks to approximate $f\left[x(t_i), u(t_i)\right]$ and construct a neural observer.

The object of this paper is to design a discrete-time neural observer for continuous time mechanical systems, which use a neural estimator and sliding mode observer sequentially, to estimate the unmeasurable velocity only based on the position measurement. The neural network will approximate the nominal nonlinear function of the mechanical system to reduce the chattering, sliding mode observer is used to assure finite time convergence. The main contribution of this paper is not to solve the chattering problems with neural networks, but to solve the infinite time convergence problem of neural observers.

III. Discrete-time Sliding Mode Neural Observer

The discrete-time neural observer proposed in this paper has the following form

$$\begin{array}{l} \hat{x}(t_{i+1}) = A\hat{x}(t_i) + B\hat{W}(t_i)\sigma\left[\hat{x}(t_i), u(t_i)\right] \\ +\left[1 - s(t_i)\right]Z(t_i) \\ \hat{y}(t_i) = C\hat{x}(t_i) \end{array} \tag{8}$$

where $\hat{x} \in \Re^{2n}$ is the state of the estimation vector, $A \in \Re^{2n\times 2n}$ is a stable fixed matrix which will be specified later. The matrix $\hat{W}^T(t_i) \in \Re^{2n}$ is the weights of the neural networks, $\sigma \in \Re^{2n}$ is a vector of sigmoid functions, C is defined in (4). The vector Z and matrices A and B are defined as

$$A = \begin{bmatrix} 0 & I_n \\ a & b \end{bmatrix}, \; Z(t_i) = \begin{bmatrix} z_1(t_i) \\ z_2(t_i) \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ I_n \end{bmatrix} \tag{9}$$

with the proper choose of the matrices $a, b \in \Re^{n\times n}$, such that the matrix A is Hurwitz stable, where $z_1, z_2 \in \Re^n$. Here s_t is a switch variable, it will switch between the neural estimator (8) and the second order sliding mode observer (11), based on the output error $e(t_i) = y - \hat{y} \in \Re^m$,

$$s(t_i) = \begin{cases} 1 & \text{if } \|e(t_i)\|^2_{Q_1} \ge \bar{\gamma} \\ 0 & \text{if } \|e(t_i)\|^2_{Q_1} < \bar{\gamma} \end{cases} \tag{10}$$

where $Q_1 = Q_1^T > 0$ is a known matrix which will be specified later, $\bar{\gamma}$ is known upper bound of the neural modeling error, which will be defined in (20).

The second order sliding mode is

$$z_1(t_i) = k_1 \|x_1 - \hat{x}_1\|^{1/2} sign[(x_1(t_i) - \hat{x}_1(t_i)] \quad (11)$$
$$z_2(t_i) = k_2 sign[x_1(t_i) - \hat{x}_1(t_i)]$$

where k_1 and k_2 are the sliding mode gains, they will be determined by the theorem in the next section. The learning algorithm (7) becomes dead-zone one

$$\hat{W}(t_{i+1}) = s(t_i)\Phi(t_i, \hat{x}_1, \hat{x}_2, \hat{W}(t_i), y) \quad (12)$$

If $\|e(t_i)\|^2_{Q_1} \geq \bar{\gamma}$, $s(t_i) = 1$, the observer is pure neural observer, (8) becomes

$$x(t_{i+1}) = A\hat{x}(t_i) + B\hat{W}(t_i)\sigma[\hat{x}(t_i), u(t_i)], \quad \hat{y}(t_i) = C\hat{x}(t_i) \quad (13)$$

With neural learning law (7) $B\hat{W}(t_i)\sigma(\hat{x}, u)$ will approximate $f(t, x, u) + \xi(t, x, u)$, and $\|e(t_i)\|^2_{Q_1}$ will be decrease.

