Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Convex relaxations for quadratic distance problems

Andrea Garulli, Alfio Masi, Antonio Vicino

Abstract— **Convex relaxations of nonconvex problems are a powerful tool for the analysis and design of control systems. An important family of nonconvex problems that are relevant to the control field is that of quadratic distance problems. In this paper, several convex relaxations are presented for quadratic distance problems which are based on the sum-of squares representation of positive polynomials. Relationships among the considered relaxations are discussed and numerical comparisons are presented, in order to highlight their degree of conservatism.**

I. Introduction

Quadratic distance problems play a key role in the analysis and synthesis of control systems. Indeed, a number of problems can be formulated as the computation of the minimum distance, in a weighted l_2-norm, from a point to a polynomial surface in a finite dimensional space. Just to mention a few examples: the computation of the l_2 parametric stability margin of a control system affected by parametric uncertainty [1], the estimation of the domain of attraction of equilibria of nonlinear systems via quadratic Lyapunov functions [2], the D-stability of real matrices [3] that plays a key role in the analysis of singularly perturbed systems [4], the computation of the region of validity of optimal linear $\mathcal{H}_\infty$ controllers for nonlinear systems [5], the characterization of the frequency plots of an ellipsoidal family of rational functions [6].

In general, quadratic distance problems are not convex. The great advances made in the last two decades in the solution of convex problems [7], [8], has motivated researchers to develop powerful tools for devising *convex relaxations* of nonconvex problems, i.e. to formulate convex problems whose solution is a bound of the optimum of the original problem. Within this context, it has been recently recognized that positivity of polynomial forms can be tackled effectively through SemiDefinite Programming problems (SDPs). A fundamental result which has been widely used states that a sufficient condition for a polynomial to be positive semidefinite is that it can be expressed as a Sum of Squares (SOS) [9], [10]. Since it is known that testing if a polynomial is an SOS is equivalent to solving a system of Linear Matrix Inequalities (LMIs) [11], [12], it is possible to generate a number of convex relaxations for problems involving positivity of polynomials, see [13] and references therein. Several SOS-based techniques for unconstrained and constrained optimization of rational functions have been presented in [14]. Convex relaxations based on the theory of moments, which can be viewed as a dual approach to the SOS paradigm, have been widely investigated, see e.g. [15], [16].

As long as quadratic distance problems are concerned, a convex relaxation based on homogeneous polynomial forms (i.e., polynomials whose terms have the same degree) has been proposed in [17]. Another family of relaxations for distance problems (not necessarily quadratic) relies on results from algebraic geometry, like the *Positivstellensatz* [12], [18]. By using these results it is possible to construct several relaxations whose degree of conservatism depends on the specific choice of the structure of the polynomial multipliers involved in the relaxation and on the degree of such polynomial.

In this paper, different convex relaxations for quadratic distance problems are reviewed and their properties are discussed. The main result is to show that the relaxation based on homogeneous forms introduced in [17] is equivalent to a relaxation based on Positivstellensatz involving a non-homogeneous polynomial of the same degree. Moreover, examples are presented in which Positivstellensatz relaxations of higher degree allow one to achieve less conservative results. Finally, numerical comparisons between the considered relaxations are reported for randomly generated quadratic distance problems.

The paper is organized as follows. Quadratic distance problems are formulated in Section II. Section III presents some basic material about the SOS representation of positive polynomials and introduces the convex relaxations. The main contributions are reported in Section IV: equivalence between two different relaxations is established and numerical comparisons among all the considered relaxations are provided. Concluding remarks are given in Section V.

II. Quadratic distance problems

Let us consider the optimization problem

$$\begin{array}{ll} \min & \xi' Q \xi \\ \text{s.t.} & \overline{w}(\xi) = 0 \end{array} \tag{1}$$

where $\xi \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix and $\overline{w}(x)$ is an n-variate polynomial of degree m. Due to the generality of the polynomial constraint, (1) is in general a nonconvex optimization problem.

Let $Q_f \in \mathbb{R}^{n\times n}$ be such that $Q = Q_f' Q_f$ and introduce the new variables $x = Q_f \xi$. Define $w(x) = \overline{w}(Q_f^{-1}x)\overline{w}(-Q_f^{-1}x)$, and consider the new optimization problem

$$\begin{array}{ll} \min & \|x\|^2 \\ \text{s.t.} & w(x) = 0 \end{array} \tag{2}$$

The authors are with Dipartimento di Ingegneria dell'Informazione, Università di Siena, via Roma 56, 53100 Siena, Italy. E-mail: {garulli,masi,vicino}@dii.unisi.it.

