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Stabilization of Multi-agent Dynamical Systems for Cyclic Pursuit Behavior

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Abstract— **This paper studies pursuit formation stability analysis and stabilization problems in target-enclosing operations by multiple dynamic agents. First, we introduce a $\mathcal{D}$-stability problem by considering the requirements for multi-agent system's transient performance, and then develop a simple diagrammatic pursuit formation stability criterion. Then, as for the formation stabilization problem when agent's dynamics and its local controller are given, we develop an optimization problem subject to LMI constraints derived based on the generalized Kalman-Yakubovich-Popov (GKYP) lemma to maximize the connectivity gain of a cyclic pursuit based on-line path generator. It provides a permissible range of gain, which guarantees the satisfaction not only of a global formation stability condition but also of a required performance specification. Finally, a constrained polynomial optimization problem is developed, in order to design agent's local controller parameters guaranteeing that a connectivity gain becomes the maximum one satisfying the global formation stability condition for a class of dynamic agents given a priori.**

I. Introduction

Several research groups have developed control strategies achieving an enclosing formation around a specific object by multiple agents [1], [2], [3]. Recently, Kim and Sugie [4] proposed a distributed on-line path planning method for target-enclosing operations by multi-agent systems based on a modified cyclic pursuit strategy. Despite its simple but particularly effective nature for target-enclosing tasks, it could be a considerable drawback in real implementations that each agent is assumed to be a point mass with full actuation. In order to overcome the above difficulty, Hara et al. [5] recently developed a distributed pursuit cooperative control scheme for multiple dynamic agents, and then presented a simple diagrammatic formation stability analysis method based on the results given in Hara et al. [6].

In this paper, we consider multiple agents in 3D space, which have common system dynamics and identical local controllers. For such multi-agent dynamical systems, this paper proposes optimization-based formation stabilization strategies for a distributed cooperative control for target-enclosing operations based on a cyclic pursuit scheme. To this end, we first briefly summarize the following conventional results presented by Hara et al. [5]: a distributed pursuit formation control mechanism. Next, we introduce a $\mathcal{D}$-stability problem by considering the requirements for the above multi-agent system's transient performance, and develop the diagrammatic pursuit formation stability criterion in Section III. Then, based on the above results, the following two kinds of optimization problems for a pursuit formation stabilization are considered:

[Problem S1] *Maximization of a connectivity gain of a cyclic pursuit based on-line path generator, which satisfies for given agent's dynamics and its local controller not only a global formation stability condition but also a required multi-agent system's performance specification.*

[Problem S2] *Optimization of agent's local controller parameters for a class of agent's dynamics given a priori, so that a given connectivity gain becomes the maximum one guaranteeing the global pursuit formation stability.*

In order to derive an optimization problem for Problem S1, we first show that the required pursuit formation stability condition can be converted to the linear matrix inequalities (LMIs), which are numerically tractable and can be solved efficiently, based on the generalized Kalman-Yakubovich-Popov (GKYP) lemma [7], [8]. Then, a concrete optimization problem subject to LMI constraint conditions for maximizing the connectivity gain is developed in Section IV. Further, in order to clearly show its distinctive features, the special case such as an optimization-based pursuit formation stabilization scheme for a class of multi-agent dynamical systems combined with PID controllers is presented in Section V-B. Finally, we develop a constrained polynomial optimization problem to solve the Problem S2 in Section VI. It provides a considerably simple and systematic optimization-based local PD controller design method, which guarantees the global pursuit formation stability.

II. System Description and Control Aim

Consider a group of n agents dispersed in 3D space as shown in Fig. 1(a). All agents are ordered from 1 to n; i.e., $P_1,\ P_2,\cdots,\ P_n$. Denote the position vectors of the target object and the agent P_i $(i = 1, 2, \cdots, n)$ in the inertial frame by $\mathbf{p}_o(t) \in \mathbb{R}^3$ and $\mathbf{p}_i(t) \in \mathbb{R}^3$, respectively. It is assumed that an agent P_i can measure the following vectors: $\mathbf{d}_i$ $(:= \mathbf{p}_i - \mathbf{p}_o)$, $\mathbf{a}_i$ $(:= \mathbf{p}_i - \mathbf{p}_{i+1})$. Define the target-fixed frame $\{\Gamma_{obj}\}$ where the origin is at the center of target object, and X_{obj}-, Y_{obj}- and Z_{obj}-axes are parallel to x-, y- and z-axes of the inertial frame, respectively. Let $\mathbf{b}_i$ denote the projected vector of $\mathbf{d}_i$ onto the X_{obj}-Y_{obj} plane in the target-fixed frame, and define the following scalars: $\theta_i = \angle(\mathbf{e}_x, \mathbf{b}_i)$, $\alpha_i = \angle(\mathbf{b}_i, \mathbf{d}_i)$ and $d_i := |\mathbf{d}_i|$ where $\mathbf{e}_x$ denotes the unit vector in the X_{obj}-direction of $\{\Gamma_{obj}\}$,

