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Robust Asymptotic Stabilization of Nonlinear Systems with Non-Hyperbolic Zero Dynamics: Part II

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Abstract— **This paper complements the accompanying work [12] by presenting a few control scenarios where the tool presented in that work can be successfully used to solve specific robust output feedback stabilization problems. The problem of robust asymptotic stabilization and regulation of nonlinear systems with non hyperbolic zero dynamics via locally Lipschitz partial state or output feedback is addressed.**

I. INTRODUCTION

This paper aims to complement the accompanying work [12] by presenting a few control scenarios where the tool presented in that work can be successfully used to solve specific robust output feedback stabilization problems. More specifically we first address the problem of robust output feedback stabilization and robust regulation of nonlinear systems in normal form whose zero dynamics are asymptotically but not exponentially stable by showing how to design locally Lipschitz regulators. In the second part of the paper we focus our attention on the robust nonlinear separation principle proposed in [17] by showing how, also in this case, the lack of exponential stability of the asymptotic attractor can be overtaken by a proper design of locally Lipschitz regulators. The key notion on which the developed theory relies, is the one of Local Exponential Reproducibility which has been introduced and characterized in terms of sufficient conditions in [12]. The proposed ideas represent a preliminary step toward the design of robust stabilization paradigms which do not only rely on high gain arguments and brute domination of interconnection terms between zero and output dynamics nor on state observability, but rather on asymptotic estimation of the stabilizing control law. In this respect it is expected that the proposed ideas can be inspiring also to approach output feedback stabilization problems for nonminimum-phase systems.

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The reader is referred to the accompanying paper [12] for key facts which are instrumental for the following analysis.

Notation For $x \in I\!R^n$, $|x|$ denotes the Euclidean norm. A class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ satisfying $|s| \leq d \Rightarrow \beta(t,s) \leq Ne^{-\lambda t}|s|$ for some positive d, N, λ is said to be a locally exponential class-$\mathcal{KL}$ function. For a smooth system $\dot{x} = f(x)$, $x \in I\!R^n$, a compact set $\mathcal{A}$ is said to be LAS($\mathcal{X}$) (respectively LES($\mathcal{X}$)), with $\mathcal{X} \subset I\!R^n$ a compact set, if it is locally asymptotically (respectively exponentially) stable with a domain of attraction containing $\mathcal{X}$. Somewhere, by $\mathcal{D}(\mathcal{A})$, we denote the domain of attraction of $\mathcal{A}$ if the latter is LAS/LES for a given dynamics.

II. ROBUST STABILIZATION AND REGULATION OF MINIMUM-PHASE NONLINEAR SYSTEMS

We consider the class of smooth nonlinear systems described in the normal form

$$\begin{array}{rcll} \dot{z} & = & f(w,z,y_1) & z \in I\!R^m \\ \dot{y}_1 & = & y_2 & \\ & \vdots & & y_i \in I\!R \\ \dot{y}_{r-1} & = & y_r & \\ \dot{y}_r & = & b(w,z,y) + a(w,z,y)u & \end{array} \tag{1}$$

with control input $u \in I\!R$ and measurable output $y_m = y_1$, in which $y = \text{col}(y_1, \ldots, y_r)$ and the variable w is thought as generated by an autonomous smooth system of the form

$$\dot{w} = s(w) \qquad w \in W \subset I\!R^s \tag{2}$$

in which W is a compact set invariant for (2). The "high frequency gain" $a(\cdot)$ in (1) is assumed bounded away form zero and, without loss of generality, such that $a(w,z,y) > 0$ for all $(w,z,y) \in W \times I\!R^m \times I\!R^r$. The previous system will be studied under the following "minimum-phase" assumption.

Assumption. There exists a compact set $\mathcal{A} \subset W \times I\!R^m$ which is LAS for the system

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z} & = & f(w,z,0) \end{array} \tag{3}$$

with a domain of attraction $\mathcal{D}(\mathcal{A})$.$\triangleleft$

In this framework the control problem we are going to address is the one of semiglobal output-feedback stabilization by means of a *locally Lipschitz* regulator precisely stated in the following.