If after time t_0, $\|e(t_i)\|^2_{Q_1} < \bar{\gamma}$, $s(t_i) = 0$. From (12) we know $\hat{W}$ is a constant matrix, $\hat{W} = \hat{W}(t_0)$ The observer (8) become pure sliding mode

$$\hat{x}(t_{i+1}) = A\hat{x}(t_i) + B\hat{W}(t_0)\sigma(\hat{x}, u) + Z(t_i), \quad \hat{y}(t_i) = C\hat{x}(t_i) \quad (14)$$

The pure neural observer (13) works only on the sampling point, it does not depend on the sample time δ. Without loss of generality, (8) can be also rewritten as

$$\begin{aligned} \hat{x}_1(k+1) &= \hat{x}_2(k) + [1 - s(k)] z_1(k) \\ \hat{x}_2(k+1) &= A_1\hat{x}(k) + \hat{W}(k)\sigma(\hat{x}(k), u) + z_2(k) \\ \hat{y}(k) &= C\hat{x}(k) \end{aligned} \quad (15)$$

with $A_1 = [a\ b]$. We define observer error $\Delta(k) = x(k) - \hat{x}(k) \in \Re^{2n}$. So the output error is $e(k) = C\Delta(k)$, which implies that

$$\begin{aligned} C^T e(k) &= C^T C\Delta - \epsilon I\Delta + \epsilon I\Delta \\ C^T e(k) &= \left(C^T C + \epsilon I\right)\Delta - \epsilon I\Delta \\ \Delta(k) &= C_\epsilon^+ e(k) + \epsilon N_\epsilon \Delta(k) \end{aligned} \quad (16)$$

where $C \in \Re^{m\times 2n}$, $C_\epsilon^+ \in \Re^{2n\times m}$, $N_\epsilon \in \Re^{2n\times 2n}$, ϵ is a small positive scalar, C_ϵ^+ and N_ϵ are defined as:

$$C_\epsilon^+ = \left(C^T C - \epsilon I\right)^{-1} C^T, \quad N_\epsilon = \left(C^T C - \epsilon I\right)^{-1} \quad (17)$$

Because sigmoid function $\sigma(\cdot)$ satisfies Lipschitz condition,

$$\tilde{\sigma}^T\tilde{\sigma} = \|\tilde{\sigma}\|^2 \leq \lambda_\sigma \|\Delta\|^2 \quad (18)$$

where $\tilde{\sigma} = \sigma(x(k)) - \sigma(\hat{x}(k))$, $\lambda_\sigma > 0$ which can be selected by users.

Adding and subtracting Ax to system (4) we have

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= A_1 x(k) + g(k, x, u) \end{aligned} \quad (19)$$

where $g(k, x, u) = f(k, x, u) - A_1 x + \xi$. According to the Stone-Weierstrass theorem [7], the smooth function $g(k, x, u)$ can be written as

$$g(k, x, u) = W^0\sigma(x, u) + \gamma$$

where W^0 is a fixed weight matrix of the neural network, γ is modeling error. Then we can rewrite (19) as

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= A_1 x(k) + W^0\sigma(x, u) + \gamma \end{aligned} \quad (20)$$

By the assumption **A1,** x is restricted to a compact set S of $x \in R^n$. By the assumption **A2,** f and ξ are uniformly bounded. Taking into account that the sigmoid function $\sigma(\cdot)$ is uniformly bounded, there exist a known positive constant $\bar{\gamma}$ such that

$$\|\gamma\|^2_{\Lambda_2} = \gamma^T\Lambda_2\gamma \leq \bar{\gamma}, \quad \Lambda_2 = \Lambda_2^T > 0 \quad (21)$$

Now (20) can be expressed in matrix form

$$x(k+1) = Ax(k) + BW^0\sigma(x, u) + \Gamma\gamma \quad (22)$$

where A and B are defined as in (9), and $\Gamma = [0\ 1]^T$.