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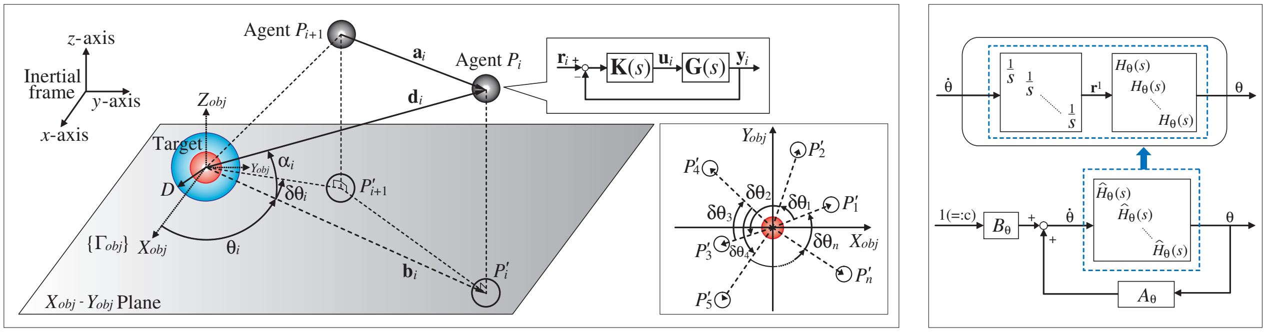


(a) Coordinate frames and notations

(b) The $\boldsymbol{\theta}$-directional formation controlled system

Fig. 1. Depiction of multi-agent dynamical systems

and $\angle(\mathbf{x}, \mathbf{y})$ denotes the counter-clockwise angle from the vector $\mathbf{x}$ to the vector $\mathbf{y}$. Then, $\mathbf{d}_i$ can be represented as $\mathbf{d}_i = [d_i \cos\theta_i \cos\alpha_i, d_i \sin\theta_i \cos\alpha_i, d_i \sin\alpha_i]^T$. Note that since $\mathbf{d}_{i+1} = \mathbf{d}_i - \mathbf{a}_i$, θ_{i+1} and $\delta\theta_i (:= \theta_{i+1} - \theta_i)$ can be calculated in a similar way.

Suppose that all agents P_i $(i = 1, 2, \cdots, n)$ have common system dynamics described by a MIMO plant:

$$\mathbf{y}_i(s) := [\theta_i(s), d_i(s), \alpha_i(s)]^T = \mathbf{G}(s)\mathbf{u}_i(s) \tag{1}$$

where $\mathbf{y}_i(s)$ is the system output, $\mathbf{u}_i(s)$ is the control input, and $\mathbf{G}(s) := \mathrm{diag}(G_\theta(s), G_d(s), G_\alpha(s))$. Also, assume that all agents are locally stabilized by an identical diagonal feedback controller $\mathbf{K}(s)$ defined as $\mathbf{K}(s) := \mathrm{diag}(K_\theta(s), K_d(s), K_\alpha(s))$ as illustrated in Fig. 1(a). Therefore, θ-directional closed-loop transfer function of each agent is described as $H_\theta(s) = \frac{G_\theta(s)K_\theta(s)}{1+G_\theta(s)K_\theta(s)}$. The d- and α-directional ones are defined in the same manner [5], [9].

Now, we consider how to form a geometric pattern for the target-enclosing operation by n agents. The detailed control objectives are formulated algebraically as follows:

(A1) $\delta\theta_i(t) \to 2\pi/n(\text{rad})$ as $t \to \infty$,
(A2) $d_i(t) \to D$ as $t \to \infty$,
(A3) $\alpha_i(t) \to \Phi(\text{rad})$ as $t \to \infty$,

where D and Φ are given by the designer. Note that, for the sake of page limitation, we mainly consider the θ-directional control scheme in this paper [9].

III. Pursuit Formation Stability Criterion

As one of the feasible simple methods which realize the required geometric formation (A1)-(A3), Hara et al. [5] proposed a distributed cooperative control scheme motivated by a cyclic pursuit strategy. In the following, we first briefly summarize the above results, and then consider the $\mathcal{D}$-stability problem for the global pursuit formation stability criterion.

Assume that n agents are randomly dispersed in 3D space at the initial time instant as depicted in Fig. 1(a), where $0 < |\delta\theta_i| < 2\pi$ for $i = 1, 2, \cdots, n$, and $\sum_{i=1}^{n} \delta\theta_i = 2\pi$. Then, the distributed θ-directional on-line path planning scheme is described as follows

$$\dot{\boldsymbol{\theta}}(t) = A_\theta \boldsymbol{\theta}(t) + B_\theta \tag{2}$$

with

$$A_\theta := \mathrm{circ}(-k_1,\ k_1,\ 0,\ 0, \cdots,\ 0) \in \mathbb{R}^{n\times n},$$
$$B_\theta := [0,\ 0, \cdots,\ 0,\ 2k_1\pi]^T \in \mathbb{R}^n,$$

where $\boldsymbol{\theta} := [\theta_1, \theta_2, \cdots, \theta_n]^T \in \mathbb{R}^n$ and *circ* denotes the circulant matrix. Thus, the overall $\boldsymbol{\theta}$-directional control scheme can be depicted as shown in Fig. 1(b), where $\hat{H}_\theta(s) := (1/s)H_\theta(s)$ and $\mathbf{r}^1 := [r_1^1, r_2^1, \cdots, r_n^1]^T \in \mathbb{R}^n$. We define $k_1 (> 0)$ as connectivity gain, and assume that $\hat{H}_\theta(s)$ is strictly proper.