The work of A. Masi is supported by the European Commission, within the project "Clearance of flight control laws using optimization", contract number AST5-CT-2006-030768.

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

where

$$w(x) = \sum_{i=0}^{m} w_{2i}(x), \tag{3}$$

and $w_{2i}(x)$, $1 \leq i \leq m$, are homogeneous polynomial forms of degree $2i$. Notice that the polynomial constraint in (2) contains only terms of even degree.

The following result can be proven [17].

Proposition 2.1: Problems (1) and (2) attain the same minimum value c_{min}. Moreover, if ξ_{min} and x_{min} are the values at which the minimum is attained in (1) and (2) respectively, then $x_{min} = Q_f \xi_{min}$.

Proposition 2.1 states that problems (1) and (2) are equivalent. Problem (2) is called a *Canonical Quadratic Distance Problem (CQDP)* [17].

The following assumptions on CQDPs are made.

Assumption 2.1: The set $\{x \in \mathbb{R}^n : \ w(x) = 0\}$ is not empty.

Assumption 2.2: There exists $\delta > 0$ such that for any $\|x\| < \delta$ it holds $w(x) > 0$.

Assumption 2.3: For any $\delta > 0$, there exist $y, z \in \mathbb{R}^n$ such that: $\|x_{min} - y\| < \delta$, $\|x_{min} - z\| < \delta$, and $w(y)w(z) < 0$.

Assumptions 2.1 and 2.2 are made without loss of generality and allow one to avoid trivial cases. Assumption 2.3 states that in any neighborhood of the optimal point x_{min}, the constraint function $w(x)$ changes sign. This assumption is not restrictive in most optimization problems of practical interest.

In the next section, several convex relaxations are presented for the CQDP (2)-(3).

III. CONVEX RELAXATIONS

In order to introduce the convex relaxations, it is necessary to recall some basic material about the SOS representation of positive polynomials. The main idea is that a polynomial form is positive semidefinite if it can be expressed as the sum of squares of suitable polynomial forms. Such sufficient condition can in turn be expressed in terms of an LMI feasibility test, as explained in the following.

Let us consider for simplicity a homogeneous polynomial form $f(x)$ in $x \in \mathbb{R}^n$ of degree $2m$. We say that $f(x)$ is positive (semidefinite) if $f(x) \geq 0 \ \forall x$. Such form can always be expressed as $f(x) = x^{\{m\}'} (F + L)\, x^{\{m\}}$, where: $x^{\{m\}} \in \mathbb{R}^d$ denotes a vector containing all monomials $x_1^{i_1} \cdots x_n^{i_n}$ for which $i_1 + \ldots + i_n = m$; $F \in \mathbb{R}^{d \times d}$ is a suitable symmetric matrix; L is a matrix belonging to the linear subspace $\mathcal{L} = \left\{ L = L' \in \mathbb{R}^{d \times d} : \ x^{\{m\}'} L x^{\{m\}} = 0 \ \ \forall x \in \mathbb{R}^n \right\}$. Let $L(\alpha)$ be a parameterization of the subspace $\mathcal{L}$, with $\alpha \in \mathbb{R}^{d_{\mathcal{L}}}$. Then, feasibility of the LMI constraint

$$F + L(\alpha) \geq 0 \tag{4}$$

implies that the homogeneous form $f(x)$ is positive. In the literature, feasibility of (4) is simply denoted by the statement "$f(x)$ is SOS", meaning that the polynomial form $f(x)$ can be expressed as a sum of squares. It is known that feasibility of (4) is only a sufficient condition for positivity of $f(x)$. Indeed, there exist polynomial forms that are positive but are not SOS (see e.g. [19], [20]). However, there are families of homogeneous forms for which positivity is equivalent to being SOS. In particular, the SOS representation is a necessary and sufficient condition for positivity in the following cases: (i) quadratic forms; (ii) two-variate homogeneous forms of any degree; (iii) three-variate homogeneous forms of degree four.

When addressing positivity of a generic polynomial form of degree $2m$ (including all lower degree terms), the above reasoning can be repeated, the only difference being that the base vector $x^{\{m\}}$ must contain all monomials in x of degree less or equal to m.

A. Relaxation based on homogeneous forms

In [17], it has been shown that CQDPs can be solved via a one-parameter family of SOS-based positivity tests. In the following, the main features of this convex relaxation are summarized.