IV. Stability Analysis

The stable learning law for the neural network is given by the following matrix differential equations

$$\begin{aligned} \hat{W}^T(k+1) &= s(k)\left[\hat{W}(k) - \eta(k) B\sigma C_\epsilon^+ e(k)\right] \\ \eta(k) &= \frac{\lambda_\sigma\left\|BW^0\right\| - k - \lambda_{\max}(A)}{1 + \|\sigma\|^2} \end{aligned} \quad (23)$$

where $C_\epsilon^+ \in \Re^{2n\times m}$, $\tilde{W}_t, \hat{W}_t \in \Re^{2n}$, $\sigma \in \Re^{2n}$, $\Lambda_3 \in \Re^{2n\times 2n}$. Here we select the initial condition W^0 as

$$k + \lambda_{\max}(A) < \lambda_\sigma\left\|BW^0\right\| < 1 + k + \lambda_{\max}(A) \quad (24)$$

where $k \geq \frac{\|\Delta(k+1)\|}{\|\Delta(k)\|} > 0$, is corresponded to the error dynamic, it is chosen big enough.

Theorem 1: Under the assumptions **A1** and **A2,** the switch policy (10) and neural training law (23), the discrete-time sliding mode neural observer (8) is designed as

$$k_2 > f^+, \quad k_1 > \left(k_2 + f^+\right)\sqrt{\frac{2}{k_2 - f^+}\frac{(1+p)}{(1-p)}} \quad (25)$$

where $f^+$is the upper bound of the neural modeling error when the weight of the neural networks is fixed as $W(t_0)$, at time $t = t_0$, $\|e(k)\|^2_{Q_1} < \bar{\gamma}$, p is some chosen constant, $0 < p < 1$. Then the observer is stable, and ensures the convergence of the estimation errors to the domain

$$|\Delta_1| < \gamma_1\delta^2, \quad |\Delta_2| < \gamma_2\delta \quad (26)$$

where γ_1 and γ_2 are some small positive constants, δ is sampling time.

Proof: The sliding mode neural observer switches between two models: (13) and (14). Now we discuss these two cases:

I) if $\|e_k\|_{Q_1}^2 \geq \bar{\gamma}$, the observer (8) becomes (13). From (20) and (13) the estimation error can be expressed as

$$\Delta_1(k+1) = \Delta_2(k) \quad (27)$$
$$\Delta_2(k+1) = A_1\Delta(k) + \tilde{W}(k)\sigma(\hat{x},u) + W^0\tilde{\sigma} + \gamma$$

where $\tilde{W}(k) = W^0 - \hat{W}(k)$. Or in matrix form,

$$\Delta(k+1) = A\Delta(k) + B\tilde{W}(k)\sigma(\hat{x},u) + BW^0\tilde{\sigma} + \Gamma\gamma \quad (28)$$

where Γ is defined in (22). We define the Lyapunov function candidate as

$$V(k) = \left\|\widetilde{W}(k)\right\|^2$$

where $\left\|\widetilde{W}(k)\right\|^2 = \sum_{i=1}^n \widetilde{w}^2(k) = tr\left\{\widetilde{W}^T(k)\widetilde{W}(k)\right\}$. Since Δ is not available, only $e(k)$ can be used in the updating law, from the updating law (23)

$$\widetilde{W}(k+1) = \widetilde{W}(k) - \eta(k)B\sigma C_\epsilon^+ e(k)$$

but We calculate the term $2\Delta_t^T PB\tilde{W}_t\sigma_x$ as

$$\begin{aligned}&\Delta V(k) = V(k+1) - V(k)\\&= \left\|\widetilde{W}(k) - \eta(k)\sigma C_\epsilon^+ e(k)^T\right\|^2 - \left\|\widetilde{W}(k)\right\|^2\\&= \eta^2(k)\left\|C_\epsilon^+ e(k)\right\|^2\|\sigma\|^2 - 2\eta(k)\left\|B\widetilde{W}(k)\sigma C_\epsilon^+ e^T(k)\right\|\end{aligned}$$

Since $\Delta(k) = C_\epsilon^+ e(k) + \epsilon N_\epsilon \Delta(k)$

$$\begin{aligned}&2\eta(k)\left\|B\widetilde{W}(k)\sigma C_\epsilon^+ e^T(k)\right\|\\&= 2\eta(k)\|\Delta(k)\|\|I - \epsilon N_\epsilon\|\left\|B\widetilde{W}(k)\sigma\right\|\end{aligned}$$

