Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Adaptive Feedforward Disturbance Rejection in Nonlinear Systems

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Abstract— **This paper investigates the problem of adaptive feedforward compensation for a class of nonlinear systems, namely that of input-to-state (and locally exponentially) convergent systems. It is shown how, under suitable assumptions, the proposed scheme succeeds in achieving disturbance rejection of a harmonic disturbance at the input of a convergent nonlinear system, with a semi-global domain of convergence. The effectiveness of the proposed solution is demonstrated by combining results from averaging analysis with techniques for semi-global stabilization. An illustrative example shows the effectiveness of the scheme.**

I. Introduction

The problem of rejecting unwanted periodic disturbances occurring in dynamical systems is a fundamental problem in control theory, with countless technological applications in control of vibrating structures [1], active noise control [2] and control of rotating mechanisms [3]. From a theoretical standpoint, any design philosophy aimed at solving the problem of periodic disturbance rejection reposes upon a specific variant of the *internal model principle*, which states that regulation can be achieved only if the controller embeds a copy of the exogenous system generating the periodic disturbance. In the classic *internal model control* (IMC), the plant is augmented with a replica of the exosystem, and the design is completed by a unit which provides stability of the closed loop (see [4] and references therein). An alternative design methodology to the one described above is provided by the so-called *adaptive feedforward compensation* (AFC), where a feedforward action is provided to offset the steady-state error induced by the exogenous disturbance in an already stable loop. The parameters of the feedforward control are computed adaptively by means of pseudo-gradient optimization, using the regulated error as a regressor [5]. In a similar methodology, referred to as *external model-based control* (EMC) [6], a stabilizing controller for the plant is computed first, and then an observer of the exosystem is designed to provide asymptotic cancellation of the disturbance.

These three design philosophies differ in the role of the stabilizing controller, which in the classic approach is embedded with the internal model itself, whereas in both AFC and EMC the stabilizer is *given* and the unit that provides disturbance rejection is placed outside the loop in an "add-on" fashion to the nominal compensator. The design of the unit that provides cancellation of the disturbance must then be carried out in such a way that stability of the loop is not affected by the adaptation mechanism or the observation error. For nonlinear systems, the possibility of "decoupling" the design of the stabilizer from that of the internal model unit would be an important methodological advancement in the theory of output regulation, as this would open the possibility of using "off-the-shelf" the wealth of techniques made available by the latest advancement in nonlinear stabilization.

In this paper, we present preliminary results aiming at setting the stage for a theory of adaptive feedforward compensation for nonlinear systems. The paper follows the seminal work of Bodson and co-workers (see, for instance, [5]), in that it provides a nonlinear equivalent of the condition for the solvability of the problem in the linear setting, and uses methods from averaging analysis to prove stability of the interconnection. In particular, we show how, under suitable assumptions, the adaptive feedforward scheme of [5] can be reinterpreted in the nonlinear setting, and applied to achieve disturbance rejection of harmonic disturbance at the input of a stable nonlinear systems, with a semi-global domain of convergence.

The paper is organized as follows: Section II gives the formulation of the problem and the standing assumptions used in the paper. In Section III, the properties of the steady-state solution of the forced uncompensated system are analyzed. The design of the controller and the proof of stability are given in Section IV and V, respectively. Finally, in Section VI an illustrative example is discussed, followed by some conclusions offered in Section VII.

II. Standing Assumptions and Problem Setup

Consider a smooth nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x, u+d)\,, \quad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$, input-matched harmonic disturbance $d \in \mathbb{R}$, and measured output $y \in \mathbb{R}$. The disturbance is generated by the following 2-dimensional autonomous LTI system

$$\begin{aligned} \dot{w} &= Sw\,, \quad w(0) = w_0 \\ d &= \Gamma w \end{aligned} \tag{2}$$

where $S = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $\omega_0 = 2\pi/T$ is a known parameter. We assume that the interconnection of system (2) and system (1) when $u = 0$ admits a unique well-defined steady-state response in the form of a continuously-differentiable mapping $x_{ss} : \mathbb{R}^2 \to \mathbb{R}^n$ satisfying $\frac{\partial x_{ss}}{\partial w} Sw =$

