

Robust Asymptotic Stabilization of Nonlinear Systems with Non-Hyperbolic Zero Dynamics: Part I

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Abstract—In this paper we present a general tool to handle the presence of zero dynamics which are asymptotically, but not locally exponentially, stable in problems of robust nonlinear stabilization by output feedback. We show how it is possible to design *locally Lipschitz* stabilizers under conditions which do not rely upon any observability assumption on the controlled plant, by thus obtaining a robust stabilizing paradigm which is not based on design of observers and separation principles. The main design idea comes from recent achievements in the field of output regulation and specifically in the design of nonlinear internal models. In this sense the results presented in this paper also complement in a non trivial way a certain number of works recently proposed in the field of output regulation by presenting meaningful conditions under which a locally Lipschitz regulator exists.

The present work is complemented by a part II paper submitted to this conference ([11]) in which possible applications of the presented tool in the context of the robust stabilization and regulation of minimum-phase nonlinear systems and robust nonlinear separation principle are presented.

Notation For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm and, for \mathcal{C} a closed subset of \mathbb{R}^n , $|x|_{\mathcal{C}} = \min_{y \in \mathcal{C}} |x - y|$ denotes the distance of x from \mathcal{C} . For \mathcal{S} a subset of \mathbb{R}^n , $\text{cl}\mathcal{S}$ and $\text{int}\mathcal{S}$ are the closure of \mathcal{S} and the interior of \mathcal{S} respectively, and $\partial\mathcal{S}$ its boundary. A class- \mathcal{KL} function $\beta(\cdot, \cdot)$ satisfying $|s| \leq d \Rightarrow \beta(t, s) \leq Ne^{-\lambda t}|s|$ for some positive d, N, λ is said to be a locally exponential class- \mathcal{KL} function. For a locally Lipschitz system of the form $\dot{z} = f(z)$ the value at time t of the solution passing through z_0 at time $t = 0$ will be written as $\phi_f(t, z_0)$ or, if the initial condition and the system are clear from the context, as $z(t)$ or $z(t, z_0)$. For a smooth system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, a compact set \mathcal{A} is said to be LAS(\mathcal{X}) (respectively LES(\mathcal{X})), with $\mathcal{X} \subset \mathbb{R}^n$ a compact set, if it is locally asymptotically (respectively exponentially) stable with a domain of attraction containing \mathcal{X} . By $\mathcal{D}(\mathcal{A})$ we denote the domain of attraction of \mathcal{A} if the latter is LAS/LES for a given dynamics.

I. THE FRAMEWORK

The main goal of this paper is to present a design tool to handle the presence of asymptotically but not necessarily exponentially stable zero dynamics in robust output-feedback stabilization problems of nonlinear systems. Although the tool we are going to present lends itself to be useful in a significant variety of control scenarios, in order to keep

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confined the discussion while maintaining a certain degree of generality, we focus our attention on the class of *smooth* systems of the form

$$\begin{aligned} \dot{x} &= f(w, x, y) \quad x \in \mathbb{R}^n \\ \dot{y} &= a(w, x, y) [\kappa Hy + B(q(w, x, y) + v)] \quad y \in \mathbb{R}^r \end{aligned} \quad (1)$$

with measurable output $y_m = Cy$, $y_m \in \mathbb{R}$, in which the triplet (H, B, C) is assumed to be prime with the pair (H, C) which is observable, κ is a positive design parameter, v is a control input and $a(w, x, y)$ a smooth real valued function with $a(w, x, y) > 0$ for all w, x and y . In the previous system the variable $w \in \mathbb{R}^s$ represents an exogenous variable which is governed by

$$\dot{w} = s(w) \quad w \in W \subset \mathbb{R}^s \quad (2)$$

with W a compact set which is *invariant* for (2). As a particular case, the signals $w(t)$ generated by (2) may be constant signals, i.e. $s(w) \equiv 0$, namely constant uncertain parameters taking value in the set W and affecting the system (1). In general, the variables w can be considered as exogenous signals which, depending on the considered control scenario, may represent references to be tracked and/or disturbances to be rejected as better explained in the accompanying paper [11].

Remark As a consequence of the fact that W is a (forward and backward) invariant set for (2), the *closed* cylinder $\mathcal{C}_{n+r} := W \times \mathbb{R}^{n+r}$ is invariant for (1),(2). Thus it is natural to regard system (1), (2) on \mathcal{C}_{n+r} and endow the latter with the relative topology. This will be done from now on by referring to system (1),(2). Analogously, the dynamics described by the first n equations of (1) and by (2) will be thought as evolving on the closed set $\mathcal{C}_n := W \times \mathbb{R}^n$ which will be endowed with the relative topology. \triangleleft

We shall study the previous system under the following “minimum-phase” assumption.