We define $\|I - \epsilon N_\epsilon\| = \beta$, from (28) we know

$$\begin{aligned}&2\eta(k)\left\|B\widetilde{W}(k)\sigma C_\epsilon^+ e^T(k)\right\| = 2\eta(k)\beta\|\Delta(k)\| *\\&\left\|\Delta(k+1) - A\Delta(k) - \Gamma\gamma - BW^0\tilde{\sigma}\right\|\\&\geq \eta(k)(\beta 2\left\|\Delta(k)^T BW^0\tilde{\sigma}\right\| - 2\Delta(k)^T(k)A\Delta(k)\\&-2\left\|\Delta^T(k)\Delta(k+1)\right\| - 2\left\|\Delta(k)^T\Gamma\gamma\right\|)\end{aligned}$$

Because

$$\begin{aligned}&-2\left\|\Delta(k)^T\Gamma\gamma\right\| \geq -\|\Delta(k)\|^2 - \|\Gamma\gamma\|^2\\&-2\Delta(k)^T(k)A\Delta(k) \geq -|\lambda_{\max}(A)|\|\Delta(k)\|^2\\&2\left\|\Delta(k)^T BW^0\tilde{\sigma}\right\| \leq \|\Delta(k)\|^2 + \left\|BW^0\right\|\|\tilde{\sigma}\|^2\\&\leq \left(1 + \lambda_\sigma\left\|BW^0\right\|\right)\|\Delta(k)\|^2\end{aligned}$$

When for any smooth system $\frac{\|\Delta(k+1)\|}{\|\Delta(k)\|} \leq k$, $k > 0$, so

$$-2\left\|\Delta^T(k)\Delta(k+1)\right\| \geq -k\|\Delta(k)\|^2$$

So

$$\begin{aligned}&\Delta V(k) \leq -\eta(k)\beta[(\lambda_\sigma\left\|BW^0\right\| - \eta(k)\|\sigma\|^2\\&-k - \lambda_{\max}(A))\|\Delta(k)\|^2 - \|\Gamma\gamma\|^2]\end{aligned}$$

If we select $\eta(k) = \frac{\lambda_\sigma\left\|BW^0\right\| - k - \lambda_{\max}(A)}{1+\|\sigma\|^2}$, from (24) we know $0 < \lambda_\sigma\left\|BW^0\right\| - k - \lambda_{\max}(A) < 1$. So

$$\Delta V(k) \leq -\eta(k)\beta\|\Delta(k)\|^2 + \eta(k)\beta\bar{\gamma} \quad (29)$$

Then we estimate the bound of $e(k)$. Using again (16) $(I - \delta N_\delta)\Delta = C_\delta^+ e_t$, we can rewrite (29) as

$$\Delta V(k) \leq -\eta(k)\beta\|e(k)\|_{Q_1}^2 + \eta(k)\beta\bar{\gamma} \quad (30)$$

Since in this case $\|e(k)\|_{Q_1}^2 \geq \bar{\gamma}$, then $\Delta V(k) \leq 0$, V is bounded. So $\|\Delta(k)\|$ and $\|W(k)\|$ are bounded. If we sum (30) from 0 up to T yields

$$V_T - V_1 \leq -\eta(k)\beta\sum_{k=1}^T e^T(k)Q_1 e(k) + \eta(k)\beta\bar{\gamma}T$$

Because $V_T \geq 0$, we have

$$\frac{1}{T}\sum_{k=1}^T e^T(k)Q_1 e(k) \leq \bar{\gamma} \quad (31)$$

So $\|e(k)\|_{Q_1}^2$ will converge into the ball with radius $\bar{\gamma}$.

