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Resilient Adaptive Control of Uncertain Time-Delay Systems - A Delay-Dependent Approach

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***Abstract*— In this article, resilient delay-dependent adaptive control algorithms are developed for closed-loop stabilization of a class of uncertain time-delay systems with time-varying state delay, nonlinear dynamical perturbation, and controller gain perturbation. The norm of the nonlinear perturbation is assumed to be bounded by a weighted norm of the state such that the upper value of the weight is unknown, and the norm of the uncertainty of the state feedback gain is assumed to be bounded by a positive constant. The results presented here can be considered as extension of previous work that assumes that the upper value of the nonlinear perturbation weight is known. Here, adaptive control schemes are developed to guarantee asymptotic stabilization of the closed-loop system when the upper bound of the state feedback gain perturbation is known and unknown.**

I. INTRODUCTION

Time delay systems are widely encountered in many real applications, such as chemical processes and communication networks. Hence, the problem of controlling time-delay systems has been investigated by many researchers in the past few decades. It has been found that controlling time-delay system can be a challenging task, especially in the presence of uncertainties and parameter variations. Several techniques have been studied in the analysis and design of time delay systems with parameter uncertainties. Such techniques include robust control [1], [2], H_∞ control [3], [4], [5], [6], and sliding mode control [7], [8], [9], [10], [11]. In the case where uncertain time-delay systems include a nonlinear perturbation, several adaptive control approaches have been introduced [12], [13], [14], [15], [16], [17]. In [12], [14], the authors developed state feedback controllers when the state vector is available for measurement and the upper bound on the delayed state perturbation vector is known. For the case where the upper bound of the nonlinear perturbation is known, more stabilizing controllers with stability conditions have been derived in [15], [13]. However, in many real control problems, the bounds of the uncertainties are unknown. For such a class of systems, the author in [16] has developed a continuous time state feedback adaptive controller to guarantee uniform ultimate boundedness for systems with partially known uncertainties. For a class of systems with multiple uncertain state delays that are assumed to satisfy the matching condition, an adaptive law that guarantees uniform ultimate boundedness has been introduced in [17]. In all of the papers discussed above, the authors investigated delay-independent stabilization and control of time-delay systems. Delay-dependent stabilization and H_∞ control of time-delay systems have been studied in [18], [19], [20], [21], [22], [23], [1]. In [1], the author discussed stabilization conditions and analyzed passivity of continuous and discrete time-delay systems with time-varying delay and norm-bounded parameter uncertainties. The results in [1] have been extended in [24] to consider designing delay-dependent adaptive controllers for a class of uncertain time-delay systems with time-varying delays in the presence of nonlinear perturbation. In [24], the nonlinear perturbation is assumed to be bounded by a weighted norm of the state vector, and for this problem adaptive controllers have been developed for the two cases where the upper bound of the weight is assumed to be known and unknown.

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An inherent assumption in the design of all of the above control algorithms is that the controller will be implemented perfectly. In [25], the authors extended the results in [24] to investigate the resilient control problem [26], [27], [28], where perturbation in controller state feedback gain is considered. It has been assumed in [25] that the nonlinear perturbation is bounded by a weighted norm of the state such that the value of the weight is known, and the norm of the uncertainty of the state feedback gain is assumed to be bounded by a positive constant. Under these assumptions, adaptive controllers were designed when the value of the upper bound of the state feedback gain perturbation is known and unknown. This paper extends the results in [25] to consider the problem where the the upper bound of the nonlinear perturbation weight is unknown. For this problem, asymptotically stabilizing adaptive controllers are derived for both cases where the upper bound of the norm of the state feedback gain perturbation is known and unknown.

The paper is organized as follows. In Section II, we define the problem statement. Then, in Section III, we present the main stability results, and finally in Section IV some concluding remarks are outlined.

Notations and Facts: In the sequel, the Euclidean norm is used for vectors. We use $W^\top$, W^{-1}, and $||W||$ to denote, respectively, the transpose of, the inverse of, and the induced norm of any square matrix W. We use $W > 0$ $(\geq, <, \leq 0)$ to denote a symmetric positive definite (positive semidefinite, negative, negative semidefinite) matrix W, and I to denote the $n \times n$ identity matrix. The symbol $\bullet$ will be used in some matrix expressions to induce a symmetric structure, that is if the matrices $L = L^\top$ and $R = R^\top$ of appropriate

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dimensions are given, then

$$\left[\begin{array}{cc} L & N \\ \bullet & R \end{array}\right] = \left[\begin{array}{cc} L & N \\ N^\top & R \end{array}\right].$$

Now, we introduce the following facts that will be used later on to establish the stability results.