Assumption There exists a compact sets $\mathcal{A} \subset \mathcal{C}_n$ and $\mathcal{X} \subset \mathcal{C}_n$, with $\mathcal{A} \subset \text{int}\mathcal{X}$, such that the set \mathcal{A} is LAS(\mathcal{X}) for the system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w, x, 0) \end{aligned} \quad (3)$$

In this framework we consider the (robust) output feedback stabilization problem which consists of designing a *locally Lipschitz* regulator of the form

$$\dot{\eta} = \varphi_k(\eta, y_m) \quad v = \rho_k(\eta, y_m) \quad \eta \in \mathbb{R}^\nu, \quad (4)$$

and, given arbitrary bounded sets $\mathcal{Y} \subset \mathbb{R}^r$ and $\mathcal{N} \subset \mathbb{R}^\nu$, a positive κ^* , such that for all $\kappa \geq \kappa^*$ and for some $\mathcal{B} \subset \mathbb{R}^\nu$

the set $\mathcal{A} \times \{0\} \times \mathcal{B}$ is LAS($\mathcal{X} \times \mathcal{Y} \times \mathcal{N}$) for the closed-loop system (1), (4).

Remark We remark how, in the previous framework, the regulator solving the problem at hand is required to be locally Lipschitz. \triangleleft

II. A BRIEF DIGRESSION ABOUT THE PROBLEM

The structure of (1) and the associated problem, apparently very specific, are indeed recurrent in a number of control scenarios in which robust non linear stabilization is involved. We defer to the second part of this paper (see [11]) the presentation of a few relevant cases where this occurs. For the time being it is interesting to note how the previous formulation presents two main peculiarities which make the problem at hand particularly challenging.

The first is that the function $q(w, x, y)$, coupling the x and y subsystem in (1), is not necessarily vanishing on the desired attractor $\mathcal{A} \times \{0\}$, namely the desired attractor $\mathcal{A} \times \{0\}$ is not necessarily forward invariant for (1) in the case $v \equiv 0$. In this respect the first crucial property required to the regulator is to be able to reproduce, through the input v , the uncertain coupling term $q(w, x, 0)$ by providing a not necessarily zero steady-state control input. This issue is intimately connected to arguments which are usually addressed in the output regulation literature (see [10], [3]), [6], [13]), in which the goal is precisely to make attractive a set, on which regulation objectives are met, which is not invariant for the open-loop system.

The second peculiarity, apparently not correlated to the previous one, relies in the fact that the set \mathcal{A} is assumed to be "only" asymptotically stable for (3) and no exponential properties are required. In this respect the study of the interconnection (1) is particularly challenging as it is not sufficient, in general, to decrease the linear asymptotic gain ([16]) between the "inputs" x and the "outputs" y of the y -subsystem (which is what one would make by increasing the value of κ since the matrix H is Hurwitz) to infer asymptotic properties in the interconnection. Indeed the presence of a not necessarily linear asymptotic gain between the "inputs" y and the "outputs" x of the x -subsystem requires a non trivial design of the input v which, intuitively, should be chosen to infer a certain (non-linear) ISS gain to the y -subsystem. Interestingly enough, we will show in the following that the two previous issues are indeed correlated, in the sense that the ability to solve the first will allow us to get rid also of the second.

The rich available literature on nonlinear stabilization already provides successful tools to solve the problem at hand if the previous two pathologies are dropped, namely if the assumption is strengthened by asking that the set \mathcal{A} is also LES(\mathcal{X}) for (3) and that the "coupling" term $q(w, x, y)$ is vanishing at $\mathcal{A} \times \{0\}$. As a matter of fact, under the previous conditions, it is a well-known fact that the set $\mathcal{A} \times \{0\}$, which is forward invariant for (1) with $v = 0$, can be stabilized by means of a large value of k as formalized in the following theorem ([14], [2]).

Theorem 1: Let \mathcal{A} be LES(\mathcal{X}) for the system (3) and $q(w, x, 0) \equiv 0$ for all $(w, x) \in \mathcal{A}$. Then for any compact set $\mathcal{Y} \subset \mathbb{R}^r$ there exists a $\kappa^* > 0$ such that for all $\kappa \geq \kappa^*$ the set $\mathcal{A} \times \{0\}$ is LES($\mathcal{X} \times \mathcal{Y}$) for (1) with $v \equiv 0$.

In the case \mathcal{A} is not exponentially stable for (3) and/or the coupling term $q(w, x, y)$ is not vanishing on the desired attractor, the problem becomes challenging and more sophisticated choices for v must be envisaged. In particular the only conclusions which can be drawn if $v \equiv 0$ is that the origin is *semiglobally practically* stable in the parameter κ , that is the trajectories of the system can be steered arbitrary close to the set $\mathcal{A} \times \{0\}$ by increasing the value of κ (see [14], [2], [10]). Even in the simpler scenario in which $q(w, x, 0) \equiv 0$ for all $(w, x) \in \mathcal{A}$, a large value of k is not sufficient to enforce the desired asymptotic behavior in the case the set \mathcal{A} fails to be exponentially stable for (3). In this case the asymptotic properties of the system have been studied in [4] by showing how the trajectories are attracted by a manifold which, only in a particular case depending on the linear approximation of the system, collapses to the origin (see Theorem 6.2 in [4]).