II) if at time $t = t_0$, $\|e(k)\|_{Q_1}^2 < \bar{\gamma}$, from (23) we know the weights become constants. (20) can be also written as

$$\begin{aligned}&x_1(k+1) = x_2(k) \quad (32)\\&x_2(k+1) = A_1 x(k) + W(t_0)\sigma(x,u) + F_t\end{aligned}$$

where F_t is neural modeling error when the weight of the neural networks is fixed as $W(t_0)$. The error dynamics is obtained from (15) and (32)

$$\begin{aligned}&\dot{\Delta}_1 = \Delta_2 - z_1 \quad (33)\\&\dot{\Delta}_2 = F_t - z_2\end{aligned}$$

We define the upper bound of F_t as f^+, that is $|F| \leq f^+$. By (11), (33) can be rewritten as

$$\Delta_1(k+1) = \Delta_2(k) - k_1|\Delta_1(k)|^{1/2}\text{sign}(\Delta_1(k)) \quad (34)$$
$$\Delta_2(k+1) = F_t - k_2\text{sign}(\Delta_1(k))$$

Let the current time $t \in [t_i, t_{i+1})$, where t_i and $t_{i+1}$are successive measurement times, $t_{i+1} - t_i = \delta$. The observer (34) may be rewritten in the continuous time as follows

$$\begin{aligned}&\dot{\Delta}_1 = \Delta_2 - k_1|\Delta_1|^{1/2}\text{sign}(\Delta_1) \quad (35)\\&\dot{\Delta}_2 = F_t - k_2\text{sign}(\Delta_1)\end{aligned}$$

We define the upper bound of F_t as f^+, that is $|F| \leq f^+$. D is some compact region around the origin O of the space Δ_1, Δ_2. By (11), (33) can be rewritten as All differential inclusions are understood in the Filippov sense [10], which means that the right hand side is enlarged in some points in order to satisfy the upper semi-continuity property. So $\dot{\Delta}_2 \in [-f^+, f^+] - k_2\text{sign}(\Delta_1)$, and $\dot{\Delta}_2 \in [-k_2 - f^+, k_2 + f^+]$ with $\Delta_1 = 0$. Using the trivial identity $\frac{d|x|}{dt} = \dot{x}sign(x)$, and computing the derivative of $\dot{\Delta}_1$ with $\Delta_1 \neq 0$ obtain

$$\ddot{\Delta}_1 \in \left[-f^+, f^+\right] - \left(\frac{1}{2}k_1 \frac{\dot{\Delta}_1}{|\Delta_1|^{1/2}} + k_2 sign(\Delta_1)\right) \quad (36)$$

At the moments $\Delta_1 = 0$, taking into account that $\dot{\Delta}_2 = F_t - k_2 \text{sign}(\Delta_1)$ and (25)

$$0 < k_2 - f^+ < \left|\dot{\Delta}_2\right| < k_2 + f^+ \quad (37)$$

Similar with our previous results in [8] and the form in [16], $\dot{\Delta}_1(t_i) \geq (k_2 - f^+) t_i$, t_i is the time intervals between the successive intersection of the trajectory with the axis $\Delta_1 = 0$. Hence $t_i \leq \frac{\dot{\Delta}_1(t_i)}{k_2 - f^+}$, the total convergence time is estimated by $T \leq \sum \frac{\dot{\Delta}_1(t_i)}{k_2 - f^+}$. Therefore, T is finite and the estimated states converge to the real states in finite time. Then the convergence of $(\hat{x}_1, \hat{x}_1) \to (x_1, x_2)$ in finite time is assured. During this time they do not leave some larger homogeneous disk

$$B_{R_0} = \left\{(\Delta_1, \Delta_2) : |\Delta_1|^{1/2} + |\Delta_2| \leq R_0\right\} \quad (38)$$

Due to the homogeneity property $M(R) = mR$ holds, where the constant $m > 0$ can be easily calculated. Thus, obviously

$$|\Delta_1| \leq mR_0\delta, \quad |\Delta_2| \leq \left(f^+ + k_2\right)\delta$$