Let $\mathcal{B}_c$ denote the boundary of the l_2 ball of radius $\sqrt{c}$, i.e. $\mathcal{B}_c = \{x : \ \|x\|^2 = c\}$. By Assumption 2.2, for sufficiently small c, $w(x) > 0$ for all $x \in \mathcal{B}_c$. Moreover, Assumption 2.3 guarantees that in any neighborhood of the intersection between $\mathcal{B}_{c_{min}}$ and $w(x) = 0$, there exist points in which $w(x) < 0$. This suggests that the solution of a CQDP can be computed via a sequence of "cutting" tests. Specifically, the solution c_{min} of problem (2) is given by

$$c_{min} = \sup \ \ \{\bar{c} \in \mathbb{R} : \ w(x) \geq 0, \ \ \forall x \in \mathcal{B}_c \ \ \forall c \in (0, \bar{c}]\}\,. \tag{5}$$

This means that c_{min} can be found by solving a family of nonnegativity tests on polynomial $w(x)$, for x belonging to a given set $\mathcal{B}_c$. Unfortunately, such tests generally amount to solving nonconvex optimization problems.

An equivalent but more compact characterization of c_{min} involves nonnegativity tests on homogeneous forms. Indeed, let us introduce the function $w(\cdot;\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that

$$w(x; c) = \sum_{i=0}^{m} \frac{\|x\|^{2(m-i)} w_{2i}(x)}{c^{m-i}}, \tag{6}$$

where $w_{2i}(x)$ are the homogeneous forms in (3). It turns out that $w(x; c)$ is a homogeneous form in x of degree $2m$ for all $c \neq 0$. In [17], it has been proven that

$$w(x) \geq 0 \ \ \forall x \in \mathcal{B}_c \quad \Longleftrightarrow \quad w(x; c) \geq 0 \ \ \forall x \in \mathbb{R}^n.$$

In other words, nonnegativity of a polynomial $w(x)$ for x belonging to a given set $\mathcal{B}_c$ can be checked by testing nonnegativity of a suitable homogeneous form. From the above discussion, it can be concluded that the solution of problem (2) is given by

$$c_{min} = \sup \ \ \{\bar{c} \in \mathbb{R} : \ w(x; c) \geq 0 \ \ \forall x \in \mathbb{R}^n \ \ \forall c \in (0, \bar{c}]\}\,. \tag{7}$$

Now the idea is to relax the inequality constraint in (7) to an SOS constraint, so that it can be formulated as an LMI. This allows one to obtain a lower bound on c_{min} via a one-parameter family of LMI feasibility tests. Indeed, let

$$\hat{c}^H_{min} = \sup \ \ \{\bar{c} \in \mathbb{R} : \ w(x; c) \text{ is SOS } \ \ \forall c \in (0, \bar{c}]\}\,. \tag{8}$$

Then, $\hat{c}^H_{min} \leq c_{min}$. In practice, the lower bound $\hat{c}^H_{min}$ can be computed within the desired precision by checking whether

$w(x; c)$ is SOS for different values of the scalar parameter c. Hereafter, we will denote (8) as an *H-relaxation* of the CQDP (2).

Some remarks on the methods described above can be made. The only source of conservativeness in the H-relaxation is due to the gap between positive homogeneous forms and SOS. An algebraic test has been devised for checking tightness of the computed lower bound. Moreover, the lower bound is known a priori to be tight in the cases when the SOS representation of homogeneous forms is equivalent to positivity.

B. Positivstellensatz relaxations

A family of convex relaxations that has been widely used in recent years for different optimization problems relevant to control system analysis and design is based on the so-called *Positivstellensatz* [12], [18], [21]. This is a fundamental result in algebraic geometry that provides a necessary and sufficient condition for unfeasibility of a set of polynomial constraints. Just to illustrate the main idea, a simplified version is stated next.

Proposition 3.1: Let $f(x)$, $g(x)$ and $h(x)$ be given polynomials in $x \in \mathbb{R}^n$. Then, the following conditions are equivalent:

1) The set $\{x \in \mathbb{R}^n : \ f(x) \geq 0,\ h(x) = 0,\ g(x) \neq 0\}$ is empty.
2) There exist two SOS polynomials $s_0(x)$, $s_1(x)$, a polynomial $t(x)$ and a nonnegative integer k such that $s_0(x) + s_1(x)f(x) + t(x)h(x) + g(x)^{2k} = 0$.

The result in Proposition 3.1 can be exploited in order to devise SOS-based convex relaxations of CQDPs.