In these critical scenarios an appropriate design of the control input v becomes unavoidable in order to compensate for the coupling term $q(w, x, y)$ which cannot be only dominated by a large value of κ . In particular, a first possible option, motivated by small gain arguments and gain assignment procedures for nonlinear systems (see [9], [8]), is to design the control v in order to assign, to the y -subsystem, a certain nonlinear ISS gain suitably identified according to small gain criterions and to the asymptotic gain of the x -subsystem in (1). This option, however, necessarily leads to design control laws which are not, in general, locally Lipschitz close to the compact attractor and, thus, which violates a basic requirement of the above problem.

An alternative option to design the control v is to be inspired by nonlinear separation principles (see, besides others, [14], [15], [2], [7], [5]), namely to design an appropriate state observer yielding an asymptotic estimate $(\hat{w}, \hat{x}, \hat{y})$ of the state variables, and to asymptotically compensate for the coupling term $q(w, x, y)$ by implementing a "certainty equivalence" control law of the form $v = -q(\hat{w}, \hat{x}, \hat{y})$. Indeed, under suitable conditions, the tools proposed in [15] would allow one to precisely fix the details and to solve the problem at hand in a rigorous way. This way of approaching the problem, though, presents a number of drawbacks which substantially limit its applicability. First, the design of the observer clearly requires the formulation of suitable observability assumptions¹ on the controlled plant, not in principle necessary for the stabilization problem to be solvable, which may be not fulfilled for a number of relevant cases. Moreover, according to the state-of-the-art of the observer design literature ([5]), the design of the observer

¹It must be noted that only *local* observability notion are potentially needed at this level as a consequence of the fact that practical stability is already guaranteed by the high-gain law κ .

may be a challenging (if not impossible) task in case of uncertain parameters affecting the observed dynamics. Finally, it is worth noting how approaching the problem according to the previous design philosophy, leads to inherently *redundant control structures*, by requiring the explicit estimate of the full state (and of possible uncertainties) in order to “only” reproduce the signal $q(w, x, y)$.

As opposite to the previous strategies, we shall show in the following how the problem at hand can be solved by means of a locally Lipschitz, not observed-based, controller.

III. THE NEW ASYMPTOTIC RESULTS

The goal of this part is to present new results regarding the solution of the robust stabilization problem formulated above. In order to ease the notation, in the following we shall drop in (1) the dependence from the variable w which, in turn, will be thought as embedded in the variable x (with the latter varying in the set \mathcal{C}_n). This, with a mild abuse of notation, will allow us to rewrite system (1) and (2) in the more compact form

$$\begin{aligned}\dot{x} &= f(x, y) \quad x \in \mathcal{C}_n \subset \mathbb{R}^{s+n} \\ \dot{y} &= a(x, y) [\kappa Hy + B(q(x, y) + v)] \quad y \in \mathbb{R}^r\end{aligned}\quad (5)$$

and system (3) as $\dot{x} = f(x, 0)$.

The existence of a locally Lipschitz regulator solving the problem at hand, will be claimed under an assumption which involves the ability of asymptotically reproducing the function $q(x(t), 0)$, where $x(t)$ is any solution of $\dot{x} = f(x, 0)$ which can be generated by taking initial conditions on \mathcal{A} , by means of a locally Lipschitz system properly defined. The following definition, which will be commented after its statement, aims to formally state the required reproducibility condition which will be then used in the forthcoming Theorem 2.

Definition 1: (LER, rLER) A triplet $(g(\cdot), h(\cdot), \mathcal{A})$, where $g: \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $h: \mathbb{R}^m \rightarrow \mathbb{R}$ are smooth functions and $\mathcal{A} \subset \mathbb{R}^m$ is a compact set, is said to be *Locally Exponentially Reproducible* (LER), if there exists a compact set $\mathcal{R} \supseteq \mathcal{A}$ which is LES for $\dot{z} = g(z)$ and, for any bounded set \mathcal{Z} contained in the domain of attraction of \mathcal{R} , there exist an integer p , *locally Lipschitz* functions $\varphi: \mathbb{R}^p \rightarrow \mathbb{R}^p$, $\gamma: \mathbb{R}^p \rightarrow \mathbb{R}$, and $\psi: \mathbb{R}^p \rightarrow \mathbb{R}^p$, with ψ a complete vector field, and a smooth function $T: \mathbb{R}^m \rightarrow \mathbb{R}^p$, such that

$$h(z) + \gamma(T(z)) = 0 \quad \forall z \in \mathcal{R}, \quad (6)$$

and for all $\xi_0 \in \mathbb{R}^p$ and $z_0 \in \mathcal{Z}$ the solution $(\xi(t), z(t))$ of

$$\begin{aligned}\dot{z} &= g(z) & z(0) &= z_0 \\ \dot{\xi} &= \varphi(\xi) + \psi(\xi)h(z) & \xi(0) &= \xi_0\end{aligned}\quad (7)$$

satisfies,

$$|(\xi(t), z(t))|_{\text{graph } T|_{\mathcal{R}}} \leq \beta(t, |(\xi_0, z_0)|_{\text{graph } T|_{\mathcal{R}}}) \quad (8)$$

where $\beta(\cdot, \cdot)$ is a locally exponentially class- $\mathcal{K}\mathcal{L}$ function.