Problem. Given arbitrary compact sets $\mathcal{Z} \in \mathcal{D}(\mathcal{A})$ and $\mathcal{Y} \in I\!R^r$, design a *locally Lipschitz* regulator of the form

$$\begin{array}{rcll} \dot{\eta} & = & \Phi(\eta, y_m) & \quad \eta \in I\!R^\nu \\ u & = & \Upsilon(\eta, y_m) & \end{array} \tag{4}$$

such that, for the closed-loop system (1), (2), (4), the set $\mathcal{A} \times \{0\} \times \mathcal{B}$ is LAS($\mathcal{Z} \times \mathcal{Y} \times \mathcal{N}$) for some compact sets $\mathcal{B}$ and $\mathcal{N}$ of $I\!R^\nu$.$\triangleleft$

It is interesting to note how, in the previous framework, both *robust output-feedback stabilization* and *output regulation* problems, which can not be easily handled with the available design tools, can be framed.

In the first case (*robust output feedback stabilization*) ([9], [11]), a possible representative and meaningful scenario to be considered in order to value the challenging aspects of the proposed problem, is the one in which the variable w models constant parametric uncertainties whose values range in the set W (in which case system (2) simplifies as $\dot{w} = 0$), the set $\mathcal{A}$ collapses to $W \times \{0\}$ (namely the origin $z = 0$ is an equilibrium for the system $\dot{z} = f(w,z,0)$ which is LAS for any possible value of the uncertainties), and the function $b(\cdot)$ in (1) is such that $b(w,0,0) = 0$ for all $w \in W$ (in which case the origin of system (1) with $u \equiv 0$ is an equilibrium point). Under these circumstances, the addressed problem boils down to a "classical" problem of stabilizing an equilibrium point (the origin) of a system in presence of parametric uncertainties. Even in this simplified scenario, though, the problem at hand is far to be easily solvable due to two main features characterizing the previous framework: the first is the absence of local exponential stability properties of the set $\mathcal{A}$ (only assumed to be LAS) while the second is the requirement that the solving regulator has to be locally Lipschitz. As a matter of fact (see also the discussion in [12]) the fact that the set $\mathcal{A}$ is only LAS (and not LES) prevents one to use, off-the-shelf, high-gain linear arguments in order to deal with the stability of system (1) due to the fact that the local asymptotic gain (see [9]) of the "zero dynamics" (3) is not, in general, linear. On the other hand, the fact that the regulator is required to be locally Lipschitz does not allow one to use control laws which are only continuous at the origin which, having in mind small gain arguments and results on gain assignment for nonlinear systems (see [10], [11]), one would adopt to handle the nonlinearity of the local asymptotic gain of (3).

The story becomes even more challenging if one look at the previous framework as a problem of *output regulation* (see [2], [13], [14], [8], [6]) in which the variable w may assume the meaning of reference signals to be tracked and/or of disturbances to be rejected generated by the autonomous system (2) which, in the output regulation literature, is usually referred to as the exosystem. In this context the measurable output y_m, which must be asymptotically steered to zero, has the role of *regulation error* and the set $\mathcal{A}$ assumes the meaning of *steady state locus* (by using the terminology introduced in [2]). The latter, according to recent developments in the field ([13]), is usually expressed as the graph of a map, namely it is assumed the existence of a smooth function $\pi : I\!R^s \to I\!R^m$, possibly *set-valued* (see [2]), such that

$$\mathcal{A} = \{(w,z) \in W \times I\!R^m \ : \ z = \pi(w)\} \,.$$

Not surprisingly, it turns out that the output regulation problem hides the same challenging design aspects highlighted above for the stabilization problem which are even worsened by the fact that the function $b(\cdot)$ in (1) is not, in general, vanishing on the desired attractor $\mathcal{A} \times \{0\}$ which thus is not necessarily forward invariant for (1) with $u \equiv 0$. In this respect, what it is required to the controller (4), is also the ability to asymptotically generate a not-zero steady state control input, namely to offset the term $b(w,z,0)/a(w,z,0)$ with $(w,z) \in \mathcal{A}$, by only processing the regulation error. Indeed, this distinguishing feature of the problem of output regulation is what motivates the key concept of internal model and the need of designing internal model-based controllers (see [2], [13], [6]). It is worth noting how, in a local setting, the problem of handling not hyperbolic zero dynamics in output regulation problems has been addressed in [7].