Fact 1: [1] Given matrices Σ_1 and Σ_2 with appropriate dimensions, it follows that

$$\Sigma_1\Sigma_2 + \Sigma_2^\top\Sigma_1^\top \ \leq \ \alpha^{-1}\,\Sigma_1\Sigma_1^\top + \alpha\,\Sigma_2^\top\Sigma_2,\ \forall\,\alpha > 0.$$

Fact 2 (Schur Complement): [1], [29] Given constant matrices Ω_1, Ω_2, Ω_3 where $\Omega_1 = \Omega_1^\top$ and $0 < \Omega_2 = \Omega_2^\top$ then $\Omega_1 + \Omega_3^\top\Omega_2^{-1}\Omega_3 < 0$ if and only if

$$\left[\begin{array}{cc} \Omega_1 & \Omega_3^\top \\ \Omega_3 & -\Omega_2 \end{array}\right] < 0 \quad or \quad \left[\begin{array}{cc} -\Omega_2 & \Omega_3 \\ \Omega_3^\top & \Omega_1 \end{array}\right] < 0.$$

II. Problem Statement

Consider the class of dynamical systems with state delay

$$\dot{x}(t) = A_o x(t) + A_d x(t-\tau) + B_o u(t) + E\left(x(t),t\right) \quad (1)$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control input, $E\left(x(t),t\right) : \Re^n \times \Re \rightarrow \Re^n$ is an unknown continuous vector function that represents a nonlinear perturbation, and τ is some unknown time-varying state delay factor satisfying $0 \leq \tau \leq \tau^+$, where the bound τ^+ is a known constant. The matrices A_o, A_d, and B_o are known real constant matrices of appropriate dimensions. The nonlinear perturbation function is defined to satisfy the following assumption.

Assumption 2.1: The nonlinear perturbation function $E\left(x(t),t\right)$ satisfies the following inequality

$$||E\left(x(t),t\right)|| \leq \theta^*\ ||x(t)||, \quad (2)$$

where θ^* is some *unknown* positive constant.
In this paper, resilient delay-dependent adaptive stabilization results are established for the system (1) when uncertainties appear in the state feedback gain of the controller.

Before we present the stability results, we start be expressing the delayed state as [1]

$$\begin{aligned} x(t-\tau) &= x(t) - \int_{-\tau}^{0} \dot{x}(t+s)ds \quad (3) \\ &= x(t) - \int_{-\tau}^{0} \left[A_o\, x(t+s) + A_d\, x(t-\tau+s)\right. \\ &\quad \left. + B_o\, u(t+s)\right] ds \\ &\quad - \int_{-\tau}^{0} E\left(x(t+s), t+s\right)\ ds. \end{aligned}$$

Hence, if we define $A_{od} = A_o + A_d$, then the system (1) can be expressed as

$$\begin{aligned} \dot{x}(t) &= A_{od}\, x(t) + A_d\, \eta(t) + B_o u(t) + E\left(x(t),t\right), \quad (4) \\ \eta(t) &= -\int_{-\tau}^{0} \left[A_o\, x(t+s) + A_d\, x(t-\tau+s)\right. \\ &\quad \left. + B_o\, u(t+s) + E\left(x(t+s), t+s\right)\right]\ ds. \end{aligned}$$

III. Main Results

In the sequel, the main design results will be presented. To stabilize the system (4), we introduce the following control law:

$$u(t) \ = \ \left(K + \Delta K\right) x(t) + \mu(t)\mathcal{I}x(t), \quad (5)$$

where $\mathcal{I} \in \Re^{m\times n}$ is a matrix whose elements are all ones, $\mu(t) \in \Re$ is adapted such that closed-loop asymptotic stabilization is guaranteed, $K \in \Re^{m\times n}$ is a state feedback gain, and $\Delta K(t) \in \Re^{m\times n}$ is the time varying uncertainty of the state feedback gain that satisfies the following assumption.

Assumption 3.1: The uncertainty of the state feedback gain satisfies the following inequality

$$||\Delta K(t)|| \leq \rho^*, \quad (6)$$

where ρ^* is some positive constant.
In this section, resilient delay-dependent stabilization results are established for the system (4) considering the following two cases:

1) The uncertainty of the state feedback gain satisfies Assumption 3.1 such that ρ^* is assumed to be *a known* positive constant.
2) The uncertainty of the state feedback gain satisfies Assumption 3.1 such that ρ^* is assumed to be *an unknown* positive constant.

A. Adaptive Control when ρ^* is Known

To stabilize the system (4) when ρ^* is known, we consider the control law (5). Let us define $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$, where $\hat{\theta}(t)$ is the estimate of θ^*, and $\tilde{\theta}(t)$ is error between the estimate and the true value of θ^*. Also, let us define $z(t) = \mu(t)x(t)$, and let the Lyapunov-Krasovskii functional for the transformed system (4) be selected as:

$$\begin{aligned} V_a(x) \ \triangleq \ & V_1(x) + V_2(x) + V_3(x) + V_4(x) + V_5(x) \\ & + V_6(x) + V_7(x) + V_8(x) + V_9(x), \quad (7) \end{aligned}$$