Due to the continuous dependence of the Filippov solutions on the graph of the differential inclusion, with sufficiently small δ the trajectories of (38) starting in D terminate in the time T in some small compact vicinity of the origin without leaving B_{R_0} on the way. Let Ω be the compact set of all points belonging to the trajectory segments, Obviously, it is an invariant set attracting the trajectories of (38). Check now that it is a globally attracting set. It is easy to see that (35) is invariant . Let Ω satisfy the inequalities

$$|\Delta_1| < a_1, \quad |\Delta_2| < a_2$$

with some discretization interval δ_0. Applying the transformation $\frac{\delta}{\delta_0}$ obtain that with arbitrary $\delta > 0$ the invariant set satisfies the inequalities

$$|\Delta_1| < \gamma_1\delta^2, \quad |\Delta_2| < \gamma_2\delta, \quad \gamma_1 = \frac{a_1}{\delta_0^2}, \gamma_2 = \frac{a_2}{\delta_0}$$

It is (26). ■

Remark 1: The discrete-time sliding mode neural observer (8) requires two design parameters: switch constant $\bar{\gamma}$ and the upper bound of neural modeling error f^+ when start the sliding mode compensation. $\bar{\gamma}$ decide when we stop neural networks learning and start sliding mode observer. How to choose these user-defined parameter is a trade-off. The bigger $\bar{\gamma}$ is, the shorter training time the neural observer has. In this case, the neural modeling error is bigger, so f^+ should be bigger. If $\bar{\gamma}$ is too small, the unmodeled dynamic prevent the condition $\|e(k)\|_{Q_1}^2 < \bar{\gamma}$ is established, so the two-stage neural observer cannot enter sliding mode compensation, the finite time convergence cannot arrive.

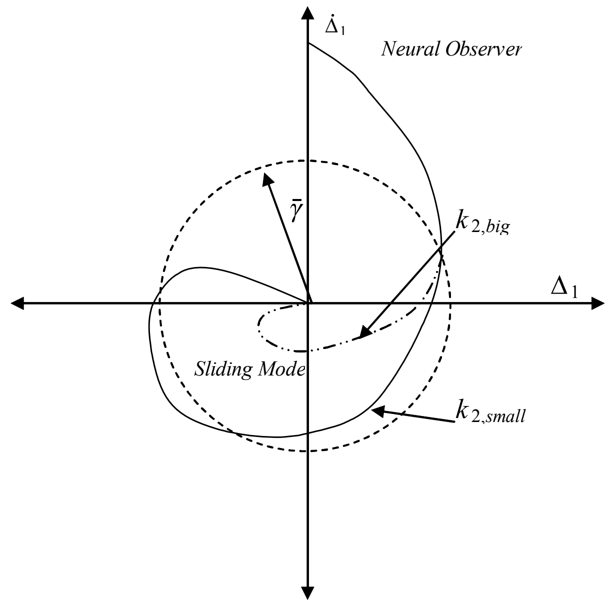

Fig. 1. Majorant curve for the finite-time convergent observer

Remark 2: Usually $f^+ > \bar{\gamma}$, because $\bar{\gamma}$ corresponds to the modeling error with optimal weight, while f^+ corresponds to the modeling error when $\|e(k)\|_{Q_1}^2 < \bar{\gamma}$. Since f^+ is unknown, how to choose k_1 and k_2 is also a trade-off problem. If we chose the sliding mode gains k_1 and k_2 very large to satisfy (25), the chattering becomes big. If we chose the sliding mode gains k_1 and k_2 smaller, (25) is not satisfied, then (36) and (37) cannot be established. Sliding mode observer will not converge, the observer error becomes bigger such that $\|e(k)\|_{Q_1}^2 > \bar{\gamma}$. Now neural training is re-started again, until $\|e(k)\|_{Q_1}^2 < \bar{\gamma}$ and enter sliding mode compensation again, see Fig.1. But, if k_1 and k_2 are too small, (25) will never be right, the two-stage neural observer cannot converge in the finite time. Another possible method is to used off-line training, such that k_1 and k_2 can be chosen small values.