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$f(x_{\rm ss}(w), \Gamma w)$. The scenario considered herein is verified, in particular, if (1) is input-to-state convergent (ISC) as defined in [7, pag. 17]. As a matter of fact, this properties implies that system (1) possess a unique T-periodic steady-state trajectory $x_{\rm ss}(t)$ whenever forced by a harmonic disturbance with the same period. Henceforth, it will be assumed that system (1) is ISC and locally exponentially convergent (LEC) (see [7, p.28]). To make these statements precise, define $\tilde{x} = x - x_{\rm ss}(w)$ and note that the dynamics of the transient behavior of system (1) (when $u = 0$) are described by the periodic system

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, w_0)\,, \quad \tilde{x}(0) = x_0 - x_{\rm ss}(w_0)$$

where $\tilde{f}(t, \tilde{x}, w_0) = f(\tilde{x} + x_{\rm ss}(w(t)), \Gamma w(t)) - f(x_{\rm ss}(w(t)), \Gamma w(t))$, and $w(t) = \exp(St)w_0$ denotes the solution of (2). In addition, we require that the convergence to the steady state is uniform in the following sense:

Assumption 2.1: There exists a smooth T-periodic function $V : [0, T) \times \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}_+$ with the following properties: for any given compact set $\mathcal{K}_w \subset \mathbb{R}^2$ there exist class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and class-$\mathcal{K}$ functions $\alpha_3(\cdot)$, $\alpha_4(\cdot)$ such that

$$\begin{aligned} \alpha_1(\|\tilde{x}\|) \le V(t, \tilde{x}, w_0) \le \alpha_2(\|\tilde{x}\|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \tilde{x}} \tilde{f}(t, \tilde{x}, w_0) \le -\alpha_3(\|\tilde{x}\|) \\ \left\| \frac{\partial V}{\partial \tilde{x}} \right\| \le \alpha_4(\|\tilde{x}\|) \end{aligned} \tag{3}$$

for any $t \in [0, T)$, $\tilde{x} \in \mathbb{R}^n$ and $w_0 \in \mathcal{K}_w$ In addition, given $\mathcal{K}_w$, there exist positive constants r, a_i, $i = 1, \ldots, 4$ such that $\forall s \in [0, r]$

$$\begin{aligned} a_1 s^2 \le \alpha_1(s)\,, \quad \alpha_2(s) \le a_2 s^2 \\ a_3 s^2 \le \alpha_3(s)\,, \quad \alpha_4(s) \le a_4 s. \end{aligned} \tag{4}$$

For the purpose of this paper, it is further assumed that the steady-state $x_{\rm ss}(w)$ is a polynomial in the components of w of finite order $m \in \mathbb{N}$, that is, $x_{\rm ss}(w) = \bar{a}_{1,0} w_1 + \bar{a}_{0,1} w_2 + \sum_{k=0}^{2} \bar{a}_{k,\,2-k} w_1^k w_2^{2-k} +, ..., + \sum_{k=0}^{m} \bar{a}_{k,\,m-k} w_1^k w_2^{m-k}$, where $\bar{a}_{i,j} \in \mathbb{R}^n$ depend on the period T. As a result, the steady-state output is a polynomial of order m as well, which reads as

$$y_{\rm ss}(w) = a_{1,0} w_1 + a_{0,1} w_2 + \sum_{k=0}^{2} a_{k,2-k} w_1^k w_2^{2-k} +, \ldots, + \sum_{k=0}^{m} a_{k,m-k} w_1^k w_2^{m-k} \tag{5}$$

where $a_{i,j} \in \mathbb{R}$.