Next, we present a simple and unified theoretical framework showing how to judge the pursuit formation stability, when a connectivity gain k_1 and an agent's dynamics $H_\theta(s)$ are given (for details, see Hara et al. [5]). It is important to note that since A_θ is a circulant matrix, it has exactly one zero eigenvalue, λ_1, while the remaining nonzero $n-1$ eigenvalues λ_i, $i = 2, 3, \cdots, n$, lie strictly in the left-half complex plane; i.e., these are located on the circumference of radius k_1 whose center is at $(-k_1, 0)$ as illustrated in Fig. 2. Note that we can disregard exactly one zero eigenvalue and determine the formation stability based on the remaining $n-1$ eigenvalues of A_θ [4], [5]. Thus, we consider hereafter only $n-1$ eigenvalues of A_θ except for one eigenvalue at the origin, if no confusion could arise.

In Fig. 1(b), the transfer function $\mathcal{G}_\theta(s)$ from c to $\boldsymbol{\theta}$ is obtained as

$$\mathcal{G}_\theta(s) = \left(\frac{1}{\hat{H}_\theta(s)} I_n - A_\theta\right)^{-1} B_\theta = \mathcal{F}_u\left(\begin{bmatrix} A_\theta & B_\theta \\ I_n & 0 \end{bmatrix}, \hat{H}_\theta(s) I_n\right) \tag{3}$$

where $\mathcal{F}_u$ denotes the upper linear fractional transformation (LFT). By considering the transfer function

$$L_\theta(s) = (sI_n - A_\theta)^{-1} B_\theta, \tag{4}$$

it follows from (3) that

$$\mathcal{G}_\theta(s) = L_\theta(\phi(s)), \quad \phi(s) := 1/\hat{H}_\theta(s) \tag{5}$$

where $\phi(s)$ is defined as a generalized frequency variable [5], [6]. For the above multi-agent system, Hara et al. [5] shows the following pursuit formation stability criterion: all nonzero poles of $\mathcal{G}_\theta(s)$ depending on k_1 are located in the

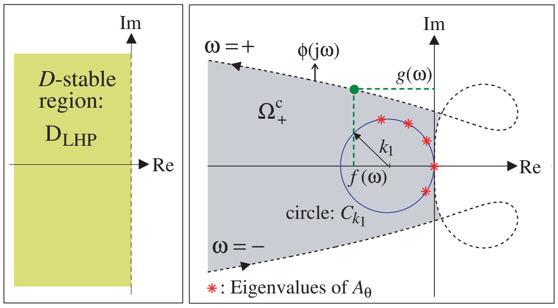
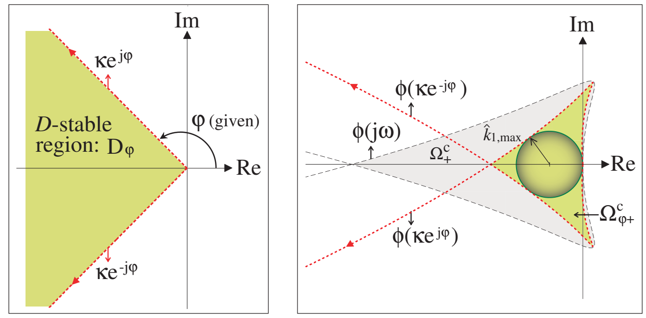


Fig. 2. The eigenvalues λ_i $(i = 1, 2, \cdots, n)$ of A_θ and the domain Ω_+^c



(a) The $\mathcal{D}$-stable region for all non-zero poles of $\mathcal{G}_\theta(s)$ (b) The domain $\Omega_{\varphi+}^c$ and the image of $\phi(\kappa e^{\pm j\varphi})$ $(\kappa \geq 0,\ 0 \leq \varphi \leq \pi)$

Fig. 3. The image of $\phi(\kappa e^{\pm j\varphi})$ and the corresponding domain $\Omega_{\varphi+}^c$

left-half plane of the complex plane (i.e., $\mathrm{D_{LHP}}$ in Fig. 2), if and only if all nonzero poles of $L_\theta(s)$ belong to the domain Ω_+^c defined as

$$\Omega_+ := \phi(\mathbb{C}_+), \quad \Omega_+^c := \mathbb{C} \backslash \Omega_+, \tag{6}$$

where $\mathbb{C}_+ = \{s \in \mathbb{C} : \mathrm{Re}[s] \geq 0\}$. Since $\Omega_+ = \{\lambda \in \mathbb{C} : \exists s \in \mathbb{C}_+ \text{ such that } \phi(s) = \lambda\}$, it follows that Ω_+^c can be alternatively expressed as $\Omega_+^c = \{\lambda \in \mathbb{C} : \forall s \in \mathbb{C}_+,\ \phi(s) \neq \lambda\}$. Although it provides a considerably simple formation stability analysis method, no explicit information about the transient performance of the designed formation control laws (see [9] for details).

In order to overcome the above problem, we consider the $\mathcal{D}$-stability problem: i.e., determining the poles of linear time-invariant system $\mathcal{G}_\theta(s)$ in a predesignated region of the complex plane, which is formulated as

> **Problem A**: ($\mathcal{D}$-stability problem) Derive a global pursuit formation stability criterion that enables us to judge whether all nonzero poles of $\mathcal{G}_\theta(s)$ in (3) belong to the $\mathcal{D}$-stable region, D_φ in Fig. 3(a), which is characterized by $\kappa e^{\pm j\varphi}$ for $\forall \kappa \geq 0$ where φ $(\pi/2 \leq \varphi \leq \pi)$ is given a priori.