To the best knowledge of the authors, it turns out that the question whether a locally Lipschitz internal model-based regulator exists, in a not-local setting, under the conditions expressed above is still unanswered. The goal of this section (see forthcoming Proposition

1) is precisely to give an answer to this point by taking advantage of the design tools described in the accompanying paper [12].

In order to simplify the notation, in the following we shall drop in (1) the dependence from the variable w which, in turn, will be thought as embedded in the variable z. This, with a mild abuse of notation, will allow us to rewrite system (1) and (2) in the more compact form

$$\begin{array}{rcll} \dot{z} & = & f(z, y_1) & z \in W \times I\!\!R^m \\ \dot{y}_1 & = & y_2 & \\ & \vdots & & y_i \in I\!\!R \\ \dot{y}_{r-1} & = & y_r & \\ \dot{y}_r & = & b(z, y) + a(z, y)u & \end{array} \tag{5}$$

and system (3) as $\dot{z} = f(z, 0)$. Furthermore, since W is invariant for (2)), system (5) evolves on the closed cylinder $W \times I\!\!R^{m+r}$ and it is natural to regard these dynamics restricted to $W \times I\!\!R^{m+r}$ and endow the latter with the relative topology. This will be done from now on by referring to system (5).

Proposition 1: Let $\mathcal{A}$ be LAS($\mathcal{Z}$) for the system $\dot{z} = f(z, 0)$ with $\mathcal{Z}$ a compact set of $\mathcal{D}(\mathcal{A})$. Let the triplet $(f(z, 0), b(z, 0)/a(z, 0), \mathcal{A})$ be rLER (see [12]). Then there exist an integer ν, a continuous function $T : \mathcal{A} \to I\!\!R^\nu$ and, for any compact set $\mathcal{Y} \subset I\!\!R^r$ and $\mathcal{N} \subset I\!\!R^\nu$, a controller of the form (4) such that the set

$$\text{graph}T \times \{0\} = \{(z, y, \eta) \in \mathcal{A} \times I\!\!R^r \times I\!\!R^\nu \; : \; y = 0\,,\; \eta = T(z)\}$$

is LAS($\mathcal{Z} \times \mathcal{Y} \times \mathcal{N}$).
The proof of the previous proposition, which is omitted for reason of space, can be obtained by a proper application of the general tool proposed in [12].

III. Output-feedback from UCO state-feedback in presence of non-hyperbolic attractors

In this part we show how the theory of robust nonlinear separation principle presented in [17], [1] can be extended with the tools developed in [12]. In particular we are interested to extend the theory of [17] by showing how to design a pure output-feedback semiglobal controller stabilizing an attractor when it is known how the latter can be asymptotically (*but not exponentially*) stabilized by means of a Uniform Completely Observable (UCO) state-feedback controller. The results we are going to discuss are (non-trivial) refinements of preliminary results presented in [15]. In the latter the claimed results were given by assuming the existence of smooth regulators, satisfying an appropriate reproducibility condition, not better characterized. Here (and in [12]) we refine that results by providing sufficient conditions for a locally Lipschitz regulator to exists (see Proposition 2 in [12]). In order to meet page-constraints, we only present the main propositions by sketching the proofs and we refer the reader to the extended journal version in preparation for further details.

Consider the smooth system

$$\begin{array}{rcll} \dot{w} & = & s(w) & w \in W \subset I\!\!R^s \\ \dot{z} & = & A(w, z, u) & z \in I\!\!R^m,\, u \in I\!\!R \\ y & = & C(w, z) & y \in I\!\!R \end{array} \tag{6}$$

in which u and y are respectively the control input and the measured output and W is a compact set which is invariant for $\dot{w} = s(w)$. As discussed in the previous section, the variables w emphasize the possible presence of parametric uncertainties and/or disturbance to be rejected and/or reference to be tracked (in the latter case the measurable output y plays more likely the role of regulation/tracking error). As done before, in order to simplify the notation, we drop the dependence of the variable w and we compact system (6) in the more convenient form

$$\begin{array}{rcll} \dot{z} & = & A(z, u) & z \in I\!\!R^m,\, u \in I\!\!R \\ y & = & C(z) & y \in I\!\!R \end{array} \tag{7}$$

which is argued to evolve on a closed invariant set $\mathcal{C}$ which is endowed with the subset topology (such a closed set being, in the form (6), the closed cylinder $\mathcal{C} := W \times I\!\!R^m$).