where

$$\begin{aligned} V_1(x) &= x^\top(t)Px(t), \quad (8) \\ V_2(x) &= r_1 \int_{-\tau}^{0}\int_{t+s}^{t} x^\top(\alpha)A_o^\top A_o x(\alpha)d\alpha ds, \quad (9) \\ V_3(x) &= r_2 \int_{-\tau}^{0}\int_{t+s-\tau}^{t} x^\top(\alpha)\ A_d^\top \\ &\qquad A_d\ x(\alpha)\ d\alpha\ ds, \quad (10) \\ V_4(x) &= r_3 \int_{-\tau}^{0}\int_{t+s}^{t} x^\top(\alpha)\ K^\top B_o^\top \\ &\qquad B_o K\ x(\alpha)\ d\alpha\ ds, \quad (11) \\ V_5(x) &= r_4 \int_{-\tau}^{0}\int_{t+s}^{t} x^\top(\alpha)\ \Delta K^\top(t) B_o^\top \\ &\qquad B_o \Delta K(t)\ x(\alpha)\ d\alpha\ ds, \quad (12) \\ V_6(x) &= r_5 \int_{-\tau}^{0}\int_{t+s}^{t} z^\top(\alpha)\ \mathcal{I}^\top B_o^\top \\ &\qquad B_o \mathcal{I}\ z(\alpha)\ d\alpha\ ds, \quad (13) \end{aligned}$$

$$V_7(x) = r_6 \int_{-\tau}^{0}\int_{t+s}^{t} E^\top(x,\alpha)\, E(x,\alpha)\, d\alpha\, ds, \quad (14)$$
$$V_8(x) = \mu^2(t), \quad (15)$$
$$V_9(x) = (1+\theta^*)\left[\tilde{\theta}(t)\right]^2, \quad (16)$$

where $r_1 > 0$, $r_2 > 0$, $r_3 > 0$, $r_4 > 0$, $r_5 > 0$ and $r_6 > 0$ are positive scalars, and $P = P^\top \in \Re^{n\times n} > 0$. It can be shown that the time derivative of the Lyapunov-Krasovskii functional is

$$\begin{aligned}\dot{V}_a(x) &= \dot{V}_1(x) + \dot{V}_2(x) + \dot{V}_3(x) + \dot{V}_4(x) + \dot{V}_5(x)\\ &\quad + \dot{V}_6(x) + \dot{V}_7(x) + \dot{V}_8(x) + \dot{V}_9(x), \quad (17)\end{aligned}$$

where

$$\dot{V}_1(x) = x^\top(t)P\dot{x}(t) + \dot{x}^\top(t)Px(t), \quad (18)$$
$$\begin{aligned}\dot{V}_2(x) &= \tau r_1 x^\top(t)A_o^\top A_o x(t)\\ &\quad - r_1\int_{-\tau}^{0} x^\top(t+s)A_o^\top A_o x(t+s)ds, \quad (19)\end{aligned}$$
$$\begin{aligned}\dot{V}_3(x) &= \tau r_2 x^\top(t)A_d^\top A_d x(t) - r_2\int_{-\tau}^{0} x^\top(t+s-\tau)\\ &\qquad A_d^\top A_d x(t+s-\tau)ds, \quad (20)\end{aligned}$$
$$\begin{aligned}\dot{V}_4(x) &= \tau r_3 x^\top(t)K^\top B_o^\top B_o K x(t)\\ &\quad - r_3\int_{-\tau}^{0} x^\top(t+s)K^\top B_o^\top B_o K x(t+s)ds, \quad (21)\end{aligned}$$
$$\begin{aligned}\dot{V}_5(x) &= \tau r_4 x^\top(t)\Delta K(t)^\top B_o^\top B_o \Delta K(t)x(t)\\ &\quad - r_4\int_{-\tau}^{0} x^\top(t+s)\Delta K^\top(t+s)B_o^\top\\ &\qquad B_o\Delta K(t+s)x(t+s)ds, \quad (22)\end{aligned}$$
$$\begin{aligned}\dot{V}_6(x) &= \tau r_5 z^\top(t)\mathcal{I}^\top B_o^\top B_o\mathcal{I}z(t)\\ &\quad - r_5\int_{-\tau}^{0} z^\top(t+s)\mathcal{I}^\top B_o^\top B_o\mathcal{I}z(t+s)ds, \quad (23)\end{aligned}$$
$$\begin{aligned}\dot{V}_7(x) &= \tau r_6\, E^\top(x,t)\, E(x,t)\\ &\quad - r_6\int_{-\tau}^{0} E^\top(x,t+s)\, E(x,t+s)\, ds, \quad (24)\end{aligned}$$
$$\dot{V}_8(x) = 2\,\mu(t)\,\dot{\mu}(t), \quad (25)$$
$$\begin{aligned}\dot{V}_9(x) &= 2\,(1+\theta^*)\,\tilde{\theta}(t)\,\dot{\tilde{\theta}}(t),\\ &= 2\,(1+\theta^*)\left[\hat{\theta}(t)-\theta^*\right]\,\dot{\hat{\theta}}(t). \quad (26)\end{aligned}$$

The next Theorem provides the main results for this case.

