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Polyhedral functions, composite quadratic functions, and equivalent conditions for stability/stabilization*

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Abstract— **Relationship between polyhedral functions and composite quadratic functions is investigated in this paper. The two composite quadratic functions considered are the pointwise maximum of quadratics and the convex hull of quadratics. It is shown that these two composite quadratic functions are universal for robust, possibly constrained, stabilization problems. In particular, a linear differential inclusion is stable (stabilizable with/without constraints) iff it admits a Lyapunov (control Lyapunov) function in these classes.**

Relationships between the existing stability/stabilization conditions derived from these functions are also investigated. It is shown that a well known stability condition in terms of matrix equalities is equivalent to a stability condition in terms of bilinear matrix inequalities (BMIs). Similar conclusions are made about conditions for stabilization of linear differential/difference inclusions and constrained control systems. This investigation provides insight into the relationship between two alternative approaches to various analysis and design problems, making it possible to transform some synthesis problems derived from polyhedral functions into LMI-based optimization problems.

Keywords: Polyhedral functions, composite quadratic functions, stability, stabilization

I. Introduction

Polyhedral functions are well established Lyapunov functions. They have been successfully applied to robust control of uncertain systems and constrained control systems (see [3], [4] and the vast reference therein). Polyhedral functions are universal Lyapunov functions in several important applications since every convex homogeneous Lyapunov function can be approximated by polyhedral functions. When polyhedral functions are applied as Lyapunov functions, various analysis and design problems can be converted into algebraic problems. Perhaps the most important application of polyhedral functions is for stability analysis of linear differential inclusions (LDIs). Different authors have worked on this problem and obtained basically equivalent algebraic conditions which are necessary and sufficient for robust stability (e.g., [3], [6], [16], [17]). Another important application is for constrained control systems, where polyhedral functions are used to search for the maximal invariant set within the state constraint by admissible control (e.g., see [19], [3], [10]).

Another popular type of Lyapunov functions are the quadratic functions. They are more numerically tractable than polyhedral functions since they usually convert analysis and design problems into optimization problems with linear matrix inequality (LMI) constraints. However, the results obtained by quadratic functions can be conservative. In recent years, significant efforts have been devoted to the development of Lyapunov functions which are derived from one or a family of quadratic functions. In [7], [22], the homogeneous polynomial functions are quadratic functions of state augumented from the original state. In [15], [20], piecewise quadratic Lyapunov functions are defined according to given partitions of state-space. Due to the quadratic-like nature of these functions, the analysis problems are able to be converted into LMIs or BMIs.

Recently, a pair of conjugate Lyapunov functions were developed and used for various analysis and design purposes, e.g., for estimation and enlargement of the domain of attraction and for evaluation of the robust nonlinear L_2 gain, in [8], [9], [11], [12], [13], [14]. One of them is obtained by taking the pointwise maximum of a family of quadratic functions and the other is the convex hull of a family of quadratic functions. They are both convex and homogeneous of degree two. Since these two functions are composed from a family of quadratic functions, they are called composite quadratic functions. When these functions are used as Lyapunov functions, the synthesis problems are converted into optimization problems with BMI constraints and can be solved with LMI-based methods.

The purpose of this paper is to clarify the relationship between the polyhedral functions and the composite quadratic functions, and the relationship between some important stability/stabilization conditions derived from them. These relationships provide new insight into several important Lyapunov functions, as well as some matrix equalities and matrix inequalities.

Notation:

- Γ^K: $= \{\gamma \in \mathbb{R}^K : \Sigma_{k=1}^K \gamma_k = 1, \gamma_k \geq 0\}$;
- $\nabla V(x)$: gradient of V at x;
- $\dot{V}(x;\zeta)$: one-sided directional derivative at x along ζ;
- co$\{S\}$: convex hull of a set S.

II. The Lyapunov functions

For a positive semidefinite function $V : \mathbb{R}^n \to [0,\infty)$, denote its 1-level set as

$$L_V := \left\{x \in \mathbb{R}^n : \ V(x) \leq 1\right\}.$$

The one sided directional derivative of $V(x)$ is defined with respect to two variables: x and a vector ζ specifying the direction of increment or motion:

$$\dot{V}(x;\zeta) := \lim_{h\downarrow 0} \frac{V(x+\zeta h) - V(x)}{h},$$

where "$h \downarrow 0$" denotes approaching 0 from right-hand side. In this section, we discuss three types of Lyapunov

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functions, the polyhedral function, the max of quadratics, the convex hull of quadratics and their relationship. Methods for computing their directional derivatives are also provided.