V. SIMULATIONS

Consider a pendulum system with Coulomb friction and external, which was studied by [8][13][19]

$$\dot{x}_1 = x_2, \quad y = x_1, \quad x_1(0) = x_2(0) = 0$$
$$\dot{x}_2 = \frac{1}{J}\tau - \frac{g}{L}\sin x_1 - \frac{V_s}{J}\hat{x}_2 - \frac{P_s}{J}\text{sign}(x_2) + v$$

where $M = 1.1$, $J = ML^2 = 0.891$, $g = 9.815$, $L = 0.9$, $V_s = 0.18$, $P_s = 0.45$, v is an uncertain external perturbation, $v = 0.5\sin 2t + 0.5\cos 5t$. Let it be driven by the twisting controller $\tau = -30\text{sign}(\theta - \theta_d) - 15\text{sign}(\dot{\theta} -$

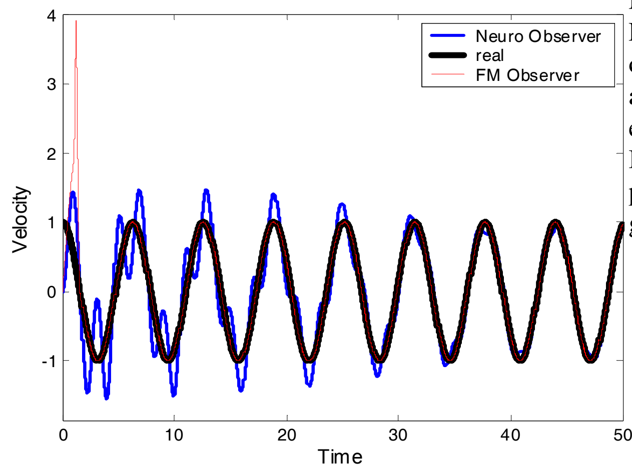

Fig. 2. Real and estimated velocity

$\dot{\theta}_d$), the reference signal is $\theta_d = \sin t$. The proposed two-stage neural observer (8) has the form

$$\hat{x}(k+1) = A\hat{x}(k) + B\hat{W}_t\sigma(\hat{x},u) + (1-s_t)Z$$
$$\hat{y} = [1\ 0]\hat{x}, \quad \hat{x}_1(0) = \hat{x}_2(0) = 0$$

Here we select $A = \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ I_2 \end{bmatrix}$, $\sigma(x,u) = \frac{0.02}{1+e^{-0.05}} - 0.2$. The weights are updated according to the learning law (23) with $KP = I_2$, $\bar{\gamma} = 2$, $W^0 = [1\ 0]$. The sliding mode observer is (11) with $k_1 = 11$ and $k_2 = 12$. Now we compare our two-stage neural observer with neural observer [17] and sliding mode observer [19]. Fig.2 shows the observer results of the velocity x_2. We can see that the two-stage neural observer switch from neural observer to sliding mode observer at $t = 1.7$, and it converges very fast with small chattering. The sliding mode observer has big chattering at all time. The neural observer does not have chattering, but has big observer error.

VI. Conclusions

Although there exist neural observer, sliding mode observer and neural sliding mode observer, discrete-time sliding mode neural observer for continuous time systems is not applied in the literature. This observer solves the infinite time convergence problem of neural observers with sliding mode compensation, and the chattering phenomenon of sliding mode observer .The stability and finite time convergence are proven. Further works will be done on multilayer neural estimator and discrete time observer.

VII. Anknowledgement

Dr. Wen Yu gratefully acknowledges the financial support of this work by the Mexican CONACyT (Consejo Nacional de Ciencia y Tecnología), grant No.50480. Dr. L.Fridman gratefully acknowledges the financial support of this work by the Mexican CONACyT, grant no.56819, and the Programa de Apoyo a Proyectos de Investigación e Innovación Tecnologíca (PAPIIT) UNAM, grant no. IN111208. Programa de Apoyo a Proyectos Institucionales para el Mejoramiento de la Ensenanza (PAPIME), UNAM, grant PE100907.