Theorem 1: *Consider system (4). If there exist matrices $0 < \mathcal{X} = \mathcal{X}^\top \in \Re^{n\times n}$, $\mathcal{Y} \in \Re^{m\times n}$, $\mathcal{Z} \in \Re^{n\times n}$, and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > \varepsilon$, $\varepsilon_5 > \varepsilon$ and $\varepsilon_6 > \varepsilon$ (where ε is an arbitrary small positive constant) such that the following LMI*

$$\begin{bmatrix} \begin{array}{c} A_{od}\mathcal{X} + \mathcal{X}A_{od} \\ +B_o\mathcal{Y} + \mathcal{Y}^\top B_o^\top \\ +\tau^+(\varepsilon_1+\varepsilon_2 \\ +\varepsilon_3+\varepsilon_4+\varepsilon_5 \\ +\varepsilon_6)\,A_dA_d^\top \end{array} & \tau^+\mathcal{X}A_o^\top & \tau^+\mathcal{X}A_d^\top & \tau^+\mathcal{Z} \\ \bullet & -\tau^+\varepsilon_1 I & 0 & 0 \\ \bullet & \bullet & -\tau^+\varepsilon_2 I & 0 \\ \bullet & \bullet & \bullet & -\tau^+\varepsilon_3 I \end{bmatrix} < 0, \quad (27)$$

has a feasible solution, and $K = \mathcal{Y}\mathcal{X}^{-1}$, and $\mu(t)$ is adapted subject to the adaptive laws

$$\begin{aligned}\dot{\mu}(t) &= Proj\left\{\alpha_1\, sgn\left(\mu(t)\right)\, ||x(t)||^2 + \alpha_2\,\mu(t)\,||x(t)||^2\right.\\ &\qquad \left. +\alpha_3\;\, sgn\left(\mu(t)\right)\,\hat{\theta}(t)\,||x(t)||^2, \mu(t)\right\}, \quad (28)\end{aligned}$$
$$\dot{\hat{\theta}}(t) = \gamma\,||x(t)||^2, \quad (29)$$

where $Proj\{\cdot\}$ [30] is applied to ensure that $|\mu(t)| \geq 1$ as follows

$$\mu(t) = \begin{cases} \mu(t) & \text{if } |\mu(t)| \geq 1 \\ 1 & \text{if } 0 \leq \mu(t) < 1 \\ -1 & \text{if } -1 < \mu(t) < 0, \end{cases}$$

and the adaptive law parameters are selected such that $\alpha_1 < -\left[||PB_o\mathcal{I}|| + \tau^+ r_4\,(\rho^*)^2\,||B_o^\top B_o|| + \rho^*||PB_o||\right]$, $\alpha_2 < -\frac{1}{2}\tau^+ r_5||\mathcal{I}^\top B_o^\top B_o\mathcal{I}||$, $\alpha_3 < -\gamma$, $\gamma > \frac{1}{2}\tau^+ r_6$ and $\hat{\theta}(0) > 1$, then the control law (5) will guarantee asymptotic stabilization of the closed-loop system.

Proof: As shown in (17), the time derivative of $V_a(x)$ is

$$\begin{aligned}\dot{V}_a(x) &= \dot{V}_1(x) + \dot{V}_2(x) + \dot{V}_3(x) + \dot{V}_4(x) + \dot{V}_5(x)\\ &\quad + \dot{V}_6(x) + \dot{V}_7(x) + \dot{V}_8(x) + \dot{V}_9(x),\\ &= x^\top(t)P\dot{x}(t) + \dot{x}^\top(t)Px(t) + \dot{V}_2(x)\\ &\quad + \dot{V}_3(x) + \dot{V}_4(x) + \dot{V}_5(x) + \dot{V}_6(x)\\ &\quad + \dot{V}_7(x) + \dot{V}_8(x) + \dot{V}_9(x). \quad (30)\end{aligned}$$

Using the system equation defined in (4) and the control law (5), we have

$$\begin{aligned}\dot{V}_a(x) &= x^\top(t)\left[PA_{od} + A_{od}^\top P + PB_oK\right.\\ &\qquad \left. +K^\top B_o^\top P\right]x(t)\\ &\quad -2x^\top(t)PA_d\int_{-\tau}^{0} A_o x(t+s)ds\\ &\quad -2x^\top(t)PA_d\int_{-\tau}^{0} A_d x(t-\tau+s)ds\\ &\quad -2x^\top(t)PA_d\int_{-\tau}^{0} B_oKx(t+s)ds\\ &\quad -2x^\top(t)PA_d\int_{-\tau}^{0} B_o\Delta K(t+s)x(t+s)ds\\ &\quad -2x^\top(t)PA_d\int_{-\tau}^{0} \mu(t+s)B_o\mathcal{I}x(t+s)ds\\ &\quad -2x^\top(t)PA_d\int_{-\tau}^{0} E(x,t+s)ds\\ &\quad +2x^\top(t)PB_o\Delta K(t)x(t)\\ &\quad +2\mu(t)x^\top(t)PB_o\mathcal{I}x(t) + 2x^\top(t)PE(x,t)\\ &\quad + \dot{V}_2(x) + \dot{V}_3(x) + \dot{V}_4(x) + \dot{V}_5(x)\\ &\quad + \dot{V}_6(x) + \dot{V}_7(x) + \dot{V}_8(x) + \dot{V}_9(x). \quad (31)\end{aligned}$$