A. The polyhedral function

Given a family of vectors $c_j \in \mathbb{R}^n, j = 1, \cdots, J$, a polyhedral function can be defined as

$$V_p(x) := \max\{x^T c_j c_j^T x : j = 1, \cdots, J\}. \quad (1)$$

The 1-level set of V_p is the polytope

$$L_{V_p} = \{x \in \mathbb{R}^n : x^T c_j c_j^T x \le 1, j = 1, \cdots, J\}. \quad (2)$$

For easy comparison with other functions, we consider a piecewise quadratic function $V_p(x)$ instead of the piecewise linear function $\max\{|c_j^T x| : j = 1, \cdots, J\}$, which has the same 1-level set as V_p. For V_p to be positive definite, J has to be no less than n and the vectors c_j's have to span the space $\mathbb{R}^n$. In this case, L_{V_p} is a compact convex set.

A polyhedral function can also be equivalently defined in terms of the vertices of a symmetric polytope. Let the vertices of the polytope L_{V_p} be $\{\pm d_k : k = 1, \cdots, K\}$. For each $x \in L_{V_p}$, there exists a $\theta \in \mathbb{R}^K$ such that $\Sigma_{k=1}^K |\theta_k| \le 1$ and $x = \Sigma_{k=1}^K \theta_k d_k$. Since the vertices appear as symmetric pairs $\pm d_k$, θ_k's can be positive or negative. If x is on the boundary of L_{V_p}, then the minimal $\Sigma_{k=1}^K |\theta_k|$ has to be 1. Since $V_p(x)$ is homogeneous of degree two, i.e., $V_p(\alpha x) = \alpha^2 V_p(x)$, we have,

$$V_p(x) = \min\left\{ \left(\Sigma_{k=1}^K |\theta_k|\right)^2 : x = \Sigma_{k=1}^K \theta_k d_k \right\}. \quad (3)$$

With this description,

$$\begin{aligned} L_{V_p} &= \text{co}\{\pm d_k : k = 1, \cdots, K\} \\ &= \{\Sigma_{k=1}^K \theta_k d_k : \ \Sigma_{k=1}^K |\theta_k| \le 1\} \end{aligned} \quad (4)$$

B. The composite quadratic functions

Let K be a positive integer. Define

$$\Gamma^K := \left\{ \gamma \in \mathbb{R}^K : \ \gamma_1 + \gamma_2 + \cdots + \gamma_K = 1, \gamma_k \ge 0 \right\}.$$

Given K positive definite matrices $P_k = P_k^{\mathrm{T}} > 0, k = 1, \cdots, K$, the max of quadratics is defined as

$$V_{\max}(x) := \max\{x^{\mathrm{T}} P_k x : k = 1, \cdots, K\}, \quad (5)$$

and the convex hull of quadratics is defined as

$$V_c(x) := \min_{\gamma \in \Gamma^K} x^{\mathrm{T}} \left(\Sigma_{k=1}^K \gamma_k P_k^{-1}\right)^{-1} x. \quad (6)$$

Sometimes it may be more convenient to call $V_{\max}$ the max function and V_c the convex hull function, as in [9].

In [9], [13], it was established that $V_{\max}$ is strictly convex, and V_c is convex and continuously differentiable with the gradient given by $\nabla V_c(x) = 2\left(\Sigma_{k=1}^K \gamma_k^*(x) P_k^{-1}\right)^{-1} x$, where

$$\gamma^*(x) = \arg\min_{\gamma \in \Gamma^K} x^{\mathrm{T}} \left(\Sigma_{k=1}^K \gamma_k P_k^{-1}\right)^{-1} x. \quad (7)$$

It follows that

$$\dot{V}_c(x;\xi) = (\nabla V_c(x))^{\mathrm{T}} \xi = 2x^{\mathrm{T}} \left(\Sigma_{k=1}^K \gamma_k^*(x) P_k^{-1}\right)^{-1} \xi. \quad (8)$$

The 1-level set of the quadratic function $V(x) = x^{\mathrm{T}} P x$ is an ellipsoid. For convenience, denote

$$\mathcal{E}(P) := \{x \in \mathbb{R}^n : x^{\mathrm{T}} P x \le 1\}.$$

It is easy to see that $L_{V_{\max}}$ is the intersection of the ellipsoids $\mathcal{E}(P_k), k = 1, \cdots, K$. In [13], it was shown that L_{V_c} is the convex hull of the union $\cup_{k=1}^K \mathcal{E}(P_k)$.

C. Relationship between polyhedral functions and composite quadratic functions

It is known that every convex function which is homogeneous of degree two can be arbitrarily closely approximated by polyhedral functions. In what follows, we show that a positive definite polyhedral function can be arbitrarily closely approximated by the max of quadratics, or by the convex hull of quadratics. The approximation by the max of quadratics is easy to verify. The approximation by the convex hull of quadratics is more useful since the convex hull function is continuously differentiable. With the approximation by a convex hull function, the sharp vertices or edges of a polytope are softened with portions of the surface of ellipsoids.