We recall (see [17]) that a function $\bar{u} : I\!\!R^m \to I\!\!R$ is said to be UCO with respect to (7) if there exist two integers n_y, n_u and a $\mathcal{C}^1$ function Ψ such that, for each solution of

$$\begin{array}{rcll} \dot{z} & = & A(z, u_0) & \\ \dot{u}_i & = & u_{i+1} & 0 = 1, \ldots, n_u - 1 \\ \dot{u}_{n_u} & = & v & \end{array} \tag{8}$$

we have, for all t where the solution makes sense,

$$\bar{u}(z(t)) = \Psi(y(t), y^{(1)}(t), \ldots, y^{(n_y)}(t), u_0(t), \ldots, u_{n_u}(t)) \tag{9}$$

where $y^{(i)}(t)$ denotes the ith derivative of y at time t.

Motivated by [17] we shall study system (7) under the following two assumptions:

a) there exist a smooth function $\bar{u} : I\!R^m \to I\!R$ and compact sets $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{Z} \in \mathcal{C}$, such that the $\mathcal{A}$ is LAS($\mathcal{Z}$) for system (7) with $u = \bar{u}(z)$;[1]
b) $\bar{u}(z)$ is UCO with respect to (7).

In this framework we shall be able to prove, under suitable reproducibility conditions specified later, that the previous two assumptions imply the existence of a locally Lipschitz dynamic output (y) feedback regulator able to asymptotically stabilize the set $\mathcal{A}$. The main theorem in this direction is detailed next. In this theorem we refer to an integer $\ell_u \geq n_u$ defined as that number such that for the system

$$\begin{array}{rcl} \dot{z} & = & A(z, \xi_0) \\ \dot{\xi}_0 & = & \xi_1 \\ & \vdots & \\ \dot{\xi}_{\ell_u} & = & u_1 \,, \end{array} \tag{10}$$

there exist smooth functions C_i such that the first n_y+1 time derivatives of y can be expressed as

$$y^{(i)} = C_i(z, \xi_0, \ldots, \xi_{\ell_u}) \qquad \forall\, i = 0, \ldots, n_y + 1\,.$$

Theorem 1: Consider system (7) and assume the existence of a compact set $\mathcal{A} \subset \mathcal{C}$ and of a smooth function $\bar{u}(z)$ such that properties (a) and (b) specified above are satisfied. Assume, in addition, that the triplets

$$(A(z, \bar{u}(z)), L^{(\ell_u+1)}_{A(z,\bar{u}(z))}\bar{u}(z), \mathcal{A}) \tag{11}$$

and

$$(A(z, \bar{u}(z)), L^{(n_y+1)}_{A(z,\bar{u}(z))}C(z), \mathcal{A}) \tag{12}$$

are rLER. Then there exist a positive o, a compact set $\mathcal{B} \subset I\!R^o$ and, for any $\mathcal{N} \subset I\!R^o$, a locally Lipschitz controller of the form

$$\begin{array}{rcl} \dot{\zeta} & = & \Phi(\zeta, y) \qquad \zeta \in I\!R^o \\ u & = & \Upsilon(\zeta, y) \end{array} \tag{13}$$

such that the set $\mathcal{A} \times \mathcal{B}$ is LAS($\mathcal{Z} \times \mathcal{N}$) for the closed-loop system (7), (13).

We refer the reader to [12], for the presentation of sufficient conditions under which triplets of the form (11), (12) can be claimed to be rLER.

As also observed in [17] (in which Theorem 1) was given in a preliminary version, this result extends Theorem 1.1 of [17] in three directions. First, note that we are dealing with stabilization of compact attractors for systems evolving on *closed sets*. This is a technical improvement on which, though, we would not like to put the emphasis. Second, note that the UCO control law $\bar{u}(z)$ is not required to be vanishing on the attractor $\mathcal{A}$ which, as a consequence, is not required to be forward invariant for the open loop system (7) with $u \equiv 0$. In this respect the proposed setting can be seen as also able to frame output regulation problems which do not fit in the framework discussed in the previous section. Finally, the previous result claims that by means of a pure locally Lipschitz output feedback controller we are able to restore the asymptotic properties of an UCO controller without relying upon exponential stability requirements of the latter and robustly with respect to uncertain parameters. The last two extensions are conceptually very much relevant and can be seen as particular application of the tools presented in [12]. Following the main laying of [17], the proof of the claim is divided in two subsections which contain results interesting on their own.