Furthermore the triplet in question is said to be *robustly Locally Exponentially Reproducible* (rLER) if it is LER and, in addition, for all locally essentially bounded $v(t)$, for all $\xi_0 \in \mathbb{R}^p$ and $z_0 \in \mathcal{Z}$ the solution $(\xi(t), z(t))$ of

$$\begin{aligned}\dot{z} &= f(z) & z(0) &= z_0 \\ \dot{\xi} &= \varphi(\xi) + \psi(\xi)[q(z) + v(t)] & \xi(0) &= \xi_0\end{aligned}\quad (9)$$

satisfies

$$|(\xi(t), z(t))|_{\text{graph } T|_{\mathcal{R}}} \leq \beta(t, |(\xi_0, z_0)|_{\text{graph } T|_{\mathcal{R}}}) + \ell(\sup_{\tau \leq t} |v(\tau)|) \quad (10)$$

where $\beta(\cdot, \cdot)$ is a locally exponentially class- $\mathcal{K}\mathcal{L}$ function and ℓ is a class- \mathcal{K} function. \triangleleft

A few words to comment the previous definition are in order. First of all note that, for a triplet (f, g, \mathcal{A}) to be LER, the key first requirement is that there exists a set \mathcal{R} which contains \mathcal{A} and which is LES for the autonomous system $\dot{z} = g(z)$. As it will be shown in Section IV (see Lemma 1), this is always the case if the set \mathcal{A} is LAS for $\dot{z} = g(z)$. According to this result, since in the framework where the previous definition will be used (see Theorem 2) the set \mathcal{A} will be assumed LAS, the existence of the set \mathcal{R} can be always considered fulfilled in our setting. The second crucial requirement characterizing the previous definition is that there exists a *locally Lipschitz* system of the form

$$\begin{aligned}\dot{\xi} &= \varphi(\xi) + \psi(\xi)u_\xi \\ y_\xi &= h(\xi)\end{aligned}\quad (11)$$

with input u_ξ and output y_ξ , such that system (7), modelling the cascade connection of the autonomous system $\dot{z} = g(z)$ with output $y_z = h(z)$ with the system (11), has a locally exponentially stable set described by $\text{graph } T|_{\mathcal{R}}$ and, on this set, the output y_ξ equals y_z (see (6)). The domain of attraction of $\text{graph } T|_{\mathcal{R}}$ is required to be of the form $\mathcal{Z} \times \mathbb{R}^p$ with \mathcal{Z} any compact set in the domain of attraction of \mathcal{R} (note that, according to the definition, system (11) is allowed to depend on the choice of \mathcal{Z}). In this respect the second requirement can be regarded as the ability, of the system (11), of *asymptotically reproducing* any output function $h(z(t))$ of system $\dot{z} = h(z)$ with initial conditions of the latter taken in \mathcal{Z} . As far as the definition of robust LER is concerned, we only note that, in addition to the previous properties, it is required that system (9) exhibits an ISS property (without any special requirement on the asymptotic gain) with respect to the exogenous input v . This enforced property will be needed in the accompanying paper [11] and does not play any special role in the next analysis.

We shall show in Section IV a meaningful sufficient condition under which a given triplet can be claimed to be rLER and, as a consequence, LER (see Lemma 2). For the time being it is worth concluding the comments to the previous definition with a couple of final observations. First, it is interesting to observe how the “output reproducibility” property required to system (11) does not hide, in principle, any kind of state observability property of the system $\dot{z} = g(z)$ with output $y_z = h(z)$. In other words system (11) must be not confused with a state observer of the z -subsystem as its role is to reproduce the output function $h(z(t))$ and not necessarily to estimate its state. Finally, it is worth stressing the crucial requirement of the definition asking system (11)

to be locally Lipschitz. In this respect it is interesting to note that, if this requirement is dropped (namely in case system (11) were allowed to be only continuous), the theoretical tools presented in [10] are sufficient to prove that any smooth triplet (g, h, \mathcal{A}) is rLER if \mathcal{A} is LAS for $\dot{z} = g(z)$. For reasons of space we omit details in this direction which, though, can be easily retrieved by the interested reader going through the proof of the forthcoming Lemma 2.

With this definition at hand, we pass to formulate the following theorem which fixes a framework where the stabilization problem previously formulated can be solved by means of a locally Lipschitz regulator.