By applying Fact 1, we have

$$\begin{aligned}
&-2x^\top(t)PA_d\int_{-\tau}^{0}A_o x(t+s)ds\\
&\le r_1^{-1}\int_{-\tau}^{0}x^\top(s)PA_dA_d^\top Px(s)ds\\
&+r_1\int_{-\tau}^{0}x^\top(t+s)A_o^\top A_o x(t+s)ds\\
&\le \tau^+ r_1^{-1}x^\top(t)PA_dA_d^\top Px(t)\\
&+r_1\int_{-\tau}^{0}x^\top(t+s)A_o^\top A_o x(t+s)ds, \qquad (32)
\end{aligned}$$

where r_1 is a positive scalar. Similarly, if r_2, r_3 and r_4 are positive scalars, we have

$$\begin{aligned}
&-2x^\top(t)PA_d\int_{-\tau}^{0}A_d x(t-\tau+s)ds\\
&\le \tau^+ r_2^{-1}x^\top(t)PA_dA_d^\top Px(t)\\
&+r_2\int_{-\tau}^{0}x^\top(t-\tau+s)A_d^\top A_d x(t-\tau+s)ds, \quad (33)
\end{aligned}$$

$$\begin{aligned}
&-2x^\top(t)PA_d\int_{-\tau}^{0}B_oKx(t+s)ds\\
&\le \tau^+ r_3^{-1}x^\top(t)PA_dA_d^\top Px(t)\\
&+r_3\int_{-\tau}^{0}x^\top(t+s)K^\top B_o^\top B_oKx(t+s)ds, \quad (34)
\end{aligned}$$

and

$$\begin{aligned}
&-2x^\top(t)PA_d\int_{-\tau}^{0}B_o\Delta K(t+s)x(t+s)ds\\
&\le \tau^+ r_4^{-1}x^\top(t)PA_dA_d^\top Px(t)+r_4\int_{-\tau}^{0}x^\top(t+s)\\
&\Delta K^\top(t+s)B_o^\top B_o\Delta K(t+s)x(t+s)ds. \qquad (35)
\end{aligned}$$

Now, let r_5 be a positive scalar, then using Fact 1 we have

$$\begin{aligned}
&-2x^\top(t)PA_d\int_{-\tau}^{0}\mu(t+s)B_o\mathcal{I}x(t+s)ds\\
&=-2x^\top(t)PA_d\int_{-\tau}^{0}B_o\mathcal{I}z(t+s)ds\\
&\le \tau^+ r_5^{-1}x^\top(t)PA_dA_d^\top Px(t)\\
&+r_5\int_{-\tau}^{0}z^\top(t+s)\mathcal{I}^\top B_o^\top B_o\mathcal{I}z(t+s)ds. \qquad (36)
\end{aligned}$$

Also, if r_6 is a positive scalar, then using Fact 1 we have

$$\begin{aligned}
&-2x^\top(t)PA_d\int_{-\tau}^{0}E(x,t+s)ds\\
&\le \tau^+ r_6^{-1}x^\top(t)PA_dA_d^\top Px(t)\\
&+r_6\int_{-\tau}^{0}E^\top(x,t+s)E(x,t+s)ds. \qquad (37)
\end{aligned}$$

Using Assumptions 2.1 and 3.1, it can be shown that

$$2\mu(t)x^\top(t)PB_o\mathcal{I}x(t)\le 2||PB_o\mathcal{I}||\,|\mu(t)|\,||x(t)||^2, \quad (38)$$

$$2x^\top(t)PB_o\Delta K(t)x(t)\le 2\,\rho^*\,||PB_o||\,||x(t)||^2, \quad (39)$$

and

$$2x^\top(t)PE(x(t))\le 2\,||P||\,\theta^*\,||x(t)||^2. \qquad (40)$$

Using equations (32)- (40) and equations (18)- (26) (with the fact that $0\le\tau\le\tau^+$) in (31), we have

$$\begin{aligned}
\dot V_a(x) \le\ & x^\top(t)\Xi x(t)\\
&+\tau^+ r_4x^\top(t)\Delta K^\top(t)B_o^\top B_o\Delta K(t)x(t)\\
&+\tau^+ r_5z^\top(t)\mathcal{I}^\top B_o^\top B_o\mathcal{I}z(t)\\
&+\tau^+ r_6E^\top(x,t)E(x,t)+2\rho^*||PB_o||\,||x(t)||^2\\
&+2||PB_o\mathcal{I}||\,|\mu(t)|\,||x(t)||^2\\
&+2\theta^*||P||\,||x(t)||^2+2\,\mu(t)\,\dot\mu(t)\\
&+2\,(1+\theta^*)\left[\hat\theta(t)-\theta^*\right]\,\dot{\hat\theta}(t), \qquad (41)
\end{aligned}$$

where

$$\begin{aligned}
\Xi\ =\ & PA_{od}+A_{od}^\top P+PB_oK+K^\top B_o^\top P\\
&+\tau^+\left(r_1^{-1}+r_2^{-1}+r_3^{-1}+r_4^{-1}+r_5^{-1}+r_6^{-1}\right)PA_dA_d^\top P\\
&+\tau^+ r_1A_o^\top A_o+\tau^+ r_2A_d^\top A_d+\tau^+ r_3K^\top B_o^\top B_oK. \qquad (42)
\end{aligned}$$