[1] By referring to (6), a meaningful case to be considered is when $\mathcal{A} = W \times \{0\}$, in which case this assumption amounts to require the existence of a state feedback stabilizer, possibly dependent on the uncertainties, able to asymptotically stabilize the origin with a certain domain of attraction.

A. Robust Asymptotic Backstepping

In this part we discuss how the UCO control law $\bar{u}$ can be robustly back-step through the chain of integrators of (8). As commented above, the forthcoming proposition extends in a not trivial way the results of [17] in the measure in which one considers the fact that $\bar{u}(z)$ is not vanishing on the attractor and that $\mathcal{A}$ is not necessarily locally exponential stable for the closed loop system.

We show that the existence of the static UCO stabilizer for (7) implies the existence of a dynamic stabilizer for (10) using the partial state ξ_i, $i = 0, \ldots, \ell_u$, and the output derivatives $y^{(i)}$, $i = 1, \ldots, n_y$. This is formally proved in the next proposition which refines and extends Theorem 3.2 of [15].

Proposition 2: Consider system (10) under the assumptions (a) and (b) previously formulated. Assume that the triplet (11) is rLER. Then there exists a positive ν, a compact set $\mathcal{R} \supset \mathcal{A}$, a continuous function $\tau : \mathcal{R} \to I\!R^{\nu+\ell_u}$, and, for any compact set $\Xi \subset I\!R^{\ell_u}$ and $\mathcal{N}' \subset I\!R^{\nu}$, a locally Lipschitz regulator of the form

$$\begin{array}{rcl} \dot{\eta} & = & \varphi(\eta, \xi, \bar{u}(z)) \qquad \eta \in I\!R^{\nu} \\ u_1 & = & \rho(\eta, \xi, \bar{u}(z))\,, \end{array} \tag{14}$$

such that the sets

$$\begin{array}{c} \text{graph}\tau := \{(z, \xi, \eta) \in \mathcal{R} \times I\!R^{\ell_u} \times I\!R^{\nu} \; : \\ (\xi, \eta) = \tau(z)\} \end{array} \tag{15}$$

and

$$\begin{array}{c} \text{graph}\; \tau|_{\mathcal{A}} := \{(z, \xi, \eta) \in \mathcal{A} \times I\!R^{\ell_u} \times I\!R^{\nu} \; : \\ (\xi, \eta) = \tau(z)\} \end{array}$$

are respectively LES($\mathcal{Z} \times \Xi' \times \mathcal{N}'$) and LAS($\mathcal{Z} \times \Xi' \times \mathcal{N}'$) for the closed-loop system (10), (14).

Proof: (sketch) Consider the change of variables

$$\begin{array}{lcl} \xi_0 \to \tilde{\xi}_0 & := & \xi_0 - \bar{u}(z) \\ \xi_i \to \tilde{\xi}_i & := & \xi_i - \frac{\partial^{(i)}\bar{u}(z)}{\partial z^i} \quad i = 1, \ldots, \ell_u \,. \end{array}$$

and note that, having defined $y_p := u - \bar{u}(z)$, $\tilde{\xi}_i = y_p^{(i)}$ and that

$$\frac{\partial^{(\ell_u+1)}\bar{u}(z)}{\partial z^{\ell_u+1}} = L^{(\ell_u+1)}_{A(z,\bar{u}(z))}\bar{u}(z) + g(z,\tilde{\xi})$$

where g is a smooth function such that $g(z,0) = 0$ for all $z \in I\!R^{s+m}$ with $\tilde{\xi} := \mathrm{col}(\tilde{\xi}_1, \ldots, \tilde{\xi}_{\ell_u})$. Consider the further change of variable

$$\begin{array}{l} \tilde{\xi}_i, \to \tilde{\xi}' := g^{-i}\tilde{\xi}_i \quad i = 0, \ldots, \ell_u - 1 \\ \tilde{\xi}_{\ell_u} \to \zeta = \tilde{\xi}_{\ell_u} - \sum_{i=0}^{\ell_u-1} \lambda_i g^{\ell_u - i}\tilde{\xi}_i \end{array}$$

where g is a positive design parameter and the λ_i's are coefficients of an Hurwitz polynomial.