To this aim, we first define the domains $\Omega_{\varphi+}$ and $\Omega_{\varphi+}^c$ in the complex plane, which are generalizations of the definitions of domains Ω_+ and Ω_+^c in (6):

$$\Omega_{\varphi+} := \phi(\mathbb{C}_{\varphi+}), \quad \Omega_{\varphi+}^c := \mathbb{C} \backslash \Omega_{\varphi+}, \tag{7}$$

where $\mathbb{C}_{\varphi+} = \{s := \kappa e^{j\vartheta} \in \mathbb{C} : -\varphi \leq \vartheta \leq \varphi,\ \forall \kappa > 0\}$. Note that the domain $\Omega_{\varphi+}^c$ includes the origin of the complex plane as shown in Fig. 3(b). Then, the key theorem which provides the stability criterion in Problem A is obtained as follows:

> *Theorem 1:* Consider the linear systems $\mathcal{G}_\theta(s)$ in (3) and $L_\theta(s)$ in (4). Also, assume that $\hat{H}_\theta(s)$ is strictly proper and stable. Then, all nonzero poles of $\mathcal{G}_\theta(s)$ are located in the $\mathcal{D}$-stable region D_φ in Fig. 3(a), if and only if all the poles of $L_\theta(s)$ belong to $\Omega_{\varphi+}^c$ in (7).

This theorem means that the $\mathcal{D}$-stability of $\mathcal{G}_\theta(s)$ in Problem A can be judged by just looking at the locations of eigenvalues of A_θ depending on k_1 in relation to a domain $\Omega_{\varphi+}^c$ determined by using $\hat{H}_\theta(s)$ (see Example 1 in [9]).

IV. Pursuit Formation Stabilization

In Theorem 1, we obtained a considerably simple diagrammatic formation stability analysis method. On the other hand, it may be required to find the maximum permissible limit of a connectivity gain k_1 satisfying the requirement presented in Problem A of the previous section. Therefore, we consider the following $\mathcal{D}$-stabilization problem in this section:

> **Problem S1**: Find the upper bound $\hat{k}_{1,\max}$ of a connectivity gain k_1 in (2) guaranteeing that all nonzero poles of $\mathcal{G}_\theta(s)$ in (3) are placed in the predesignated $\mathcal{D}$-stable region, D_φ, illustrated in Fig. 3(a).

From Theorem 1 and Figs. 2 and 3, the condition, which guarantees all the eigenvalues of A_θ with $k_1 \leq \hat{k}_1$ belong to the domain $\Omega_{\varphi+}^c$ characterized by $\kappa e^{\pm j\varphi}$, is derived as

$$(f_\varphi(\kappa) + \hat{k}_1)^2 + g_\varphi^2(\kappa) > \hat{k}_1^2, \ \hat{k}_1 > 0, \ \forall \kappa > 0, \tag{8}$$

where $f_\varphi(\kappa)$ and $g_\varphi(\kappa)$ are defined, respectively, as

$$f_\varphi(\kappa) := \mathrm{Re}[\phi(\kappa e^{j\varphi})], \quad g_\varphi(\kappa) := \mathrm{Im}[\phi(\kappa e^{j\varphi})].$$

Then, the following inequality condition which is equivalent to (8) can easily be derived [5]:

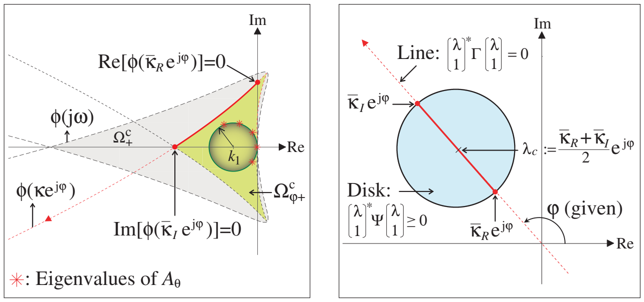
$$\begin{bmatrix} \hat{H}_\theta(\kappa e^{j\varphi}) \\ 1 \end{bmatrix}^* \Pi \begin{bmatrix} \hat{H}_\theta(\kappa e^{j\varphi}) \\ 1 \end{bmatrix} < 0, \ \Pi := \begin{bmatrix} 0 & -\hat{k}_1 \\ -\hat{k}_1 & -1 \end{bmatrix} \tag{9}$$

where $\forall \kappa > 0$ and φ is given by the designer. Therefore, the optimization problem to find the upper bound of the connectivity gain k_1 satisfying the requirement in Problem S1 can be formulated as follows:

$$\hat{k}_{1,\max} := \arg \max_{\hat{k}_1, \kappa} \hat{k}_1 \tag{10}$$

subject to (9) and $\hat{k}_1 > 0$. Hence, if one sets k_1 in (2) as $0 < k_1 \leq \hat{k}_{1,\max}$, then all nonzero poles of $\mathcal{G}_\theta(s)$ in (3) are placed in the predesignated $\mathcal{D}$-stable region D_φ. However, since the constraint condition (9) should be checked for $\forall \kappa > 0$, the above optimization problem may not be easily solved.