To guarantee that $x^\top(t)\Xi x(t)<0$, it sufficient to show that $\Xi<0$. Let us introduce the linearizing terms, $\mathcal{X}=P^{-1}$, $\mathcal{Y}=K\mathcal{X}$, and $\mathcal{Z}=\mathcal{X}^\top K^\top B_o^\top$. Also, let $\varepsilon_1=r_1^{-1}$, $\varepsilon_2=r_2^{-1}$, $\varepsilon_3=r_3^{-1}$, $\varepsilon_4=r_4^{-1}$, $\varepsilon_5=r_5^{-1}$ and $\varepsilon_6=r_6^{-1}$. Now, by pre-multiplying and post-multiplying Ξ by $\mathcal{X}$, we have

$$\begin{aligned}
\mathcal{X}\Xi\mathcal{X}\ =\ & A_{od}\mathcal{X}+\mathcal{X}A_{od}^\top+B_o\mathcal{Y}+\mathcal{Y}^\top B_o^\top\\
&+\tau^+\left(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6\right)A_dA_d^\top\\
&+\tau^+\varepsilon_1^{-1}\mathcal{X}A_o^\top A_o\mathcal{X}+\tau^+\varepsilon_2^{-1}\mathcal{X}A_d^\top A_d\mathcal{X}\\
&+\tau^+\varepsilon_3^{-1}\mathcal{Z}\mathcal{Z}^\top. \qquad (43)
\end{aligned}$$

By invoking the Schur complement of (43), we arrive at the LMI (27) which guarantees that $\Xi<0$, and consequently $x^\top(t)\Xi x(t)<0$. Now, we need to show that the remaining terms of (41) are negative definite. Using the definition of $z(t)=\mu(t)x(t)$, we know that

$$\begin{aligned}
&\tau^+ r_5z^\top(t)\mathcal{I}^\top B_o^\top B_o\mathcal{I}z(t)\\
&\le \tau^+ r_5\,||\mathcal{I}^\top B_o^\top B_o\mathcal{I}||\,\mu^2(t)\,||x(t)||^2. \qquad (44)
\end{aligned}$$

Also, using Assumptions 2.1 and 3.1 , we have

$$\tau^+ r_6E^\top(x,t)E(x,t)\le\tau^+ r_6\left(\theta^*\right)^2\,||x(t)||^2, \qquad (45)$$

and

$$\begin{aligned}
&\tau^+ r_4x^\top(t)\Delta K^\top(t)B_o^\top B_o\Delta K(t)x(t)\\
&\le \tau^+ r_4\left(\rho^*\right)^2\,||B_o^\top B_o||\,||x(t)||^2. \qquad (46)
\end{aligned}$$

Now, using (44)- (46), the adaptive laws (28)- (29), and the fact that $|\mu(t)| \geq 1$, equation (41) becomes

$$\begin{aligned}\dot{V}_a(x) \;\; \leq \;\; & x^\top(t)\Xi x(t) + \tau^+ r_4\,(\rho^*)^2\;||B_o^\top B_o||\;||x(t)||^2 \\ & +\tau^+ r_5\,||\mathcal{I}^\top B_o^\top B_o \mathcal{I}||\;\mu^2(t)\;||x(t)||^2 \\ & +\tau^+ r_6\,(\theta^*)^2\;||x(t)||^2 + 2\rho^*||PB_o||\;||x(t)||^2 \\ & +2||PB_o\mathcal{I}||\;|\mu(t)|\;||x(t)||^2 \\ & +2\theta^*||P||\;||x(t)||^2 + 2\alpha_1\;|\mu(t)|\;||x(t)||^2 \\ & +2\alpha_2\;\mu^2(t)\;||x(t)||^2 \\ & +2\alpha_3\;|\mu(t)|\;\hat{\theta}(t)\;||x(t)||^2 \\ & +2\gamma\;|\mu(t)|\;\hat{\theta}(t)\;||x(t)||^2 - 2\gamma\;\theta^*\;||x(t)||^2 \\ & +2\gamma\;\theta^*\;\hat{\theta}(t)\;||x(t)||^2 - 2\gamma\;(\theta^*)^2\;||x(t)||^2 \quad (47)\end{aligned}$$

It can be shown that $\dot{V}_a(x) < 0$ if the adaptive law parameters α_1, α_2, and α_3 are selected as stated in Theorem 1, and γ is selected to satisfy the following two conditions: $\gamma > \frac{1}{2}\tau^+ r_6$ and $||P|| - \gamma + \gamma\hat{\theta}(t) < 0$. Hence, we need to select γ such that

$$\gamma \;>\; max\left\{\frac{1}{2}\tau^+ r_6\;,\;\frac{||P||}{1-\hat{\theta}(t)}\right\}. \qquad (48)$$

It is clear that when $\hat{\theta}(t) > 1$, we only need to ensure that $\gamma > \frac{1}{2}\tau^+ r_6$. Note that from equation (29), $\hat{\theta}(t) > 1$ can be easily ensured by selecting $\hat{\theta}(0) > 1$ and $\gamma > \frac{1}{2}\tau^+ r_6$ to guarantee that $\hat{\theta}(t)$ in equation (29) is monotonically increasing. Hence, we guarantee that

$$\dot{V}_a(x) \;\; \leq \;\; x^\top(t)\Xi x(t), \qquad (49)$$

where $\Xi < 0$. Hence, $\dot{V}_a(x) < 0$ which guarantees asymptotic stabilization of the closed-loop system. ■