Theorem 2: Let \mathcal{A} be LAS(\mathcal{X}) for the system $\dot{x} = f(x, 0)$ for some compact set $\mathcal{X} \subset \mathcal{C}$. Assume, in addition, that the triplet $(f(x, 0), q(x, 0), \mathcal{A})$ is LER. Then there exist a locally Lipschitz regulator of the form (4), a compact set $\mathcal{R} \supseteq \mathcal{A}$, a continuous function $\tau : \mathcal{R} \rightarrow \mathbb{R}^{\nu}$, and, for any compact set $\mathcal{Y} \subset \mathbb{R}^r$ and $\mathcal{N} \subset \mathbb{R}^{\nu}$, a positive constant κ^* , such that for all $\kappa \geq \kappa^*$ the set

$$\text{graph } \tau \times \{0\} = \{(x, y, \eta) \in \mathcal{R} \times \mathbb{R}^r \times \mathbb{R}^{\nu} : y = 0, \eta = \tau(x)\} \quad (12)$$

is LES($\mathcal{X} \times \mathcal{Y} \times \mathcal{N}$) for (5), (4) and the set

$$\text{graph } \tau|_{\mathcal{A}} \times \{0\} = \{(x, y, \eta) \in \mathcal{A} \times \mathbb{R}^r \times \mathbb{R}^{\nu} : y = 0, \eta = \tau(x)\} \quad (13)$$

is LAS($\mathcal{X} \times \mathcal{Y} \times \mathcal{N}$) for (5), (4).

IV. SUFFICIENT CONDITIONS FOR ASYMPTOTIC REPRODUCIBILITY

The main goal of this section is to present a number of results which are useful to test when a triplet (g, h, \mathcal{A}) is rLER (and thus LER). As also commented just after its statement in Section III, the definition relies upon two requirements, the first being the existence of a compact set $\mathcal{R} \supseteq \mathcal{A}$ which is LES for $\dot{z} = g(z)$, while the second asking the existence of locally Lipschitz functions (φ, ψ, γ) and of a smooth function T such that system (9) presents the ISS property (10).

Towards this end we start with a result which claims that the first requirement for a triplet (g, h, \mathcal{A}) to be rLER, namely the existence of a set \mathcal{R} which is LES for $\dot{z} = g(z)$, is automatically guaranteed if the set \mathcal{A} is LAS for $\dot{z} = g(z)$. Thus, put in the context of Theorem 2, the first requirement of the definition is not restrictive at all. Details of this fact are reported in the following lemma whose proof can be found in [12].

Lemma 1: Consider system

$$\dot{z} = g(z) \quad z \in \mathbb{R}^m \quad (14)$$

evolving on an invariant closed set $\mathcal{C} \subset \mathbb{R}^m$. Let $\mathcal{A} \subset \mathcal{C}$ be a compact set which is LAS with domain of attraction $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{C}$. For any compact set $\mathcal{S} \subset \mathcal{C}$ such that $\mathcal{A} \subset \text{int}\mathcal{S}$, there exists a compact set \mathcal{R} satisfying $\mathcal{A} \subseteq \mathcal{R} \subset \mathcal{S}$ which is LES for (14) with domain of attraction $\mathcal{D}(\mathcal{R}) \equiv \mathcal{D}(\mathcal{A})$.

We pass now to analyze the second crucial requirement behind the definition of rLER, namely the existence of locally Lipschitz functions (φ, ψ, γ) and of a smooth function T such that condition (10) is satisfied for system (9). In this respect we present, in the next lemma, a meaningful condition which is sufficient for a triplet to be rLER. Such a condition relies upon the existence of an open bounded set which is *invariant* for $\dot{z} = g(z)$. We postpone to the statement of the lemma the comments to the given condition and the presentation of an additional result which aims to make the condition applicable also to cases where the bounded invariant open set is not guaranteed to exist.

Lemma 2: Let \mathcal{O} be an open bounded set which is invariant for $\dot{z} = g(z)$ with $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$ a C^∞ function. Let \mathcal{A} be LAS for $\dot{z} = g(z)$ and let $h : \mathbb{R}^m \rightarrow \mathbb{R}$ be a smooth function. If there exists a $c > 0$ such that

$$\dim \Omega(z) = c \quad \forall z \in \mathcal{O} \quad (15)$$

where

$$\Omega(z) := \sum_{k=0}^{\infty} \text{span} \frac{\partial}{\partial z} L_{g(z)}^k h(z) \quad (16)$$

then the triplet (g, h, \mathcal{A}) is rLER.

The proof of this lemma is sketched in Section V.