In order to overcome the above difficulty, we then convert the inequality condition (9) to LMIs by using the GKYP lemma [7], [8]. Note that LMIs are numerically tractable and



(a) The image of $\phi(\kappa e^{\pm j\varphi}), \forall\kappa \geq 0$, where φ(rad) is given a priori

(b) The set of complex numbers determined by $\mathbf{\Lambda}(\Gamma, \Psi)$

Fig. 4. The image of $\phi(\kappa e^{j\varphi})$ and the set of complex numbers $\mathbf{\Lambda}(\Gamma, \Psi)$

can be solved efficiently. Now, we assume that the state-space realization of $\hat{H}_\theta(s)$ is given by

$$\hat{H}_\theta(s) = C_h(sI - A_h)^{-1}B_h + D_h. \tag{11}$$

We then characterize the restricted range of κ to check the constraint condition (9) within a framework of the GKYP lemma. For example, if $\phi(s)(= 1/\hat{H}_\theta(s))$ and φ are given, $\phi(\kappa e^{j\varphi})$ is readily obtained (see Figs. 3 and 4). Then, $\bar{\kappa}_R$ and $\bar{\kappa}_I$ $(0 < \bar{\kappa}_R < \bar{\kappa}_I)$ satisfying $\mathrm{Re}[\phi(\bar{\kappa}_R e^{j\varphi})] = 0$ and $\mathrm{Im}[\phi(\bar{\kappa}_I e^{j\varphi})] = 0$, respectively, are found via simple calculations (if exist). Now one can see from Fig. 4(a) that it is sufficient to check the condition (9) in the range $\bar{\kappa}_R \leq \kappa \leq \bar{\kappa}_I$. In this case, the set of complex numbers $\mathbf{\Lambda}(\Gamma, \Psi)$ corresponding to the above-mentioned range of κ is defined as

$$\mathbf{\Lambda}(\Gamma,\Psi) := \left\{ \lambda \in \mathbb{C} : \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Gamma \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = 0, \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Psi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \geq 0 \right\} \tag{12}$$

with

$$\Gamma := \begin{bmatrix} 0 & \tan\varphi - j \\ \tan\varphi + j & 0 \end{bmatrix}, \ \Psi := \begin{bmatrix} -1 & \lambda_c \\ \bar{\lambda}_c & -\bar{\kappa}_R\bar{\kappa}_I \end{bmatrix}, \tag{13}$$

and $\lambda_c = \frac{\bar{\kappa}_R + \bar{\kappa}_I}{2} e^{j\varphi}$ (see Fig. 4(b)). Therefore, under the setting (12) with (13), the upper bound $\hat{k}_{1,\max}$ of k_1 mentioned in Problem S1 can easily be obtained by solving the following optimization problem subject to LMI constraint conditions:

Optimization problem for Problem S1: For given $\hat{H}_\theta(s) \sim (A_h, B_h, C_h, D_h)$ and φ $(\pi/2 \leq \varphi \leq \pi)$, solve

$$\hat{k}_{1,\max} := \arg \max_{\hat{k}_1, P\in\mathbf{H}_n, Q\in\mathbf{H}_n} \hat{k}_1, \tag{14}$$

where $\mathbf{H}_n$ denotes $n \times n$ Hermitian matrix, subject to $\hat{k}_1 > 0$ and

$$Q > 0, \ M^*ZM < 0, \tag{15}$$

where $Z = \mathrm{diag}(\Gamma \otimes P + \Psi \otimes Q, \Pi)$ with Π in (9) and Γ and Ψ in (12)-(13), and

$$M := \begin{bmatrix} A_h & B_h \\ I & 0 \\ C_h & D_h \end{bmatrix}.$$

Note that the LMI constraint condition in (15), which is equivalent to (9), is derived based on the GKYP lemma [7], [8]. If one sets k_1 in (2) as $0 < k_1 \leq \hat{k}_{1,\max}$, then all nonzero poles of $\mathcal{G}_\theta(s)$ in (3) are placed in the predesignated $\mathcal{D}$-stable region (D_φ) in Fig. 3(a) (see Example 1 in [9]).

V. Pursuit Formation Stabilization: Maximization of Connectivity Gain for PID Controller Case

In this section, we introduce a class of multi-agent dynamical systems locally stabilized by PID controllers, and then present how to stabilize this pursuit formation controlled systems based on the result given in Section IV. In order to make our idea clear, we set $\varphi = \pi/2$ (i.e., $\mathcal{D}$-stable region is $\mathrm{D_{LHP}}$ in Fig. 2), since the extension to the general case $\pi/2 < \varphi < \pi$ is trivial.

A. A class of multi-agent dynamical systems locally stabilized by PID controllers

Assume that θ-directional agent dynamics is given as

$$G_\theta(s) = \frac{\zeta}{s(s+\xi)} \tag{16}$$

where $\zeta > 0$. Then, the PID controller $K_{\mathrm{PID}}(s)$ such as

$$K_{\mathrm{PID}}(s) = k_p\left(1 + \frac{1}{t_i s} + t_d s\right), \tag{17}$$

where $k_p > 0$, $t_i > 0$ and $t_d > 0$ is introduced to stabilize (16). Hence, it follows from $G_\theta(s)$ and $K_{\mathrm{PID}}(s)$ that

$$H_\theta(s) = \frac{t_d t_i s^2 + t_i s + 1}{(t_i/\zeta k_p)s^3 + (\xi t_i/\zeta k_p + t_d t_i)s^2 + t_i s + 1}. \tag{18}$$

Let $\tilde{s} = t_i s$. Then, (18) can be modified as

$$H_\theta(\tilde{s}) = \frac{a\tilde{s}^2 + \tilde{s} + 1}{b\tilde{s}^3 + (a+c)\tilde{s}^2 + \tilde{s} + 1} \tag{19}$$

where $a := t_d/t_i (> 0)$, $b := 1/(\zeta k_p t_i^2) (> 0)$, $c := \xi/(\zeta k_p t_i)$. Therefore, without loss of generality, the following form of the generalized frequency variable $\phi(s)$ can be considered hereafter:

$$\phi(s) = \frac{1}{\hat{H}_\theta(s)} = \frac{s}{H_\theta(s)} = \frac{bs^4 + (a+c)s^3 + s^2 + s}{as^2 + s + 1}. \tag{20}$$

Note that $H_\theta(s)$ is stable if and only if $a + c > b$.