Proposition 1: Given a set of vectors $d_k \in \mathbb{R}^n, k = 1, \cdots, K$, that span $\mathbb{R}^n$, and let V_p be defined as in (3). Define

$$V_{c,\delta}(x) = \min_{\gamma \in \Gamma^K} x^{\mathrm{T}} \left(\Sigma_{k=1}^K \gamma_k (d_k d_k^T + \delta I)\right)^{-1} x$$

Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\frac{1}{1+\varepsilon} V_p(x) \le V_{c,\delta}(x) \le V_p(x) \quad \forall x \in \mathbb{R}^n. \quad (9)$$

D. Directional derivative of the max function

Consider

$$V_{\max} = \max\{x^T P_k x : k = 1, \cdots, K\},$$

with $P_k = P_k^{\mathrm{T}} \ge 0$ for all k. For $x \in \mathbb{R}^n$, define

$$I_{\max}(x) := \{k : x^T P_k x = V_{\max}(x)\}.$$

Lemma 1: For a vector $\xi \in \mathbb{R}^n$, the directional derivative of $V_{\max}$ at x along ξ is

$$\dot{V}_{\max}(x;\xi) = \max\{2x^T P_k \xi : k \in I_{\max}(x)\}. \quad (10)$$

E. Directional derivative of polyhedral functions

The formula (10) can also be used to compute directional derivative of a polyhedral function $V_p(x)$ by letting $P_k = c_k c_k^T$. In some important applications, such as stability/stabilization of LDI, and stabilization of systems with input and state constraints, the directional derivative at the vertices play a key role.

Let $V_p(x)$ be a polyhedral function with two equivalent descriptions:

$$V_p(x) = \max\{x^{\mathrm{T}} c_j c_j^{\mathrm{T}} x : j = 1, \cdots, J\} \quad (11)$$

and

$$V_p(x) = \min\left\{ \left(\Sigma_{k=1}^K |\theta_k|\right)^2 : x = \Sigma_{k=1}^K \theta_k d_k \right\}. \quad (12)$$

Define

$$\theta^*(x) := \arg\min\left\{ \left(\Sigma_{k=1}^K |\theta_k|\right)^2 : x = \Sigma_{k=1}^K \theta_k d_k \right\}. \quad (13)$$

Lemma 2: Let $V_p(x)$ be given by (11) and (12). Given vectors $\xi_k \in \mathbb{R}^n, k = 1, \cdots, K$. Let

$$\alpha = \max\{\dot{V}_p(d_k;\xi_k) : \ k = 1, \cdots, K\}.$$

Then

$$\dot{V}_p(x;\Sigma_{k=1}^K\theta_k^*(x)\xi_k) \le \alpha V_p(x) \ \ \forall\, x\in\mathbb{R}^n \quad (14)$$

Applying Lemma 2 to the case where $\xi_k = Ad_k$ or $\xi_k = Ad_k + Bu_k$, we have

Corollary 1: Let $V_p(x)$ be given by (12). For $A \in \mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times m}$, let

$$\alpha_1 = \max\{\dot{V}_p(d_k;Ad_k) : k=1,\cdots,K\},$$
$$\alpha_2 = \max\{\dot{V}_p(d_k;Ad_k+Bu_k) : k=1,\cdots,K\}.$$

Then

$$\dot{V}_p(x;Ax) \le \alpha_1 V_p(x), \quad \forall\, x\in\mathbb{R}^n,$$
$$\dot{V}_p(x;Ax+B\Sigma_{k=1}^K\theta_k^*(x)u_k) \le \alpha_2 V_p(x), \quad \forall\, x\in\mathbb{R}^n.$$

The above corollary will be used for deriving algebraic conditions for stability and stabilization of LDI by focusing on the vertices. For stabilization, a feedback law $u(x) = \Sigma_{k=1}^K\theta_k^*(x)u_k$ is constructed from the control at the vertices and the representation of x with respect to the vertices.

Although the two forms of $V_p(x)$ in (11) and (12) are equivalent, the relationship between c_j's and d_k's can be complicated. When $V_p(x)$ is given by (12) in terms of the vertices d_k, it would be desirable to compute the directional derivative at the vertices without converting to the other form. The following proposition makes it convenient to characterize the directional derivative at the vertices and will be readily applied for deriving algebraic conditions.