A first way to proceed, proposed in [12], is to notice that emptiness of the set

$$\mathcal{E}_c = \{x \in \mathbb{R}^n : \ c - \|x\|^2 \geq 0,\ w(x) = 0,\ \|x\|^2 - c \neq 0\} \tag{9}$$

implies that c is a lower bound to the solution c_{min} of (2). By choosing in Proposition 3.1 $s_0(x) = 0$, $t(x) = (\|x\|^2 - c)p(x)$, and $k = 1$, it turns out that a sufficient condition for emptiness of set (9) is that there exist a polynomial $p(x)$ such that $\|x\|^2 - c + p(x)w(x)$ is SOS. Hence,

$$\begin{aligned} \hat{c}^{P}_{min} = \sup\{&\bar{c} \in \mathbb{R} : \ \exists p(x) \text{ s.t.} \\ &\|x\|^2 - c + p(x)w(x) \text{ is SOS } \ \forall c \in (0, \bar{c}]\} \end{aligned} \tag{10}$$

is a lower bound of c_{min}. It is worth observing that (10) is a convex problem, whose constraint is an LMI in both c and the coefficients of the multiplier polynomial $p(x)$. Hence, it can be solved via a single SDP and does not require a search over the parameter c. Problem (10) will be denoted as a *P-relaxation* of the CQDP. On the other hand, the conservatism of the P-relaxation depends not only on the specific choice of the Positivstellensatz multipliers, but also on the degree r of the polynomial $p(x)$. It will be shown in Section IV that the degree of $p(x)$ which allows one to obtain a tight lower bound can be very high, thus requiring the solution of large SDPs.

A more general relaxation based on Positivstellensatz can be obtained by following a reasoning similar to that adopted in Section III-A to derive the H-relaxation. Let us define the set

$$\mathcal{E}'_c = \{x \in \mathbb{R}^n : \ c - \|x\|^2 = 0,\ w(x) = 0\}. \tag{11}$$

Then, one can write the solution of the CQDP (2) as

$$c_{min} = \sup \ \{\bar{c} \in \mathbb{R} : \ \mathcal{E}'_c \text{ is empty } \ \ \forall c \in (0, \bar{c}]\}. \tag{12}$$

By applying Proposition 3.1, a sufficient condition for emptiness of the set $\mathcal{E}'_c$ turns out to be the existence of polynomials $p(x)$ and $t(x)$ such that

$$p(x)(\|x\|^2 - c) + t(x)w(x) - 1 \text{ is SOS.} \tag{13}$$

Therefore, one can compute a lower bound to c_{min} as

$$\begin{aligned} &\hat{c}^{GP}_{min} = \sup\{\bar{c} \in \mathbb{R} : \ \exists p(x), t(x) \text{ s.t.} \\ &p(x)(\|x\|^2 - c) + t(x)w(x) - 1 \text{ is SOS } \ \ \forall c \in (0, \bar{c}]\}. \end{aligned} \tag{14}$$

Since the SOS condition in (13) is not linear in both c and the coefficients of $p(x)$, the computation of the lower bound $\hat{c}^{GP}_{min}$ requires a search over c and the solution of a one-parameter family of SDPs. We will denote the convex relaxation (14) as a *GP-relaxation*. It will be shown in Section IV that the H-relaxation is a special instance of the GP-relaxation.

Notice that a GP-relaxation requires to fix the degrees of both polynomials $p(x)$ and $t(x)$. In order to restrict the family of GP-relaxations, we will choose such degrees so that $\deg\{p(x)\} = \deg\{t(x)\} + \deg\{w(x)\} - 2 = \deg\{t(x)\} + 2(m-1)$. This implies that the maximum degree of the SOS constraint (13) is always $k = \deg\{t(x)\} + 2m$. To highlight this, we will denote by GP_s a GP-relaxation of degree $s = k - 2m = \deg\{t(x)\}$.

IV. Relationships between different relaxations

In this section, relationships between the SOS-based relaxations introduced in Section III are investigated.

A. Equivalence between relaxations H and GP_0

The main result of this paper is to establish the equivalence between the H-relaxation and the GP_0-relaxation.

Theorem 4.1: Let $c < c_{min}$. Then, the following statements are equivalent:

a) $w(x; c)$ is SOS
b) there exist a polynomial $p(x)$ of degree $2(m-1)$ and a scalar t_0 such that $q(x) = p(x)(\|x\|^2 - c) + t_0 w(x) - 1$ is SOS.