Next, we characterize the domains Ω_+ and Ω_+^c defined as (6) in the complex plane. These regions are partitioned by the image of $\phi(j\omega)$ in (20) where $\omega \in \mathbb{R}$. Define $f(\omega) := \mathrm{Re}\,[\phi(j\omega)]$ and $g(\omega) := \mathrm{Im}\,[\phi(j\omega)]$ as follows:

$$f(\omega) = \frac{\omega^4(-ab\omega^2 + b - c)}{(1 - a\omega^2)^2 + \omega^2}, \tag{21}$$

$$g(\omega) = \frac{(a^2 + ac - b)\omega^5 + (1 - 2a - c)\omega^3 + \omega}{(1 - a\omega^2)^2 + \omega^2}. \tag{22}$$

Then, the image of $\phi(j\omega)$ yields six types of diagrams. See Fig. 6 in [9], which is omitted because of page limitation. In the following, the pursuit formation stabilization problem for the above-multi-agent dynamical systems is considered.

B. GKYP lemma based pursuit formation stabilization

In this subsection, the special case of the pursuit formation stabilization scheme developed in Section IV is presented to clearly show its distinctive features:

Problem S1': For given $G_\theta(s)$, $K_{\rm PID}(s)$ (i.e., a, b and c in (20) are given) and $\varphi = \pi/2$, how to find the upper bound $k_{1,\max}$ of a connectivity gain $k_1(>0)$, which guarantees nonzero $n-1$ eigenvalues of A_θ with $k_1(\le k_{1,\max})$ belong to Ω^c_+ depicted in Fig. 6 of [9].

Now, one can easily find from Fig. 2 that if a given k_1 satisfies that

$$(\phi(j\omega)+k_1)^*(\phi(j\omega)+k_1) > k_1^2,\ \forall\omega\in\mathbb{R}\backslash\{0\}, \tag{23}$$

then nonzero $n-1$ eigenvalues λ_i $(i=2,3,\cdots,n)$ of A_θ are placed in the domain Ω^c_+. Then, similarly to (9), it is equivalent to (23) that

$$\begin{bmatrix}\hat{H}_\theta(j\omega)\\ 1\end{bmatrix}^* \Pi \begin{bmatrix}\hat{H}_\theta(j\omega)\\ 1\end{bmatrix} < 0,\ \Pi := \begin{bmatrix}0 & -k_1\\ -k_1 & -1\end{bmatrix} \tag{24}$$

for $\forall\omega\in\mathbb{R}\backslash\{0\}$. In (24), $\hat{H}_\theta(s)$ is proper as shown in (20), and thus has the state-space realization given by $\hat{H}_\theta(s)\sim(A_h,B_h,C_h,D_h)$ as shown in (11). For the above problem, the frequency-domain inequality (FDI) specification (24) can easily be checked by using the GKYP lemma, which transforms a FDI in a finite (or semi-infinite) frequency range into a set of LMIs as mentioned in optimization problem in Section IV. It means that checking the FDI in (24) within a given frequency range specified in Fig. 6 of [9] can be converted to the search for matrices $P,Q\in\mathbf{H}_n$ satisfying the LMIs in (15). In this problem setting, $\Psi\in\mathbf{H}_2$ is set as defined in Fig. 6 of [9]. On the other hand, $\Gamma\in\mathbf{H}_2$ is set as $\Gamma := \begin{bmatrix}0&1\\1&0\end{bmatrix}$ since the continuous-time setting is considered in this paper [7].

Based on the above results, the upper bound $k_{1,\max}$ of a connectivity gain k_1 is readily obtained by just solving the following constrained optimization:

Optimization problem for Problem S1': For given $G_\theta(s)$ and $K_{\rm PID}(s)$, solve

$$k_{1,\max} := \arg \max_{k_1, P\in\mathbf{H}_n, Q\in\mathbf{H}_n} k_1 \tag{25}$$

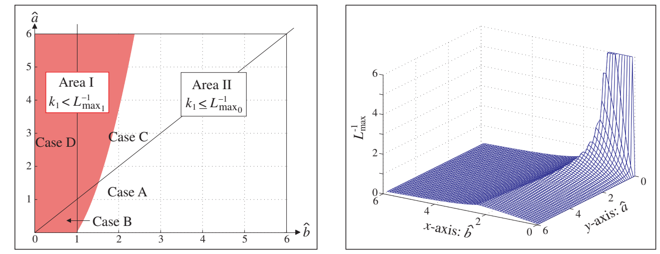
subject to $k_1>0$ and LMI constraints in (15) with $\Gamma := \begin{bmatrix}0&1\\1&0\end{bmatrix}$ and $\Psi\in\mathbf{H}_2$ in Fig. 6 of [9].

In the following section, we present an optimization-based design method for dynamic agent's local controller.