Proposition 2: Let $V_p(x)$ be given by (12). Suppose that each d_k is a vertex, i.e, it cannot be expressed as the convex combination of $\pm d_j, j\neq k$. Then for a vector $\xi\in\mathbb{R}^n$,

$$\dot{V}_p(d_k;\xi) = \min\left\{2\lambda_k + 2\Sigma_{j\neq k}|\lambda_j| : \xi = \Sigma_{j=1}^K\lambda_j d_j\right\} \quad (15)$$

III. Stability of linear differential inclusions

Consider a linear differential inclusion (LDI)

$$\dot{x}\in \text{co}\{A_i x : i=1,\cdots,N\}. \quad (16)$$

The LDI is said to be exponentially stable with a convergence rate $\eta > 0$ if there exists a constant α such that for every possible solution $x(\cdot)$, we have $|x(t)| < \alpha e^{-\eta t}|x(0)|$ for all $t>0$, where $|\cdot|$ denotes any kind of norm.

It is established in the literature that asymptotic stability of LDI implies exponential stability. Furthermore, the LDI is exponentially stable with convergence rate η, if and only if there exists a convex Lyapunov function $V(x)$, homogeneous of degree two, such that

$$\dot{V}(x;A_i x) < -2\eta V(x) \qquad \forall\, x\in\mathbb{R}^n, i=1,\cdots,N. \quad (17)$$

It is recently established in [9] that the stability of (16) is equivalent to the stability of the dual system

$$\dot{y}\in \text{co}\{A_i^T y : i=1,\cdots,N\}, \quad (18)$$

with the same convergence rate. In particular, (17) is satisfied if and only if

$$\dot{V}^*(y;A_i^T y) < -2\eta V^*(y) \qquad \forall\, y\in\mathbb{R}^n, i=1,\cdots,N. \quad (19)$$

where V^* is the conjugate function of V.

A. Stability condition derived from polyhedral functions

In the literature, necessary and sufficient conditions for the stability of the LDI (16) have been obtained by different authors (e.g., [3], [16], [17]). In [16], the condition was obtained by using Fakars' Lemma and in [3], the condition was obtained by using the discrete-time Euler approximating system. These conditions are similar and basically equivalent. A closely related result was derived in [19] for achieving asymptotic stability on polytopes under state and input constraints. The algebraic condition in [19] can be readily applied to obtain stability condition for LDIs by removing the state and input constraints.

Theorem 1: [3], [16], [17] The LDI (16) is exponentially stable with a convergence rate η if and only if there exist an integer $K\ge n$, an $n\times K$ matrix D of rank n and $K\times K$ matrices

$$\Lambda_i = (\lambda_{ijk})_{j,k=1}^K, \ \ i=1,\cdots,N,$$

satisfying

$$\lambda_{ikk} + \Sigma_{j\neq k}|\lambda_{ijk}| < -\eta, \ \ \forall\, i,k, \quad (20)$$

and the matrix equalities

$$A_i D = D\Lambda_i \qquad i=1,\cdots,N. \quad (21)$$

(Here λ_{ijk} is the element of Λ_i at the j-th row, k-th column.) By duality, A_i in (21) can be replaced with A_i^{T}.

B. Stability condition derived from composite quadratic functions

Using the max function, a sufficient condition for the stability of LDI can be easily established using the S procedure (see [5], [9]). The condition takes the form of a family of BMIs. A set of dual matrix inequalities are obtained in [9] for stability and various performance via the conjugate relationship between max functions and convex hull functions. In what follows, we summarize the main results from [9] on stability.

Theorem 2: Let $V_{\max}$ and V_c be the max function and the convex hull function constructed from $P_k = P_k^T > 0, k = 1,\cdots,K$ by (5) and (6). Denote $Q_k = P_k^{-1}$.

1. If there exist $\gamma_{ijk}\ge 0, i=1,\cdots,N, j,k=1,\cdots,K$ and $\eta>0$ satisfying
$$A_i^{\mathrm{T}}P_k + P_kA_i < \Sigma_{j\neq k}\gamma_{ijk}(P_j-P_k) - 2\eta P_k, \ \ \forall\, i,k, \quad (22)$$
then $\dot{V}_{\max}(x;A_i x) \le -2\eta V_{\max}(x)$ for all x, i.
2. If there exist $\gamma_{ijk}\ge 0, i=1,\cdots,N, j,k=1,\cdots,K$ and $\eta>0$ satisfying
$$A_iQ_k + Q_kA_i^{\mathrm{T}} < \Sigma_{j\neq k}\gamma_{ijk}(Q_j-Q_k) - 2\eta Q_k, \ \ \forall\, i,k, \quad (23)$$
then $\dot{V}_c(x;A_i x) \le -2\eta V_c(x)$ for all x, i.

Based on the above theorem, LMI-based algorithms have been derived for evaluation of the convergence rate. Although there is no guarantee that the global optimal solution can be obtained, extensive numerical examples have shown the effectiveness of the algorithm. Technically speaking, increasing the number K would lead to less conservative results.