References

[1] F.Abdollahi, H.A.Talebi, R.V.Patel, A Stable Neural Network-Based Observer With Application to Flexible-Joint Manipulators, *IEEE Transactions on Neural Networks*, Vol.17 , No.1, 118 - 129, 2006

[2] J. Alvarez, Y. Orlov, and L. Acho, An invariance principle for discontinuos dynamic systems with application to coloumb friction oscilator, *J. Dyna. Syst. Messure. Control*. vol.122. pp. 687-690, 2000.

[3] F. Abdollhi, H. A. Talebi, and R. V. Patel, A stable neural network-based observer with application to flexible-joint manipulators, *IEEE Trans. on Neural Networks*, vol. 17, No. 1 2006.

[4] A. N. Atassi andH. K. Khalil, Separation results for the stabilization of nonlinear systems using different hig-gain observer design, *Syst. Control Lett.*, vol. 39, pp. 183–191, 2000

[5] J.P. Barbot, M. Djemai, and T. Boukhobza, Sliding Mode Observers, *Sliding Mode Control in Engineering*, W. Perruquetti and J. P. Barbot, Eds. New York: Marcel Dekker,103-130, 2002.

[6] I.Chairez, A.Poznyak, T. Poznyak, "New Sliding-Mode Learning Law for Dynamic Neural Network Observer," *IEEE Transactions on Circuits and Systems II: Express Briefs*, Vol.53 , No.12, 1338 - 1342, 2006

[7] G. Cybenco, Approximation by superposition of sigmoidal functions, *Math. Control. Signals and Syst,* 303-314, 1989

[8] J. Davila, L. Fridman and A. Levant. Second- Order Sliding Mode Observer for Mechanical Systems, *IEEE Trans. on Aut. Control*, Vol. 50. No. 11. 1785-1789,2005.

[9] C. Edwards and S. K. Spurgeon, Robust output tracking using a sliding mode controller/observer scheme, *Int. J. of Control*, vol 64, pp. 967-983. 1996.

[10] A.F. Filippov, *Differential equations with discontinuous Right Hand Sides*. Dordrecht, The Netherlands: Kluwer, 1988.

[11] Y. H.Kim , F. L.Lewis , C. T.Abdallah, Non linear observer design using dynamic recurren neural networks, *Proccedings of the 35th Conference on decision Control,* 1996.

[12] H.K.Khalil, *Nonlinear Systems*, Prentice Hall NJ 07458, 1996

[13] A. Levant, Robust exact differentiator via sliding mode thechnique, *Automatica*, vol. 34, no. 3. pp. 379-384, 1998.

[14] D.G.Luemberger, Observing the state of linear systems, *IEEE Transactions on Military Electron* 1964, 8 74-90.

[15] R.Marino and P.Tomei, Adaptive observer with arbitrary exponential rate of convergence for nonlinear system, *IEEE Trans. Automat. Contr.*, vol.40, 1300-1304, 1995.

[16] Y. Orlov, L. Aguilar, and J. C. Cadiou, Switched chattering control vs backlash/friction phenomena in electrical servo motors, *Int. J. Control*. vol. 76. no.9-10. 959-967, 2003.

[17] A. S. Poznyak, W. Yu, Robust asymptotic neuro-observer with time delay term, *Int. J. Robust and Nonlinear Control,* Vol.10, No2, 535-559, 2000

[18] A. S. Poznyak, "Stochastic output noise effects in sliding mode estimation," *Int. J. Control*, vol. 76, pp. 986-999. 2003.

[19] Y.X.Su , P.CMuller and C.H.Zheng ,A simple nonlinear observer for a class of uncertain mechanical systems, *IEEE Transactions on Automatic Control,* Vol.52, No.7, 1340-1345, 2007.

[20] J.C. Willems, Least squares optimal control and algebraic Riccati equations, *IEEE Transactions on Automatic Conrtrol,* Vol.16, 621-634, 1971

[21] W.Yu, M.A. Moreno, X.Li, "Observer Based Neuro Identifier," *IEE Proceedings - Control Theory and Applications*, Vol.147, No.2, 145-152, 2000.