Proof: a) $\Rightarrow$ b). Let $w(x; c)$ be SOS for some $c < c_{min}$. Then, let us choose

$$p(x) = \sum_{i=0}^{m-1} p_{2i}(x) \tag{15}$$

where

$$p_{2i}(x) = t_0 \sum_{j=0}^{i} \left(\frac{\|x\|^{2(i-j)} w_{2j}(x)}{c^{(i-j+1)}} \right) - \frac{\|x\|^{2i}}{c^{i+1}}. \tag{16}$$

With this choice of the polynomial multiplier $p(x)$, it turns out that $q(x)$ is a homogeneous form. Indeed, by writing

$q(x) = \sum_{i=0}^{m} r_{2i}(x)$, where the $r_{2i}(x)$ are homogeneous forms of degree $2i$, it can be observed that

$$\begin{array}{l} r_0(x) = t_0 w_0 - t_0 w_0 + 1 - 1 = 0 \\ r_{2i}(x) = t_0 w_{2i}(x) \\ +t_0 \left(\sum_{j=0}^{i-1} \frac{\|x\|^{2(i-1-j)} w_{2j}(x)}{c^{(i-j)}} - \frac{\|x\|^{2(i-1)}}{c^i} \right) \|x\|^2 \\ -t_0 \left(\sum_{j=0}^{i} \frac{\|x\|^{2(i-j)} w_{2j}(x)}{c^{(i-j)}} \right) + \frac{\|x\|^{2i}}{c^i}, \\ \text{for } i = 1, 2, \ldots, m-1. \end{array} \quad (17)$$

By noticing that

$$\begin{array}{l} -t_0 \left(\sum_{j=0}^{i} \frac{\|x\|^{2(i-j)} w_{2j}(x)}{c^{(i-j)}} \right) \\ = -t_0 w_{2i}(x) - t_0 \left(\sum_{j=0}^{i-1} \frac{\|x\|^{2(i-j)} w_{2j}(x)}{c^{(i-j)}} \right). \end{array} \quad (18)$$

one gets $r_{2i}(x) = 0$ for $i = 0, 1, \ldots, m-1$. Therefore the polynomial $q(x)$ boils down to

$$\begin{array}{rl} q(x) & = r_{2m}(x) = t_0 \left(\sum_{i=0}^{m} \frac{\|x\|^{2(m-i)} w_{2i}(x)}{c^{m-i}} \right) - \frac{\|x\|^{2m}}{c^m} \\ & = t_0 w(x;c) - \frac{\|x\|^{2m}}{c^m}. \end{array} \quad (19)$$

Since $w(x;c)$ is SOS, there exists a positive semidefinite symmetric matrix Q such that $w(x;c) = x^{\{m\}'} Q x^{\{m\}}$. Let D be a positive definite diagonal matrix such that $\|x\|^{2m} = x^{\{m\}'} D x^{\{m\}}$. Being $w(x;c) = w_0 \frac{\|x\|^{2m}}{c^m} + \sum_{i=1}^{m} \frac{\|x\|^{2(m-i)} w_{2i}(x)}{c^{m-i}}$, one can write $Q = \frac{w_0}{c^m} D + \Delta$, for a suitable symmetric matrix Δ. Then, one has

$$t_0 w(x;c) - \frac{\|x\|^{2m}}{c^m} = x^{\{m\}'} \left(\frac{t_0 w_0 - 1}{c^m} D + t_0 \Delta \right) x^{\{m\}} \quad (20)$$

By selecting t_0 such that $t_0 \geq \frac{w_0+1}{w_0}$, and recalling that $w_0 > 0$ (due to Assumption 2.2), one has

$$\frac{t_0 w_0 - 1}{c^m} D + t_0 \Delta \geq \frac{w_0}{c^m} D + \Delta = Q \geq 0$$

and therefore by (19) and (20) one can conclude that $p(x)(\|x\|^2 - c) + t_0 w(x) - 1$ is SOS.

b)$\Rightarrow$a). Now, let us assume that $q(x)$ is SOS for some polynomial $p(x)$ of degree $2(m-1)$ and a constant t_0. Let us write $p(x)$ as

$$p(x) = \sum_{i=0}^{m-1} p_{2i}(x) + \sum_{i=0}^{2m-2} \delta_i(x), \quad (21)$$

where the homogeneous forms $p_{2i}(x)$ are the same as in (16), and the $\delta_i(x)$ are homogeneous forms of degree i. By substituting (21) into $q(x)$, one has that

$$q(x) = t_0 w(x;c) - \frac{\|x\|^{2m}}{c^m} + \left(\sum_{i=0}^{2m-2} \delta_i(x) \right) \left(\|x\|^2 - c \right).$$