However, it may not be necessary to pick a very large K. Examples show that $K = 2, 3$ would result in significant improvement as compared to quadratic functions ($K = 1$).

In many examples, it is observed that, with a fixed K, the maximal convergence rate obtained from the max function and that from the convex hull function are different. This difference in evaluation of convergence rate actually shows that the pair of conjugate functions complement each other, since we can always pick the better result.

Since the matrix inequalities in (22) and (23) are obtained via S procedure and duality, it is possible that they would give conservative conditions for stability, even with respect to the specific Lyapunov functions. However, with the help of the necessary and sufficient condition established by the polyhedral functions, we will show in the next section that the conditions (22) and (23) are not conservative, if K is allowed to be any integer.

C. Equivalent stability conditions

Let $K \geq n$ be a given positive integer. Consider V_p constructed from $2K$ vertices $\pm d_k, k = 1, \cdots, K$, and $V_{\max}, V_c$ constructed from K matrices $P_k = P_k^T > 0$. We are interested in the relationship between the matrix equalities (21) and the matrix inequalities (22) and (23).

Lemma 3: Given matrices $A_i, i = 1, \cdots, N$, and a number $K \geq n$. Suppose that there exist a positive number $\eta > 0$, an $n \times K$ matrix D of rank n and $K \times K$ matrices Λ_i, satisfying

$$\lambda_{ikk} + \Sigma_{j \neq k} |\lambda_{ijk}| < -\eta, \ \ \forall\, i, k, \tag{24}$$

$$A_i D = D \Lambda_i \ \ \forall\, i. \tag{25}$$

Let d_k be the kth column of D. Then

$$A_i d_k d_k^T + d_k d_k^T A_i^T \leq -2\eta d_k d_k^T + \Sigma_{j \neq k} |\lambda_{ijk}| (d_j d_j^T - d_k d_k^T) \quad \forall\, i, k. \tag{26}$$

Furthermore, there exist matrices $Q_k = Q_k^{\mathrm{T}} > 0$, and numbers $\gamma_{ijk} \geq 0, j, k = 1, \cdots, K, i = 1, \cdots, N$, satisfying

$$A_i Q_k + Q_k A_i^T < -2\eta Q_k + \Sigma_{j \neq k} \gamma_{ijk} (Q_j - Q_k), \ \forall\, i, k. \tag{27}$$

Recall from Proposition 1 that a polyhedral function can be arbitrarily closely approximated by a convex hull function $V_c(x)$ constructed from $P_k = (d_k d_k^{\mathrm{T}} + \varepsilon I)^{-1}$, we may expect that same convergence rate should be ensured with a $V_c(x)$ close enough to $V_p(x)$. However, Lemma 3 does not come as a direct consequence of the approximation in Proposition 1 and Theorem 2. This is because by Theorem 2, the matrix inequalities only give a sufficient condition (as a result of the S procedure) for stability with certain convergence rate. On the contrary, Lemma 3 indicates that the matrix inequality condition derived from the S procedure is not conservative as V_c approaches a polyhedral function.

Combining Theorems 1, 2 and Lemma 3, we have the following result.

Theorem 3: The following statements are equivalent:

1. The LDI is exponentially stable with convergence rate η.
2. There exist an integer $K \geq n$, an $n \times K$ matrix D of rank n and $K \times K$ matrices Λ_i, satisfying
$$\lambda_{ikk} + \Sigma_{j \neq k} |\lambda_{ijk}| < -\eta, \ \ \forall\, i, k, \tag{28}$$
$$A_i D = D \Lambda_i \ \ \forall\, i. \tag{29}$$
3. There exist an interger K, an $n \times K$ matrix D of rank n and numbers $\gamma_{ijk} \geq 0, j, k = 1, \cdots, K, i = 1, \cdots, N$, satisfying
$$A_i d_k d_k^T + d_k d_k^T A_i^T \leq -2\eta d_k d_k^T + \Sigma_{j \neq k} \gamma_{ijk} (d_j d_j^T - d_k d_k^T), \ \forall\, i, k. \tag{30}$$
4. There exist an integer K, $n \times n$ matrices $Q_k = Q_k^T > 0, k = 1, \cdots, K$, and numbers $\gamma_{ijk} \geq 0, j, k = 1, \cdots, K, i = 1, \cdots, N$, satisfying
$$A_i Q_k + Q_k A_i^T < -2\eta Q_k + \Sigma_{j \neq k} \gamma_{ijk} (Q_j - Q_k), \ \ \forall\, i, k. \tag{31}$$

Furthermore, for a fixed number $K \geq n$, items 2, 3 are equivalent and they both imply item 4. By duality, A_i can be replaced with A_i^T in items 2,3,4.