VI. Pursuit Formation Stabilization: Local PD Controller Design

In this section, it is assumed that only an agent's dynamics $G_\theta(s)$ in (16) is given a priori. Then, we consider how to design the PD controller

$$K_{\rm PD}(s) = \hat{k}_p(1+\hat{t}_d s),\ \hat{k}_p>0,\ \hat{t}_d>0, \tag{26}$$



(a) Areas I and II determined via Theorem 2 (I)-(II)

(b) The plot of $L^{-1}_{\max}$ in the $\hat{a}$-$\hat{b}$ plane

Fig. 5. The plot of $L^{-1}_{\max}$.

so that we can get the largest value of a connectivity gain k_1 which guarantees the formation stability. In the following, we will show that the above problem can be reduced to a constrained polynomial optimization problem. Our interest is again restricted to the case of $\varphi=\pi/2$ in order to avoid the notation complexity, but the generalization is trivial.

Problem S2: For a given $G_\theta(s)$ in (16), find the PD controller's gains $\hat{k}_p$ and $\hat{t}_d$ in (26), and its corresponding upper bound $k_{1,\max}$ of a connectivity gain which satisfies the following global pursuit formation stability condition: all nonzero poles of $\mathcal{G}_\theta(s)$ belong to the $\mathcal{D}$-stable region, $\rm D_{LHP}$, defined in Fig. 2.

Before we proceed, the generalized frequency variable $\phi(s)(=1/\hat{H}_\theta(s)=s/H_\theta(s))$ is defined from (16) and (26) as

$$\phi(s)=\frac{\hat{a}s^3+\hat{b}s^2+s}{s+1},\ \hat{a}:=\frac{1}{\zeta\hat{k}_p\hat{t}_d^2}(>0),\ \hat{b}:=\frac{\xi}{\zeta\hat{k}_p\hat{t}_d}+1. \tag{27}$$

Then, in order to develop an optimization problem for the above problem, we first consider the following inequality condition which is derived from (23) and (27):

$$L(\omega) := -2{\rm Re}[\hat{H}_\theta(j\omega)] = \frac{2(\hat{a}\omega^2+\hat{b}-1)}{\hat{a}^2\omega^4+(\hat{b}^2-2\hat{a})\omega^2+1} < \frac{1}{k_1} \tag{28}$$

for $\forall\omega\in\mathbb{R}\backslash\{0\}$. The condition (28) implies that if a given k_1^{-1} is bigger than the maximum value of $L(\omega)$ (except at $\omega=0$), then nonzero $n-1$ eigenvalues λ_i $(i=2,3,\cdots,n)$ of A_θ are placed in the domain Ω^c_+ defined via (27) (refer to [5]). From the above observations, the following key result which specifies the maximum permissible limit of a gain $k_1(>0)$ is obtained, which is an alternative algebraic formation stabilization method [5].

Theorem 2: Let $L_{\max_0}$ and $L_{\max_1}$ be defined, respectively, as

$$L_{\max_1}(\hat{a},\hat{b}) := \frac{2(\hat{a}\hat{\omega}+\hat{b}-1)}{\hat{a}^2\hat{\omega}^2+(\hat{b}^2-2\hat{a})\hat{\omega}+1},$$
$$L_{\max_0}(\hat{b}) := 2(\hat{b}-1),$$

where

$$\hat{\omega} := \frac{1-\hat{b}}{\hat{a}}+\frac{\hat{b}}{\hat{a}^2}\sqrt{\hat{a}(\hat{a}-\hat{b}+1)}. \tag{29}$$

Suppose that for given $\hat{a}$ and $\hat{b}$, an connectivity gain $k_1(>0)$ in (2) satisfies the following condition:

(I) $k_1 < L^{-1}_{\max_1}(\hat{a}, \hat{b})$, if $\hat{\omega}$ is a positive real number and $L_{\max_1}(\hat{a}, \hat{b}) \geq L_{\max_0}(\hat{b})$,
(II) $k_1 \leq L^{-1}_{\max_0}(\hat{b})$, otherwise.

Then, nonzero $n-1$ eigenvalues λ_i $(i = 2, 3, \cdots, n)$ of A_θ are placed in the domain Ω^c_+.

The region where the constraint conditions given in Theorem 2 (I) are satisfied is illustrated in Fig. 5(a) (Area I). It means that if given $\hat{a}$ and $\hat{b}$ of (27) exist in Area I, then the maximum value of k_1 should be determined by $k_1 < L^{-1}_{\max_1}$. On the other hand, if $\hat{a}$ and $\hat{b}$ exist in Area II, then $k_1 \leq L^{-1}_{\max_0}$. Figure 5(b) illustrates the plot of $L^{-1}_{\max}$ which is set as $L^{-1}_{\max} = L^{-1}_{\max_1}$ in Area I and $L^{-1}_{\max} = L^{-1}_{\max_0}$ in Area II. Based on the results in Theorem 2, the following optimization-based agent design method can easily be formulated:

Optimization problem for Problem S2: In order to determine the system parameters $\hat{a}$ and $\hat{b}$ of $H_\theta(s)$ in Area I, solve

$$\min_{(\hat{a},\hat{b})\in\mathcal{S}} L^{-1}_{\max_1}(\hat{a}, \hat{b}) \tag{30}$$

subject to

$$\hat{w} \text{ is a positive real number} \tag{31}$$

$$L_{\max_1}(\hat{a}, \hat{b}) \geq L_{\max_0}(\hat{b}) \tag{32}$$

where an agent's dynamics in the form of (16) is given, and $\mathcal{S}$ denotes a predefined set of $\hat{a}$ and $\hat{b}$. Then, the PD controller's gains $\hat{k}_p > 0$ and $\hat{t}_d > 0$ in (26) are obtained from $\hat{a}^* = 1/(\zeta\hat{k}_p\hat{t}_d^2)$ and $\hat{b}^* = \xi/(\zeta\hat{k}_p\hat{t}_d) + 1$ where $\hat{a}^*$ and $\hat{b}^*$ denote optimal values. Further, the maximum of a connectivity gain k_1 is obtained as $L^{-1}_{\max_1}(\hat{a}^*, \hat{b}^*)$.