This means that there exists a positive semidefinite symmetric matrix M, such that

$$q(x) = \begin{pmatrix} 1 \\ x \\ x^{\{2\}} \\ \vdots \\ x^{\{m\}} \end{pmatrix}' M \begin{pmatrix} 1 \\ x \\ x^{\{2\}} \\ \vdots \\ x^{\{m\}} \end{pmatrix}. \quad (22)$$

Let us partition matrix M as

$$M = \begin{pmatrix} M_{0,0} & M_{0,1} & M_{0,2} & \ldots & M_{0,m} \\ M_{1,0} & M_{1,1} & M_{1,2} & \ldots & M_{1,m} \\ M_{2,0} & M_{2,1} & M_{2,2} & \ldots & M_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{m,0} & M_{m,1} & M_{m,2} & \ldots & M_{m,m} \end{pmatrix}$$

so that, by grouping terms of the same degree in (22) one gets the relationships

$$\begin{array}{l} t_0 w(x;c) - \frac{\|x\|^{2m}}{c^m} + \delta_{2m-2}(x)\|x\|^2 = \\ \quad = x^{\{m\}'} M_{m,m} x^{\{m\}} \end{array} \quad (23)$$

$$\begin{array}{l} \delta_{2(m-1-i)}(x)\|x\|^2 - c\, \delta_{2(m-i)}(x) = \\ \quad = \sum_{j=0}^{2i} x^{\{m-j\}'} M_{m-j,m-2i+j} x^{\{m-2i+j\}} \\ \quad \text{for } i = 1, \ldots, m \end{array} \quad (24)$$

where it has been assumed $x^{\{j\}} = 0$ for any negative j and $\delta_{-1}(x) = 0$, to obtain a more compact notation.

By adding equation (23) and all equations (24) multiplied by $\frac{\|x\|^{2i}}{c^i}$, one obtains

$$\begin{array}{l} t_0 w(x;c) - \frac{\|x\|^{2m}}{c^m} + \delta_{2m-2}(x)\|x\|^2 + \\ \sum_{i=1}^{m} \frac{\|x\|^{2i}}{c^i} \left\{ \delta_{2(m-1-i)}(x)\|x\|^2 - c\, \delta_{2(m-i)}(x) \right\} = \\ = t_0 w(x;c) - \frac{\|x\|^{2m}}{c^m} = \\ = x^{\{m\}'} M_{m,m} x^{\{m\}} + \\ \sum_{i=1}^{m} \frac{\|x\|^{2i}}{c^i} \sum_{j=0}^{2i} x^{\{m-j\}'} M_{m-j,m-2i+j} x^{\{m-2i+j\}}. \end{array} \quad (25)$$

Let us assume for the sake of exposition that m is even (the case when m is odd is analogous). Then, the right hand side term of (25) can be rewritten as

$$t_0 w(x;c) - \frac{\|x\|^{2m}}{c^m} = v_e(x)' M_e v_e(x) + v_o(x)' M_o v_o(x) \quad (26)$$

where

$$v_e(x) = \begin{pmatrix} \frac{\|x\|^m}{c^{m/2}} \\ \frac{\|x\|^{m-2}}{c^{(m-2)/2}} x^{\{2\}} \\ \vdots \\ \frac{\|x\|^2}{c} x^{\{m-2\}} \\ x^{\{m\}} \end{pmatrix}, \quad v_o(x) = \begin{pmatrix} \frac{\|x\|^{m-1}}{c^{(m-1)/2}} x \\ \frac{\|x\|^{m-3}}{c^{(m-3)/2}} x^{\{3\}} \\ \vdots \\ \frac{\|x\|^3}{c^{3/2}} x^{\{m-3\}} \\ \frac{\|x\|}{c^{1/2}} x^{\{m-1\}} \end{pmatrix}$$

and

$$M_e = \begin{pmatrix} M_{0,0} & M_{0,2} & \ldots & M_{0,m} \\ & M_{2,2} & \ldots & M_{2,m} \\ & & \ddots & \vdots \\ & & & M_{m,m} \end{pmatrix}$$

$$M_o = \begin{pmatrix} M_{1,1} & M_{1,3} & \ldots & M_{1,m-1} \\ & M_{3,3} & \ldots & M_{3,m-1} \\ & & \ddots & \vdots \\ & & & M_{m-1,m-1} \end{pmatrix}.$$