Consider the following three optimization problems for evaluation of the convergence rate.

$$\eta_1^* = \sup_{D, \Lambda_i} \eta, \quad s.t. \ (28), (29), \ D \text{ has full row rank} \tag{32}$$

$$\eta_2^* = \sup_{d_k, \gamma_{ijk} \geq 0} \eta, \quad s.t. \ (30), [d_1 \ \cdots \ d_K] \text{ has full row rank} \tag{33}$$

$$\eta_3^* = \sup_{Q_k > 0, \gamma_{ijk} \geq 0} \eta, \quad s.t. \ (31) \tag{34}$$

Problems (32) and (33) determine the optimal convergence rate using polyhedral functions and (34) determines that using the convex hull of quadratics. Problem (33) appears to be semidefinite programming but the rank 1 constraint on $d_k d_k^T$ is hard to deal with. Based on Theorem 3, we can conclude that $\eta_1^* = \eta_2^* \leq \eta_3^*$ for a fixed K since items 2 and 3 implies item 4. This means that, even though there always exists a polyhedral function to verify the stability of an LDI, the number K needed may be larger than that needed for a max function or a convex hull function. For example, for an LDI $\dot{x} = \mathrm{co}\{A_1 x, A_2 x\}$ with

$$A_1 = \begin{bmatrix} -10 & -0.2 & -0.2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -10 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

it was found in [21] that 2×5101 vertices ($K = 5101$) are required for the polyhedral function to verify the stability, while by solving (34) with $K = 5$, stability is verified with convergence rate $\eta_3^* = 0.0018$.

IV. Robust Stabilization of LDI

Consider an open loop linear differential inclusion

$$\dot{x} \in \mathrm{co}\{A_i x + B_i u : \ i = 1, \cdots, N\}, \ x \in \mathbb{R}^n, u \in \mathbb{R}^m. \tag{35}$$

If a linear state feedback $u = Fx$ is applied, the closed-loop system is also an LDI:

$$\dot{x} \in \text{co}\{(A_i + B_i F)x : \; i = 1, \cdots, N\}, \quad x \in \mathbb{R}^n. \quad (36)$$

The stabilization condition can be readily obtained by replacing A_i in (29), (30) and (31) with $A_i + B_i F$, where F is a design parameter. In the case of linear output feedback $u = Fy$ where $y = Cx$, we need to replace A_i with $A_i + B_i FC$.

Here we would like to consider nonlinear state feedback design which offers more degree of freedom and enhanced performance. A necessary and sufficient condition for robust stabilization by nonlinear state feedback was provided in [1] via polyhedral Lyapunov functions. A sufficient condition was provided in [11] via convex hull functions. The following theorem establishes a connection between these conditions.

Theorem 4: The following statements are equivalent:

1. The LDI (35) is exponentially stablizable with convergence rate η by nonlinear state feedback.
2. There exist an integer $K \geq n$, an $n \times K$ matrix D of rank n, an $m \times K$ matrix U, and $K \times K$ matrices $\Lambda_i, i = 1, \cdots, N$, satisfying
$$\lambda_{ikk} + \Sigma_{j \neq k} |\lambda_{ijk}| < -\eta, \;\; \forall \, i, k, \quad (37)$$
$$A_i D + B_i U = D\Lambda_i \;\; \forall \, i. \quad (38)$$
3. There exist an integer K, $n \times n$ matrices $Q_k = Q_k^T > 0$, $m \times n$ matrices $Y_k, k = 1, \cdots, K$, and numbers $\gamma_{ijk} \geq 0, j, k = 1, \cdots, K, i = 1, \cdots, N$, satisfying
$$A_i Q_k + B_i Y_k + Q_k A_i^T + Y_k^T B_i^T$$
$$< -2\eta Q_k + \Sigma_{j \neq k} \gamma_{ijk}(Q_j - Q_k), \;\; \forall \, i, k. \quad (39)$$

For a given number $K \geq n$, item 2 implies item 3.

With the matrices U, D satisfying (37), (38), a nonlinear feedback law $u(x)$ can be constructed from the columns of U, $u_k, k = 1, \cdots, K$, and the columns of D, $d_k, k = 1, \cdots, K$, as follows:

$$u(x) = \Sigma_{k=1}^K \theta_k^*(x) u_k, \quad (40)$$

where $\theta^*(x)$ is an optimal representation of x with respect to d_i's, as defined in (13). Consider $V_p(x)$ given in terms of the vertices $\pm d_k, k = 1, \cdots, K$:

$$V_p(x) = \min \left\{ \left(\Sigma_{k=1}^K |\theta_k| \right)^2 : x = \Sigma_{k=1}^K \theta_k d_k \right\}.$$