The system in the new coordinates reads as

$$\begin{array}{rcl} \dot{z} & = & A(z,\bar{u}(z)) + \tilde{A}(z, C\tilde{\xi}_v) \\ \dot{\tilde{\xi}}_v & = & gH\tilde{\xi}_v + B\zeta \\ \dot{\zeta} & = & u_1 - L^{(\ell_u+1)}_{A(z,\bar{u}(z))}\bar{u}(z) + g'_g(z,\tilde{\xi}_v,\zeta) \end{array} \quad (16)$$

where $B = \mathrm{col}(0, \ldots, 0, 1)$, H is a Hurwitz matrix, $\tilde{\xi}_v := \mathrm{col}(\tilde{\xi}'_0, \ldots, \tilde{\xi}'_{\ell_u-1})$ and $g'_g(\cdot)$ is a smooth function such that $g'_g(z,0,0) = 0$ for all $z \in I\!R^n$. From these facts, the results follows by taking $u_1 = -\kappa\zeta + v$ and by properly adapting the tool presented in [12]. ■

B. Extended Dirty Derivatives Observer

In this part we present a result which allows one to obtain a pure output feedback stabilizer once a partial state-feedback stabilizer (namely a stabilizer processing the output and a certain number of its time derivative) is known. Along the lines pioneered in [5] and [17], the idea is to substitute the knowledge of the time derivatives of the output with appropriate estimates provided by a "dirty derivative observer" (by using the terminology of [17]). In our context, though, we propose an "extended" dirty derivative observer, where the adjective "extended" is to emphasize the presence of a dynamic extension of the classical observer structure motivated by the need of handling the presence of possible not exponentially stable attractors in the partial-state feedback loop and the fact that, on this attractor, the measured output is not necessarily vanishing.

The setting in which we address the problem (which integrates with the analysis of the previous subsection in the framework of Theorem 1) is the one in which we know, for the system (7), a stabilizer of the form

$$\begin{array}{rcl} \dot{\varsigma} & = & \bar{\varphi}(\varsigma, y, y^{(1)}, \ldots, y^{(n_y)}) \quad \varsigma \in I\!R^d \\ u & = & \bar{\rho}(\varsigma, y, y^{(1)}, \ldots, y^{(n_y)}) \end{array} \quad (17)$$

such that the following facts hold true:

a) there exists a compact set $\mathcal{R} \supset \mathcal{A}$ and a continuous function $\tau : \mathcal{R} \to I\!R^d$ such that the sets graphτ and graph $\tau|_{\mathcal{A}}$ are respectively LES($\mathcal{Z} \times \mathcal{H}$) and LAS($\mathcal{Z} \times \mathcal{H}$) for the closed-loop system (7), (17) for some compact set $\mathcal{H} \subset I\!R^d$;
b) there exist smooth functions C_i, $i = 0, \ldots, n_y + 1$, such that the output derivatives $y^{(i)}$ of the closed-loop system (7), (17) can be expressed as
$$y^{(i)} = C_i(z, \varsigma) \qquad i = 0, \ldots, n_y + 1\,;$$
c) the following holds
$$\bar{\rho}(\varsigma, y, y^{(1)}, \ldots, y^{(n_y)})\Big|_{\mathrm{graph}\tau} = \bar{u}(z)$$

Within the previous framework we are able to prove the following proposition which, along with Proposition 2, immediately yields Theorem 1.

Proposition 3: Consider system (7) and assume the existence of a dynamic regulator of the form (17) such that the previous properties (a)-(c) are satisfied. Assume, in addition, that the triplet (12) is rLER. Then there exist a positive o, a compact set $\mathcal{B} \subset I\!R^o$ and, for any compact set $\mathcal{N} \subset I\!R^o$, an output feedback controller of the form (13) such that the set $\mathcal{A} \times \mathcal{B}$ is LAS($\mathcal{Z} \times \mathcal{N}$) for the closed-loop system (7), (13).