Being $M \geq 0$, one has also $M_e \geq 0$ and $M_o \geq 0$. Then, from (26) one has that $t_0 w(x;c)$ is SOS. Being $c \leq c_{min}$

one has that $w(x;c) \geq 0$ and hence also $t_0 \geq 0$. Therefore, $w(x;c)$ is SOS.$\diamondsuit$

B. Conservatism of the H-relaxation

Theorem 4.1 has shown that a given value of c is feasible for the H-relaxation if and only if it is feasible also for the GP_0-relaxation. This implies that the conservatism level of the two relaxations is the same, i.e. $\hat{c}_{min}^H = \hat{c}_{min}^{GP_0}$.

Let us now consider the GP_s-relaxation for $s > 0$. An interesting issue to address is whether there exist CQDPs for which this relaxation provides less conservative results with respect to the H-relaxation (or equivalently the GP_0 one). By (7), the H-relaxation is conservative only if there exist values of $c \leq c_{min}$ such that the homogenous form $w(x;c)$ is positive semidefinite but cannot be written as a sum of squares. In the following, some examples of CQDPs constructed in order to satisfy this requirement are presented.

Example 4.1: Consider a CQDP with $m = 3$ and $n = 3$, such that

$$w(x) = 1 - \|x\|^6 + 10 f_{ns}(x), \quad (27)$$

where $f_{ns}(x)$ is a nonnegative homogeneous form which is not SOS [17]. Let us consider for example the form

$$f_{ns}(x) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6 \quad (28)$$

introduced in [22]. One has that

$$w(x;c) = \|x\|^6 \left(\frac{1}{c^3} - 1 \right) + 10 f_{ns}(x).$$

Being $w(x;1) = 10 f_{ns}(x) \geq 0$, by (7) it holds $c_{min} \geq 1$. Moreover, since $w(x) = 0$ for $x_1 = x_2 = x_3 = \frac{1}{\sqrt{3}}$, one can conclude that $c_{min} = 1$. However, by construction, $w(x;1)$ is not SOS and therefore $\hat{c}_{min}^H < 1$ will be returned by the H-relaxation. Indeed, the relaxations H and GP_0 return $\hat{c}_{min}^H = \hat{c}_{min}^{GP_0} = 0.9851$. If the GP_2 relaxation is considered one gets $\hat{c}_{min}^{GP_2} = 1 = c_{min}$. This means that for every $c \leq 1$ it is possible to find a polynomial $t(x)$ of degree 2 and a polynomial $p(x)$ of degree 6 such that (13) holds. This is not surprising, because it is known that $f_{ns}(x)(x_1^2 + x_2^2 + x_3^2)$ is SOS [12].
It is also interesting to analyze the performance of the P-relaxation. Table I reports the values of $\hat{c}_{min}^P$ as a function of the degree r of the multiplier polynomial $p(x)$. It turns out that problem (10) returns a tight lower bound if $r = 10$. Therefore, the P-relaxation is also less conservative than the H-relaxation for this particular example.

r	0	2	4	6	8	10
$\hat{c}_{min}^P$	0	0.0037	0.0593	0.2198	0.7037	1

TABLE I $\hat{c}_{min}^P$ as a function of r computed for Example 4.1.

Example 4.2: Consider the CQDP with $n = 4$ and $m = 2$, such that

$$w(x) = 1 - \|x\|^4 + 10 f_{ns}(x)$$

and

$$f_{ns}(x) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_4^4 - 4x_1 x_2 x_3 x_4. \quad (29)$$

This is a nonnegative form which is not SOS [19] and hence the same reasoning as in Example 4.1 can be repeated. In this case, by applying the H-relaxation one gets $\hat{c}_{min}^H = 0.8633$, while the GP_2-relaxation reaches the true optimum. The P-relaxation achieves $\hat{c}_{min}^P = 0.9350$ for $r = 10$ (see Table II). With the computing resources employed for the numerical examples, it has not been possible to tackle problems with $r \geq 12$, corresponding to an LMI of size $d \geq 495$ with more than 10^5 free variables (all computations have been performed by using SOSTOOLS [23] on a PC with processor Intel Xeon 5150).

r	0	2	4	6	8	10
$\hat{c}_{min}^P$	0.0002	0.0182	0.1037	0.2950	0.6485	0.9350

TABLE II $\hat{c}_{min}^P$ as a function of r computed for Example 4.2.