B. Adaptive Control when ρ^* is Unknown

To stabilize the system (4) when ρ^* is unknown, the control law (5) is considered. Before we present the stability results for this case, let us define $\tilde{\rho}(t) = \hat{\rho}(t) - \rho^*$, where $\hat{\rho}(t)$ is the estimate of ρ^*, and $\tilde{\rho}(t)$ is error between the estimate and the true value of ρ^*. Here, the following Lyapunov-Krasovskii functional is used

$$V_b(x) \;\; = \;\; V_a(x) \; + \; V_{10}(x), \qquad (50)$$

where $V_a(x)$ is defined in equations (7), and $V_{10}(x)$ is defined as

$$V_{10}(x) = (1+\rho^*)\,[\tilde{\rho}(t)]^2\,, \qquad (51)$$

where its time derivative is

$$\dot{V}_{10}(x) = 2\;(1+\rho^*)\,\tilde{\rho}(t)\;\dot{\tilde{\rho}}(t). \qquad (52)$$

Since $\tilde{\rho}(t) = \hat{\rho}(t) - \rho^*$, then $\dot{\tilde{\rho}}(t) = \dot{\hat{\rho}}(t)$. Hence, equation (52) becomes

$$\dot{V}_{10}(x) = 2\;(1+\rho^*)\,[\hat{\rho}(t) - \rho^*]\;\dot{\hat{\rho}}(t). \qquad (53)$$

The next Theorem provides the main results for this case.

Theorem 2: *Consider system (4). If there exist matrices $0 < \mathcal{X} = \mathcal{X}^\top \in \Re^{n\times n}$, $\mathcal{Y} \in \Re^{m\times n}$, $\mathcal{Z} \in \Re^{n\times n}$, and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > \varepsilon$, $\varepsilon_5 > \varepsilon$ and $\varepsilon_6 > \varepsilon$ (where ε is an arbitrary small positive constant) such that the LMI (27) has a feasible solution, and $K = \mathcal{Y}\mathcal{X}^{-1}$, and $\mu(t)$ is adapted subject to the adaptive laws*

$$\begin{aligned}\dot{\mu}(t) \;\; = \;\; & Proj\left\{\beta_1\;sgn\,(\mu(t))\;\;||x(t)||^2 + \beta_2\;\mu(t)\;||x(t)||^2\right. \\ & +\beta_3\;\;sgn\,(\mu(t))\;\;\hat{\theta}(t)\;||x(t)||^2 \\ & \left.+\beta_4\;\;sgn\,(\mu(t))\;\;\hat{\rho}(t)\;||x(t)||^2, \mu(t)\right\}, \qquad (54)\end{aligned}$$

$$\dot{\hat{\theta}}(t) \;\; = \;\; \sigma\;||x(t)||^2, \qquad (55)$$

$$\dot{\hat{\rho}}(t) \;\; = \;\; \varsigma\;||x(t)||^2, \qquad (56)$$

where $Proj\{\cdot\}$ [30] is applied to ensure that $|\mu(t)| \geq 1$ as follows

$$\mu(t) = \begin{cases} \mu(t) & \textit{if } |\mu(t)| \geq 1 \\ 1 & \textit{if } 0 \leq \mu(t) < 1 \\ -1 & \textit{if } -1 < \mu(t) < 0, \end{cases}$$

and the adaptive law parameters are selected such that $\beta_1 < -[||PB_o\mathcal{I}||]$, $\beta_2 < -\frac{1}{2}\tau^+ r_5||\mathcal{I}^\top B_o^\top B_o\mathcal{I}||$, $\beta_3 < -\sigma$, $\beta_4 < -\varsigma$, $\sigma > \frac{1}{2}\tau^+ r_6$, $\varsigma > \frac{1}{2}\tau^+ r_4||B_o^\top B_o||$, $\hat{\theta}(0) > 1$ and $\hat{\rho}(0) > 1$, then the control law (5) will guarantee asymptotic stabilization of the closed-loop system.

Proof: The time derivative of $V_b(x)$ is

$$\dot{V}_b(x) \;\; = \;\; \dot{V}_a(x) \; + \; \dot{V}_{10}(x). \qquad (57)$$

Following the steps used in the proof of Theorem 1 and using equation (53), it can be shown that

$$\begin{aligned}\dot{V}_b(x) \;\; \leq \;\; & x^\top(t)\Xi x(t) \\ & +\tau^+ r_4 x^\top(t)\Delta K^\top(t) B_o^\top B_o \Delta K(t) x(t) \\ & +\tau^+ r_5 z^\top(t)\mathcal{I}^\top B_o^\top B_o \mathcal{I} z(t) \\ & +\tau^+ r_6 E^\top(x,t)E(x,t) + 2\rho^*||PB_o||\;||x(t)||^2 \\ & +2||PB_o\mathcal{I}||\;|\mu(t)|\;||x(t)||^2 \\ & +2\theta^*||P||\;||x(t)||^2 + 2\;\mu(t)\;\dot{\mu}(t) \\ & +2\;\;(1+\theta^*)\left[\hat{\theta}(t) - \theta^*\right]\;\dot{\hat{\theta}}(t) \\ & +2\;\;(1+\rho^*)\,[\hat{\rho}(t) - \rho^*]\;\;\dot{\hat{\rho}}(t), \qquad (58)\end{aligned}$$