Remark Condition (15)-(16) must be not confused with a (local) uniform observability property for the system $\dot{z} = g(z)$ with output $y_z = h(z)$ which can be obtained by limiting the sum in (16) to $k \leq m$ and asking that $c = m$. \triangleleft

The requirement about the existence of the *invariant* set \mathcal{O} which characterizes the previous result may limit in a substantial way the class of systems for which the robust reproducibility condition can be checked through (15). In this respect we present a result which can be useful, besides others, to overtake this limitation. In short, we shall show that the property of a triplet (g, h, \mathcal{A}) to be rLER is guaranteed if the pair (g, h) is immersed, in a neighborhood of \mathcal{A} , into an immersing pair (\tilde{g}, \tilde{h}) through a smooth function τ , and the triplet $(\tilde{g}, \tilde{h}, \tau(\mathcal{A}))$ is rLER. The details of the result are presented in the next lemma whose proof is omitted for reasons of space.

Lemma 3: Let $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $h : \mathbb{R}^m \rightarrow \mathbb{R}$ be given smooth functions. Let \mathcal{A} be a compact set which is LAS for the system $\dot{z} = g(z)$ and let \mathcal{S} be a compact set such that $\mathcal{A} \subset \text{int}\mathcal{S}$. Assume that the pair (g, h) is immersed, through a C^1 map $\tau : \mathbb{R}^m \rightarrow \mathbb{R}^{\tilde{m}}$, into a smooth pair $\tilde{g} : \mathbb{R}^{\tilde{m}} \rightarrow \mathbb{R}^{\tilde{m}}$ and $\tilde{h} : \mathbb{R}^{\tilde{m}} \rightarrow \mathbb{R}$, with respect to \mathcal{S} , namely

$$\begin{aligned} \frac{\partial \tau}{\partial z} g(z) &= \tilde{g}(\tau(z)) \\ h(z) &= \tilde{h}(\tau(z)) \end{aligned} \quad \forall z \in \mathcal{S}. \quad (17)$$

Assume, in addition, that \tilde{g} is a complete vector field. If $(\tilde{f}, \tilde{q}, \tau(\mathcal{A}))$ is rLER then (f, q, \mathcal{A}) is such.

We present now a simple example to show how the previous result can be effectively used to overtake the limitation imposed in Lemma 2 by the existence of the invariant bounded set.

Example. Consider the system (3) in the form

$$\begin{aligned} \dot{w} &= 0 & w \in W &:= \{w \in \mathbb{R} : \underline{w} \leq w \leq \bar{w}\} \\ \dot{x} &= -x^3 + w & z \in \mathbb{R} \end{aligned} \quad (18)$$

with “output” $h(w, x) = x$, defined on the closed cylinder $\mathcal{C} := W \times \mathbb{R}$, for which the set $\mathcal{A} := \{(w, x) \in \mathcal{C} : x = \sqrt[3]{w}\}$ is LAS with $\mathcal{D}(\mathcal{A}) = \mathcal{C}$. If one is willing to check the robust exponential reproducibility of the triplet (g, h, \mathcal{A}) with $g = \text{col}(0, -x^3 + w)$ the previous lemma cannot be applied as such due to the absence of a bounded invariant set containing \mathcal{A} (indeed finite escape times occur in backward time for initial conditions outside \mathcal{A}). To this purpose, however, lemma 3 helps as shown in the following.

Let \mathcal{S} be a compact set of the form

$$\mathcal{S} = \{(w, x) \in \mathcal{C} : \sqrt[3]{\underline{w}} - \frac{1}{2} \leq x \leq \sqrt[3]{\bar{w}} + \frac{1}{2}\}$$

and note that, by Lemma 1, there exists a compact set $\mathcal{R} \subseteq \mathcal{S}$, $\mathcal{A} \subseteq \mathcal{R}$, which is LES for (18) with $\mathcal{D}(\mathcal{R}) = \mathcal{C}$.

Pick a smooth function $a : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ such that $a(s) = 1$ for all $\sqrt[3]{\underline{w}} - \frac{1}{2} \leq s \leq \sqrt[3]{\bar{w}} + \frac{1}{2}$ and $a(s) = 0$ for all $s \leq \sqrt[3]{\underline{w}} - 1$ and $s \geq \sqrt[3]{\bar{w}} + 1$, and consider the “immersing” system

$$\dot{w} = 0 \quad \dot{\tilde{x}} = a(\tilde{x})[-\tilde{x}^3 + \tilde{w}] \quad (19)$$

with “output” $\tilde{h}(\tilde{w}, \tilde{x}) = \tilde{x}$. System (18) is immersed into system (19) with respect to the set \mathcal{S} through any smooth function $\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\tau(w, x) = \text{col}(w, x) \quad \forall (w, x) \in \mathcal{S}.$$