On the other hand, once the system parameters $\hat{a}^*$ and $\hat{b}^*$ of $\hat{H}_\theta(s)$ which maximize a connectivity gain are determined via the above optimization problem, then it is possible to apply the $\mathcal{D}$-stabilization strategy presented in Section IV to find a $\mathcal{D}$-stabilizing connectivity gain k_1. Note that, if we intend to design $\hat{a}$ and $\hat{b}$ in Area II, these values can be determined from the condition presented in Theorem 2 (II).

It is also important to note that one can add additional constraint condition denoted by $\mathcal{S}$ in the above optimization problem. For example, the additional constraint conditions such that

(c1) *all the poles of $H_\theta(s)$ are located in a predesignated region in the complex plane; i.e., the $\mathcal{D}$-stabilization problem for each agent,*
(c2) *the predefined ranges of $\hat{a}$ and $\hat{b}$, which are set based on the desirable ranges of $\hat{k}_p$ and $\hat{t}_d$,*

are considered. In order to derive a numerical formulation for the constraint condition (c1), we introduce the following notations: let $\lambda_i(H_\theta(s))$ denote the ith pole of the system $H_\theta(s)$, and $\lambda_{\max}(H_\theta(s))$ be the pole whose real part is greater than those of other poles, i.e.,

$$\text{Re}[\lambda_{\max}(H_\theta(s))] = \max_i\{\text{Re}[\lambda_i(H_\theta(s))],\ \forall i\}.$$

The above-mentioned constrained polynomial optimization problem for Problem S2 subject to additional constraints (c1) and (c2) can easily be solved through the constrained particle swarm optimization scheme [10]. Refer to Example 2 in [9].

VII. CONCLUSION

In this paper, we have presented novel formation stability analysis and formation stabilization schemes for a distributed cooperative control based on a cyclic pursuit strategy. As for the formation stability analysis, we introduced a $\mathcal{D}$-stability problem in multi-agent dynamical systems, and then developed a simple diagrammatic pursuit formation stability criterion. Then, as for the formation stabilization problem when agent's dynamics and its local controller are given, we developed an optimization problem subject to LMI constraints to maximize the connectivity gain of a cyclic pursuit based on-line path generator, which satisfies not only a global formation stability condition but also a required multi-agent system's performance specification. In this case, the LMIs are derived based on the generalized Kalman-Yakubovich-Popov (GKYP) lemma. Finally, a constrained polynomial optimization problem was developed in order to design agent's local PD controller parameters guaranteeing that a given connectivity gain becomes the maximum one satisfying the global formation stability condition for a class of dynamic agents given a priori.

REFERENCES

[1] K. Kobayashi, K. Otsubo and S. Hosoe, "Design of decentralized capturing behavior by multiple robots," in IEEE Workshop on Distributed Intelligent Systems: Collective Intelligence and its applications, pp. 463–468, 2006.
[2] J. A. Marshall, M. E. Broucke and B. A. Francis, "Pursuit formations of unicycles," *Automatica*, vol. 42, pp. 3–12, 2006.
[3] R. Sepulchre, D. A. Paley and N. E. Leonard, "Group coordination and cooperative control of steered particles in the plane," in Group Coordination and Cooperative Control, pp. 217–232, Springer-Verlag: Lecture Notes in Control and Information Sciences, 2006.
[4] T.-H. Kim and T. Sugie, "Cooperative control for target-capturing task based on a cyclic pursuit strategy," *Automatica*, vol. 43, no. 8, pp. 1426–1431. 2007.
[5] S. Hara, T.-H. Kim and Y. Hori, "Distributed formation control for target-enclosing operations based on a cyclic pursuit strategy," in Proc. of the 17th IFAC World Congress, Seoul, Korea, July 6-11, 2008.
[6] S. Hara, T. Hayakawa and H. Sugata, "Stability analysis of linear systems with generalized frequency variables and its application to formation control," In Proc. of the 46th IEEE Conference on Decision and Control, pp. 1459–1466, 2007.
[7] S. Hara, T. Iwasaki and D. Shiokata, "Robust PID control using generalized KYP synthesis: direct open-loop shaping in multiple frequency ranges," *IEEE Control Systems Magazine*, vol. 26, no. 1, pp. 80–91, 2006.
[8] T. Iwasaki and S. Hara, "Generalized KYP lemma: Unified frequency domain inequalities with design applications," *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 41–59, 2004.
[9] T.-H. Kim and S. Hara, "Stabilization of multi-agent dynamical systems for cyclic pursuit behavior," Technical Reports, The University of Tokyo, 2008. (available at http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html)
[10] I. Maruta, T.-H. Kim and T. Sugie, "Synthesis of fixed-structure H_∞ controllers via constrained particle swarm optimization," in Proc. of the 17th IFAC World Congress, Seoul, Korea, July 6-11, 2008.