where Ξ is defined in equation (42). Using the linearization procedure and invoking the Schur complement (as in the proof of Theorem 1), it can be shown that Ξ is guaranteed to be negative definite whenever the LMI (27) has a feasible solution. Using the adaptive laws (54)- (56) in (58) and the fact that $|\mu(t)| \geq 1$, we get

$$\begin{aligned}\dot{V}_b(x) \;\; \leq \;\; & x^\top(t)\Xi x(t) + \tau^+ r_4\,(\rho^*)^2\;||B_o^\top B_o||\;||x(t)||^2 \\ & +\tau^+ r_5\;||\mathcal{I}^\top B_o^\top B_o\mathcal{I}||\;\mu^2(t)\;||x(t)||^2 \\ & +\tau^+ r_6\,(\theta^*)^2\;||x(t)||^2 + 2\rho^*||PB_o||\;||x(t)||^2 \\ & +2||PB_o\mathcal{I}||\;|\mu(t)|\;||x(t)||^2 \\ & +2\theta^*||P||\;||x(t)||^2 + 2\beta_1\;|\mu(t)|\;||x(t)||^2 \\ & +2\beta_2\;\mu^2(t)\;||x(t)||^2 \\ & +2\beta_3\;|\mu(t)|\;\hat{\theta}(t)\;||x(t)||^2 \\ & +2\beta_4\;|\mu(t)|\;\hat{\rho}(t)\;||x(t)||^2 \\ & +2\sigma\;|\mu(t)|\;\hat{\theta}(t)\;||x(t)||^2 - 2\sigma\;\theta^*\;||x(t)||^2 \\ & +2\sigma\;\theta^*\;\hat{\theta}(t)\;||x(t)||^2 - 2\sigma\;(\theta^*)^2\;||x(t)||^2\end{aligned}$$

$$+2\varsigma\ |\mu(t)|\ \hat{\rho}(t)\ ||x(t)||^2 - 2\varsigma\ \rho^*\ ||x(t)||^2$$
$$+2\varsigma\ \rho^*\ \hat{\rho}(t)\ ||x(t)||^2 - 2\varsigma\ (\rho^*)^2\ ||x(t)||^2. \quad (59)$$

Arranging terms of equation (59), it can be shown that $\dot{V}_b(x) < 0$ if the adaptive law parameters β_1, β_2, β_3, and β_4 are selected as stated in Theorem 2, and σ and ς are selected to satisfy the following conditions: $\sigma > \frac{1}{2}\tau^+ r_6$, $2||P|| - \sigma + \sigma\hat{\theta}(t) < 0$, $\varsigma > \frac{1}{2}\tau^+ r_4 ||B_o^\top B_o||$, and $||PB_o|| - \varsigma + \varsigma\hat{\rho}(t) < 0$. Hence, we need to select σ and ς such that

$$\sigma \quad > \quad max\left\{\frac{1}{2}\tau^+ r_6\ ,\ \frac{||P||}{1-\hat{\theta}(t)}\right\}, \quad (60)$$

$$\varsigma \quad > \quad max\left\{\frac{1}{2}\tau^+ r_4 ||B_o^\top B_o||\ ,\ \frac{||PB_o||}{1-\hat{\rho}(t)}\right\}. \quad (61)$$

It is clear that when $\hat{\theta}(t) > 1$ and $\hat{\rho}(t) > 1$, we only need to ensure that $\sigma > \frac{1}{2}\tau^+ r_6$ and $\varsigma > \frac{1}{2}\tau^+ r_4||B_o^\top B_o||$. Note that from equations (55)- (56), $\hat{\theta}(t) > 1$ and $\hat{\rho}(t) > 1$ can be easily ensured by selecting $\hat{\theta}(0) > 1$ and $\hat{\rho}(0) > 1$ and σ and ς as stated in Theorem 2 to guarantee that $\hat{\theta}(t)$ and $\hat{\rho}(t)$ are monotonically increasing. Hence, we guarantee that

$$\dot{V}_b(x) \quad \leq \quad x^\top(t)\ \Xi\ x(t), \quad (62)$$

where $\Xi < 0$. Hence, $\dot{V}_b(x) < 0$ which guarantees asymptotic stabilization of the closed-loop system. ■

IV. CONCLUSION

In this paper, we investigated the problem of designing resilient delay-dependent adaptive controllers for a class of uncertain time-delay systems with time-varying delays and a nonlinear perturbation when perturbations also appear in the state feedback gain of the controller. It is assumed that the nonlinear perturbation is bounded by a weighted norm of the state vector such that the upper bound of the weight is unknown. It is also assumed that the norm of the uncertainty of the state feedback gain is bounded by a positive constant. For the two cases when this positive constant is known and unknown, adaptive control schemes have been developed to guarantee asymptotic closed-loop stabilization results.

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