By Proposition 2, conditions (37), (38) ensure that

$$\dot{V}_p(d_k; A_i d_k + B_i u_k) \leq -2\eta \quad \forall \, i, k. \quad (41)$$

By Corollary 1 and (40), we have

$$\dot{V}_p(x; A_i x + B_i u(x)) \leq -2\eta V_p(x) \quad \forall \, x, i. \quad (42)$$

The construction of feedback law under condition of item 3 is outlined below (see [11]). For $x \in \mathbb{R}^n$, let

$$\gamma^*(x) = \arg \min_{\gamma \in \Gamma^K} x^{\rm T} \left(\Sigma_{k=1}^K \gamma_k Q_k \right)^{-1} x. \quad (43)$$

Define

$$Y(\gamma^*) = \sum_{k=1}^K \gamma_k^* Y_k, \quad Q(\gamma^*) = \sum_{k=1}^K \gamma_k^* Q_k, \quad (44)$$

$$F(\gamma^*) = Y(\gamma^*) Q(\gamma^*)^{-1}. \quad (45)$$

Then the feedback law is $u(x) = F(\gamma^*(x))x$.

V. Constrained stabilization

Let $G \in \mathbb{R}^{p \times n}$. Denote

$$\mathcal{L}(G) = \{x : |Gx|_\infty \leq 1\} = \{x : |g_i x| \leq 1, i = 1, \cdots, p\},$$

where g_i is the ith row of G.

Consider an open-loop linear system,

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \quad (46)$$

where the state and input have to satisfy the following constraints

$$|u(t)|_\infty \leq 1, \;\; x(t) \in \mathcal{L}(G) \;\; \forall \, t > 0. \quad (47)$$

One basic problem is to find a set X_0 inside $\mathcal{L}(G)$ so that for each initial state $x_0 \in X_0$, there exist a control satisfying the bound $|u(t)|_\infty \leq 1$ to keep $x(t) \in X_0$ for all t. Such a set X_0 is called a controlled invariant set. If in addition $\lim_{t \to \infty} x(t) = 0$ with a certain convergence rate η, for all $x_0 \in X_0$, we say that X_0 is controlled invariant with convergence rate η. In [10], attempt was made to find the maximal controlled invariant set inside $\mathcal{L}(G)$ for discrete-time systems. It was shown that the maximal controlled invariant set is convex. For continuous-time systems, the maximal controlled invariant set is also convex and thus can be arbitrarily approximated by polytopes and convex hull of ellipsoids. In what follows, we give conditions for controlled invariance of polytopes and of convex hull of ellipsoids. The relationship between the conditions will also be examined.

Theorem 5: The polytope $L_{V_p} = \text{co}\{\pm d_k : k = 1, \cdots, K\}$, with $K \geq n$, is controlled invariant with convergence rate η under the constraints (47) if and only if there exist an $n \times K$ matrix D of rank n, an $m \times K$ matrix U, and a $K \times K$ matrix $\Lambda = \{\lambda_{jk}\}_{j,k=1}^K$, satisfying

$$\lambda_{kk} + \Sigma_{j \neq k} |\lambda_{jk}| < -\eta, \;\; \forall \, k, \quad (48)$$
$$AD + BU = D\Lambda, \quad (49)$$
$$|u_{ij}| \leq 1 \quad \forall \, i, j, \quad (50)$$
$$|g_i d_k| \leq 1, \quad \forall \, i, k, \quad (51)$$

where d_k is the kth column of D.

Similar constrained control problem was addressed in [19], where linear control law $u = Fx$ was considered. In this case, $U = FD$ depends on D and is much more restricted than that in Theorem 5.

To derive conditions for controlled invariance of the convex hull of ellipsoids, recall that, for a matrix $G \in \mathbb{R}^{p \times n}$ and a matrix $P = P^{\rm T} > 0$,

$$\mathcal{E}(P) \subset \mathcal{L}(G) \Leftrightarrow \begin{bmatrix} 1 & g_i \\ g_i^{\rm T} & P \end{bmatrix} \geq 0, i = 1, \cdots, p. \quad (52)$$

Theorem 6: Let V_c be constructed from $P_k = P_k^{\rm T} > 0$ and let $Q_k = P_k^{-1}$. Suppose there exist $m \times n$ matrices $Y_k, k = 1, \cdots, K$, and numbers $\gamma_{jk} \geq 0, j, k = 1, \cdots, K$, satisfying

$$AQ_k + BY_k + Q_k A^T + Y_k^T B^T$$
$$< -2\eta Q_k + \Sigma_{j \neq k} \gamma_{jk}(Q_j - Q_k), \;\; \forall \, k \quad (53)$$

$$g_i Q_k g_i^{\mathrm{T}} \leq 1 \;\; \forall i, k \tag{54}$$

$$\begin{bmatrix} 1 & Y_{k\ell} \\ Y_{k\ell}^{\mathrm{T}} & Q_k \end{bmatrix} \geq 0 \;\; \forall k, \ell, \tag{55}$$

where $Y_{k\ell}$ is the ℓ-th row of Y_k, then $L_{V_c} \subset \mathcal{L}(G)$ is invariant with convergence rate η under the feedback law $u(x) = F(\gamma^*(x))x$, where $F(\gamma^*(x))$ is given by (45). Furthermore, $|u(x)|_\infty \leq 1$ for all $x \in L_{V_c}$.