Furthermore the bounded set \mathcal{O} defined as

$$\mathcal{O} = \{(w, \tilde{x}) \in \mathcal{C} : \sqrt[3]{\underline{w}} - 1 \leq \tilde{x} \leq \sqrt[3]{\bar{w}} + 1\},$$

which is open with respect to the subset topology induced by \mathcal{C} , is invariant for (19). According to lemma 3, the triplet (g, h, \mathcal{A}) is rLER if the triplet $(\tilde{g}, \tilde{h}, \mathcal{A})$, with $\tilde{g} = \text{col}(0, a(\tilde{x})[-\tilde{x}^3 + \tilde{w}])$, is such. But, as $\mathcal{O} \supset \mathcal{A}$ is invariant for (19) and \tilde{g} is complete, Lemma 2 can be applied. In this specific case

$$\Omega(\tilde{w}, \tilde{x}) = \text{span} \begin{pmatrix} 0 & 1 \\ a(\tilde{x}) & * \end{pmatrix}$$

where $*$ is a junk term, from which it follows that

$$\dim \Omega(\tilde{w}, \tilde{x}) \equiv 2 \quad \forall (\tilde{w}, \tilde{x}) \in \mathcal{O}$$

which implies that the triplet $(\tilde{g}, \tilde{h}, \mathcal{A})$ is rLER. \triangleleft

V. SKETCH OF THE PROOF OF LEMMA 2

Since \mathcal{A} is LAS for $\dot{z} = g(z)$ it turns out, by Lemma 1, that there exists a compact set \mathcal{R} satisfying $\mathcal{O} \supset \mathcal{R} \supseteq \mathcal{A}$ which is LES and $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{A})$. Note also that $\mathcal{O} \subseteq \mathcal{D}(\mathcal{R})$, because \mathcal{O} is backward invariant for $\dot{z} = g(z)$.

Let $Z(t, z)$ denote the flow of $\dot{z} = g(z)$. Since by construction g is backward complete and C^∞ on \mathcal{O} , $Z(t, z)$

is defined for all $(t, z) \in (-\infty, 0) \times \mathbb{R}^n$ and gives rise to a C^∞ function. Also, for each non positive t , $z \mapsto Z(t, z)$ is a diffeomorphism. Set $y(t, z) := h(Z(t, z))$. For each z , the function $t \in (-\infty, 0] \mapsto y(t, z)$ is bounded. Also, for each $t \in (-\infty, 0]$ and any $k \in \mathbb{N}$, we have

$$\frac{d^k y(t, z)}{dt^k} = L_g^k h(Z(t, z)).$$

Let $(F, G) \in \mathbb{R}^{p \times p} \times \mathbb{R}^{p \times 1}$ be a controllable pair with F Hurwitz and define $T : \mathcal{O} \rightarrow \mathbb{R}^p$ as

$$T(z) := \int_{-\infty}^0 e^{-Fs} G y(s, z) ds. \quad (20)$$

This map is continuous and satisfies

$$L_g T(z) = F \tau(z) + G q(z) \quad \forall z \in \mathcal{O}.$$

Observe that

$$\frac{\partial Z}{\partial z}(0, z) = I, \quad \frac{\partial^2 Z}{\partial z^2}(0, z) = 0$$

$$\frac{\partial^2 Z}{\partial s \partial z}(s, z) = \frac{\partial g}{\partial z}(Z(s, z)) \frac{\partial Z}{\partial z}(s, z)$$

and

$$\frac{\partial^3 Z}{\partial s \partial z^2}(s, z) = \frac{\partial g}{\partial z}(Z(s, z)) \frac{\partial^2 Z}{\partial z^2}(s, z) + \left(\frac{\partial^2 g}{\partial z^2}(Z(s, z)) \times \frac{\partial Z}{\partial z}(s, z) \right) \frac{\partial Z}{\partial z}(s, z).$$

Also, if z is in \mathcal{O} then $Z(s, z)$ is in the compact set $\text{Cl}(\mathcal{O})$ for all s . It follows that $\left| \frac{\partial^2 Z}{\partial s \partial z}(s, z) \right|$ and $\left| \frac{\partial^3 Z}{\partial s \partial z^2}(s, z) \right|$ do not grow in s faster than $e^{F_1|s|}$ and $e^{2F_1|s|}$, respectively with F_1 a bound of $\left| \frac{\partial g}{\partial z} \right|$ on $\text{Cl}(\mathcal{O})$. So, when the eigenvalues of F have real part strictly smaller than $\ell = -2F_1$, the function τ is C^2 on \mathcal{O} .

Finally observe that

$$\frac{\partial \tau}{\partial z}(z) = \int_{-\infty}^0 e^{-Fs} G \frac{\partial y}{\partial z}(s, z) ds. \quad (21)$$

In [10] (see Proposition 2), it was shown that if $p \geq 2m+2$ there exists a set $\mathcal{S} \subset \mathcal{C}$ of zero Lebesgue measure such that if $\sigma(F) \in \{\zeta \in \mathcal{C} : \Re(\zeta) < \ell\} \setminus \mathcal{S}$ then there exists a class- \mathcal{K} function $\varrho(\cdot)$ such that

$$|h(z_1) - h(z_2)| \leq \varrho(|T(z_1) - T(z_2)|) \quad \forall z_1, z_2 \in \mathcal{R}. \quad (22)$$