C. Numerical comparisons

Examples 4.1-4.2 have shown that it is possible to construct quadratic distance problems for which the H-relaxation is more conservative than the GP_s relaxation with $s > 0$, and also than the P-relaxation, provided that a multiplier $p(x)$ of sufficiently high degree is chosen.

In order to compare the performance of the proposed relaxations, families of CQDPs have been randomly generated for different values of n and m. First, a set of $N = 10000$ CQDPs with $n = m = 2$ has been considered. The coefficients of the polynomial $w(x)$ in (3) have been randomly generated from a uniform distribution in $[-1, 1]$. Assumptions 2.1 and 2.2 have been fulfilled by choosing the coefficients so that $w(0) > 0$ and $w(x) = 0$ for $x = (1 \ \ 1)'$. Being $n = 2$, it is known that for every c, $w(x;c) \geq 0$ if and only if $w(x;c)$ is SOS. Therefore in this case the relaxations H and GP_0 always return the optimal value c_{min}. Conversely, the P-relaxation is in general conservative, and its conservatism reduces as the degree r of $p(x)$ is increased. Table III reports the number $N_P(r)$ of CQDPs for which $\hat{c}_{min}^P$ is strictly less than c_{min}, for different values of r. The average relative error of these $N_P(r)$ CQDPs

$$\epsilon_P(r) = \frac{1}{N_P(r)} \sum_{i=1}^{N_P(r)} \frac{c_{min,i} - \hat{c}_{min,i}^P(r)}{c_{min,i}}$$

is also provided. Even if this family of problems is relatively simple, it can be observed that the P-relaxation may require a very high degree of the multiplier $p(x)$ in order to reach the true optimum. Moreover, in the case when the optimum is not reached the average relative error $\epsilon_P(r)$ is not negligible.

r	0	2	4	6	8
$N_P(r)$	9651	1160	564	56	22
$\epsilon_P(r)$	0.9981	0.3854	0.2163	0.2824	0.3879

r	10	12	14	16	
$N_P(r)$	8	8	6	6	
$\epsilon_P(r)$	0.7219	0.6472	0.6784	0.6341	

TABLE III $N_P(r)$ and $\epsilon_P(r)$ for 10000 CQDPs with $(n, m) = (2, 2)$.

A second set of experiments used has concerned 10000 randomly generated CQDPs with $(n,m)=(4,3)$. We observed that the H-relaxation always reaches the true optimum. This has been verified by applying the algorithm for checking tightness proposed in [17]. Table IV shows the number of CQDP for which the P-relaxation returns conservative results and the corresponding relative error, as a function of r.
Similar results have been obtained on 10000 randomly generated CQDPs with $(n,m)=(5,2)$. Once again, the H-relaxation always provides the true optimum. The performance of the P-relaxation is reported in Table V.
These numerical experiments show that the conservatism of the P-relaxation tends to increase with n and m, as it is necessary to consider polynomial multipliers of higher degree in order to approach the true optimum.

r	0	2	4
$N_P(r)$	10000	9999	8467
$\epsilon_P(r)$	0.9995	0.9633	0.4110

TABLE IV $N_P(r)$ and $\epsilon_P(r)$ for 10000 CQDPs with $(n,m)=(4,3)$.

r	0	2	4
$N_P(r)$	10000	10000	9670
$\epsilon_P(r)$	0.9992	0.8931	0.3755

TABLE V $N_P(r)$ and $\epsilon_P(r)$ for 10000 CQDPs with $(n,m)=(5,2)$.

V. Conclusions

In this paper, different convex relaxations have been proposed and analyzed for an important class of nonconvex optimization problems relevant to the control field. It has been shown that the considered family of quadratic distance problems (CQDPs) can be solved either by applying a relaxation based on homogeneous forms or by adopting a special instance of the Positivstellensatz, with the same conservatism level. Moreover, the only source of conservatism for such relaxations is due to the gap between positive semidefinite forms and sums-of-squares. Numerical experiments have shown that Positivstellensatz relaxations of higher degree allow one to reduce this gap, at the price of a higher computational burden. It has also been highlighted that different choices of the polynomial multipliers generate different structures of the SOS constraints, introducing a trade off between the conservatism level and the degree of the resulting polynomial constraint (or equivalently the size of the corresponding LMI).

Ongoing work concerns both theoretical and practical aspects. On one side, it is useful to pursue theoretical results highlighting relationships between different relaxations, in order to understand which relaxation should be used for specific classes of optimization problems. Numerical experiments are also necessary in order to assess the actual level of conservatism of the proposed relaxations on problems arising from real-world applications.

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