Recall from Theorem 4 that (53) ensures invariance of L_{V_c} in the absence of input and state constraints. Now the constraint (54) ensures that $\mathcal{E}(P_k) \subset \mathcal{L}(G)$ for each k, thus $L_{V_c} = \mathrm{co}\{\cup_{k=1}^K \mathcal{E}(P_k)\} \subset \mathcal{L}(G)$ since $\mathcal{L}(G)$ is convex. Meanwhile, the constraint (55) ensures that $|u(x)|_\infty \leq 1$ for all $x \in L_{V_c}$. This can be seen as follows. Multiplying (55) on the right and left with $\mathrm{diag}\{1, Q_k^{-1}\}$ and let $F_k = Y_k Q_k^{-1}$, we have $\begin{bmatrix} 1 & F_{k\ell} \\ F_{k\ell}^{\mathrm{T}} & Q_k^{-1} \end{bmatrix} \geq 0$ for all k, ℓ. By (52), this is equivalent to $\mathcal{E}(Q_k^{-1}) \subset \mathcal{L}(F_k)$, and to

$$|F_k x|_\infty \leq 1 \quad \forall x \in \mathcal{E}(Q_k^{-1}). \tag{56}$$

Recalling from the proof of Theorem 1 in [11] that, if $x \in L_{V_c}$, then $u(x) = F(\gamma^*(x))x = \sum_{k=1}^K \gamma_k^* F_k x_k$ for certain $x_k \in \mathcal{E}(Q_k^{-1})$. Since $|F_k x_k|_\infty \leq 1$ for all $x_k \in \mathcal{E}(Q_k^{-1})$, we have $|u(x)|_\infty \leq 1$.

Since the condition in Theorem 6 is only sufficient for the controlled invariance of L_{V_c}, it is not certain that the maximal controlled invariant set can be determined via the condition. The following result shows that, each controlled invariant polytope can be arbitrarily approximated by the convex hull of ellipsoids satisfying conditions in Theorem 6.

Theorem 7: Given an integer $K \geq n$ and an $n \times K$ matrix D of rank n. Suppose there exist a $m \times K$ matrix U, and a $K \times K$ matrix Λ, satisfying (48) through (51). Let $M = \max\{g_i g_i^{\mathrm{T}} : \; i = 1, \cdots, p\}$. For $\varepsilon > 0$, let $Q_k = (d_k d_k^{\mathrm{T}} + \varepsilon I)/(1 + \varepsilon M)$, $Y_k = u_k d_k^{\mathrm{T}}/(1 + \varepsilon M)$. Then there exists a sufficiently small ε such that Q_k and Y_k satisfy (53) to (55).

VI. Equivalent stability conditions for linear difference inclusions

Consider a linear difference inclusion

$$x(k+1) \in \mathrm{co}\{A_i x(k) : i = 1, \cdots, N\}. \tag{57}$$

We have the following result.

Theorem 8: The following statements are equivalent:

1. The LDI (57) is asymptotically stable.
2. There exist a number $K \geq n$, an $n \times K$ matrix D of rank n and $K \times K$ matrices Λ_i satisfying
$$\Sigma_{j=1}^K |\lambda_{ijk}| < 1 \;\; \forall i, k, \tag{58}$$
$$A_i^T D = D \Lambda_i \quad \forall i. \tag{59}$$
3. There exist an integer $K > 0$, matrices $P_k = P_k^T > 0$ and numbers $\gamma_{ijk} \geq 0, i = 1, \cdots, N; j, k = 1, \cdots, K$ satisfying $\Sigma_{j=1}^K \gamma_{ijk} < 1$ for all i, k and
$$A_i^T P_k A_i < \Sigma_{j=1}^K \gamma_{ijk} P_j \quad \forall i, k. \tag{60}$$

When K is fixed, item 2 implies item 3.

VII. Conclusions

This paper clarifies the relationship between the polyhedral functions and the composite quadratic functions, and the relationship between some important stability conditions derived from them. The results provide new insight into three types of important Lyapunov functions, as well as the relationship between some matrix equalities and matrix inequalities.

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