By Proposition 3 of [10], (22) implies the existence of a continuous $\gamma : \mathbb{R}^p \rightarrow \mathbb{R}$ satisfying

$$\gamma \circ T(z) + h(z) = 0 \quad \forall z \in \mathcal{R}. \quad (23)$$

Choose now, as candidate $\varphi(\cdot)$ and $\psi(\cdot)$ the functions

$$\varphi(\xi) = F\xi \quad \psi(\xi) = G$$

and note that, with this choice, system (9) is given by

$$\begin{aligned} \dot{z} &= g(z) \\ \dot{\xi} &= F\xi + G[h(z) + v]. \end{aligned} \quad (24)$$

It turns out that the function T in (20) satisfies

$$\frac{\partial T}{\partial z} g(z) = FT(z) + Gh(z) \quad \forall z \in \mathcal{O}$$

From these facts, since F is Hurwitz, it follows that for any $\mathcal{Z} \in \mathcal{D}(\mathcal{R}) \subset \mathcal{O}$ the set $\text{graph } T|_{\mathcal{R}}$ is LES($\mathcal{Z} \times \mathbb{R}^p$) for system (24) if $v \equiv 0$ and that an estimate of the form (10) holds, for any bounded v .

The previous facts prove the Lemma with the remarkable exception that the function γ is only guaranteed to be continuous. In the remaining part of the proof consists in showing that if (15) holds then the function γ is also locally Lipschitz.

First of all, it is observed that the function $y(t, z)$ defined above is such that:

- For any z in \mathcal{O} , any $v \notin \Omega_z^\perp$ and any $T < 0$, there exists $t_-(z, v, T)$ and $t_+(z, v, T)$ in $[T, 0]$ satisfying :

$$\frac{\partial y}{\partial z}(t, z) \cdot v \neq 0 \quad \forall t \in (t_-(z, v, T), t_+(z, v, T)) .$$

- For any z in \mathcal{O} and any v in Ω_z^\perp ,

$$\frac{\partial y}{\partial z}(t, z) \cdot v = 0 \quad \forall t \leq 0 .$$

The dimension of Ω_z being c at each point of \mathcal{O} , it is possible to choose, at any $z_j \in \mathcal{O}$, integers k_1, \dots, k_c such that

$$\Omega_{z_j} = \sum_{i=1}^c \text{span}\left\{ \frac{\partial L_g^{k_i} h}{\partial z}(z_j) \right\}$$

and to choose a local coordinate pair (U_j, Φ_j) in which the first c coordinate functions coincide with $L_g^{k_1}, \dots, L_g^{k_c}$. Split the vector $x = \Phi_j(z)$ of new coordinates in two parts, x_a collecting the c first coordinates and x_b collecting the $n - c$ last ones, and define, for each x in $\Phi_j(U_j)$,

$$\bar{h}_j(x_a, x_b) = h(\Phi_j^{-1}(x)), \quad \bar{y}_j(t, x_a, x_b) = y(t, \Phi_j^{-1}(x)) .$$

In these coordinates, the properties above can be rewritten as:

- For each pair (x_a, x_b) in $\Phi_j(U_j)$, there exist t_- and t_+ satisfying :

$$\frac{\partial \bar{y}_j}{\partial x_a}(t, x_a, x_b) \neq 0 \quad \forall t \in (t_-, t_+) . \quad (25)$$

- For each pair (x_a, x_b) in $\Phi_j(U_j)$:

$$\frac{\partial \bar{y}_j}{\partial x_b}(t, x_a, x_b) = 0 \quad \forall t \leq 0 . \quad (26)$$

Using these properties and mimicking certain arguments of [1], it is possible to conclude that the function $\bar{\tau}_j$ defined as

$$\begin{aligned} \bar{\tau}_j(x_a, x_b) &= T(\Phi_j^{-1}(x_a, x_b)) \\ &= \int_{-\infty}^0 e^{-Fs} G \bar{y}_j(s, x_a, x_b) ds , \end{aligned}$$

is such that

$$\text{rank}\left(\frac{\partial \bar{\tau}_j}{\partial x_a}(x_a, x_b)\right) = c \quad \forall (x_a, x_b) \in \Phi_j(U_j)$$

while $\frac{\partial \bar{\tau}_j}{\partial x_b}(x_a, x_b) = 0$, $\frac{\partial \bar{q}_j}{\partial x_b}(x_a, x_b) = 0$ for all $(x_a, x_b) \in \Phi_j(U_j)$.

From these properties and the fact that $T(z)$ is C^2 it is possible to show the existence of a number L_j such that, for any arbitrary pair (z_1, z_2) in $U_j \times U_j$

$$|h(z_1) - h(z_2)| \leq L_j |T(z_1) - T(z_2)| \quad \forall (z_1, z_2) \in V_z .$$

This implies that the function $\varrho(\cdot)$ in (22) is linear near zero and this in turn guarantees that the (uniquely defined) function $\gamma(\cdot)$ in (23) is locally Lipschitz.

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