Proof: (Sketch) As candidate controller, we consider a system of the form

$$\begin{array}{rcl} \dot{\varsigma} & = & \bar{\varphi}_\ell(\varsigma, y, \hat{y}_1, \ldots, \hat{y}_{n_y}) \\ \dot{\hat{y}}_i & = & \hat{y}_{i+1} + L^{i+1}\lambda_i(\hat{y}_0 - y) \\ \dot{\hat{y}}_{n_y} & = & L^{n_y+1}\lambda_{n_y}(\hat{y}_0 - y) + v \\ u & = & \bar{\rho}_\ell(\varsigma, y, \hat{y}_1, \ldots, \hat{y}_{n_y}) \end{array}$$

in which v is a control input to be designed, L is a positive design parameters, the λ_i's are the coefficients of an Hurwitz polynomial and $\bar{\varphi}_\ell(\cdot)$ and $\bar{\rho}_\ell$ are appropriate saturated versions of the functions $\bar{\varphi}(\cdot)$ and $\bar{\rho}(\cdot)$ of (17) satisfying $\bar{\varphi}_\ell(s) = \bar{\varphi}(s)$ if $\|\bar{\varphi}(s)\| \leq \ell$, $\|\bar{\varphi}_\ell(s)\| \leq \ell$ for all s, and $\bar{\rho}_\ell(s) = \bar{\rho}(s)$ if $\|\bar{\rho}(s)\| \leq \ell$, $\|\bar{\rho}_\ell(s)\| \leq \ell$ for all s, with ℓ a design parameter.

Let now $y_d = \mathrm{col}(y, y^{(1)}, \ldots, y^{(n_y)})$, $\hat{y} = \mathrm{col}(\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_{n_y})$ and consider the change of variables $\hat{y} \mapsto e = D_L(y_d - \hat{y})$ where $D_L =$

$\text{diag}(L^{n_y}, L^{n_y-1}, \ldots, 1)$. In this coordinate setting, by denoting $x = \text{col}(\, z \quad \varsigma \,)$, the overall closed-loop system reads as

$$\begin{array}{rcl} \dot{x} & = & f(x) + \Delta(x, e) \\ \dot{e} & = & LHe + B(q(x) + v) \end{array} \tag{18}$$

in which H is a Hurwitz matrix in observability canonical form, $B = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix}^{\mathrm{T}}$ $\dot{x} = f(x)$ is a compact representation of the system (7), (17), $q(x) = C_{n_y+1}(z, \varsigma)$ and $\Delta(x, e)$ is a suitably defined function such that $\Delta(x, 0) = 0$ for all $x \in \text{graph}\tau$ provided that the variable ℓ is properly fixed. From these facts, by Proposition 4 (by which it can be proved that (12) rLER $\Rightarrow$ $(f(x), q(x), \text{graph}\ \tau|_{\mathcal{A}})$ LER), and by the results in [12], the result follows. ■

APPENDIX

A. Auxiliary Results

Proposition 4: Consider a system of the form

$$\begin{array}{rclr} \dot{x}_1 & = & f_1(x_1, x_2) & x_1 \in I\!R^{n_1} \\ \dot{x}_2 & = & f_2(x_1, x_2) & x_2 \in I\!R^{n_2} \end{array} \tag{19}$$

and assume that there exist a compact set $\mathcal{A} \subset I\!R^{n_1}$ and a smooth function $\tau : I\!R^{n_1} \to I\!R^{n_2}$ such that the set

$$\text{graph}\ \tau|_{\mathcal{A}} = \{(x_1, x_2) \in \mathcal{A} \times I\!R^{n_2} \quad : \quad x_2 = \tau(x_1)\}$$

is LES for (19). Let $q : I\!R^{n_1} \times I\!R^{n_2} \to I\!R$ be a smooth function. Then the triplet $(f, q,\ \text{graph}\ \tau|_{\mathcal{A}})$ is LER if the triplet $(f_1(x_1, \tau(x_1)),\ q(x_1, \tau(x_1)), \mathcal{A})$ is rLER.

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