The control problem considered in this paper consists in finding a control law that provides asymptotic cancellation of the disturbance d while maintaining boundedness of the internal trajectories of the closed-loop system. Letting the solutions of (2) be parameterized by the initial condition w_0, the problem is cast in the semi-global output stabilization framework as follows:

Semi-global Periodic Output Stabilization Problem: Given the parameterized family of T-periodic systems

$$\begin{aligned} \dot{x} &= f(x, \Gamma e^{St} w_0 + u) \\ y &= Cx \end{aligned} \tag{6}$$

find a parameterized family of T-periodic controllers

$$\begin{aligned} \dot{\eta} &= g_\kappa(t, \eta, y)\,, \quad \eta(0) = \eta_0 \\ v &= h_\kappa(t, \eta, y) \end{aligned} \tag{7}$$

with $\eta \in \mathbb{R}^\nu$ and $g_\kappa(\cdot,\cdot,\cdot)$, $h_\kappa(\cdot,\cdot,\cdot)$ smooth functions of their arguments, such that for any given compact sets $K_x \subset \mathbb{R}^n$ and $K_w \subset \mathbb{R}^2$ there exist a compact set $K_\eta \subset \mathbb{R}^\nu$ and a selection κ^* of the parameter vector κ such that for any $w_0 \in K_w$ the trajectories of the closed-loop system

$$\begin{aligned} \dot{x} &= f(x, \Gamma e^{St} w_0 + h_{\kappa^*}(t, \eta, Cx)) \\ \dot{\eta} &= g_{\kappa^*}(t, \eta, Cx) \end{aligned}$$

originating within $(x_0, \eta_0) \in K_x \times K_\eta$, are bounded and satisfy $\lim_{t\to\infty} y(t) = 0$.

III. STRUCTURE OF THE STEADY-STATE OUTPUT

Due to the standing assumptions, the periodic steady-state output $y_{\rm ss}(w(t)) = y_{\rm ss}(\exp(St)w_0)$ can be expanded in a finite Fourier series bearing the contribution of harmonics of order at most m of the fundamental tone at frequency ω_0. Let $0 \le p \le m$ and $0 < d \le m$ denote arbitrary *even* and *odd* integers, respectively. Then, the following result holds, whose proof needs to be omitted for space reasons.

Proposition 1: Any odd term in $y_{\rm ss}(w(t))$ of the form $a_{d,p} w_1^d(t) w_2^p(t)$ can be represented as

$$a_{d,p}\, w_1^d(t)\, w_2^p(t) \;= \Gamma e^{St} \begin{pmatrix} n_{d,p}(w_0) & 0 \\ 0 & n_{d,p}(w_0) \end{pmatrix} w_0 + h_{d,p}(t, w_0) \tag{8}$$

whereas odd terms of the form $a_{p,d} w_1(t)^p w_2(t)^d$ can be given the representation

$$a_{p,d}\, w_1^p(t)\, w_2^d(t) = \Gamma e^{St} \begin{pmatrix} 0 & n_{p,d}(w_0) \\ -n_{p,d}(w_0) & 0 \end{pmatrix} w_0 + h_{p,d}(t, w_0) \tag{9}$$

where $n_{d,p}(w_0)$, $n_{p,d}(w_0)$, $h_{d,p}(t, w_0)$ and $h_{p,d}(t, w_0)$ are smooth functions. In particular, $h_{d,p}(t, w_0)$ and $h_{p,d}(t, w_0)$, which represent the contributions of harmonics different from the fundamental, are T-periodic and vanish in $w_0 = 0$. Moreover, even terms in the polynomial (5) do not yield any contribution to the fundamental harmonic of $y_{\rm ss}(w(t))$. Letting $\bar{\Gamma} := (\Gamma \;\; \Gamma \;\; \ldots \;\; \Gamma) \in \mathbb{R}^{1\times(2m-2)}$ and using equations (8) and (9), simple manipulations show that the steady-state output (5) can be written as

$$y_{\rm ss}(e^{St} w_0) = r_0(w_0) + \Gamma e^{St} r_1(w_0) w_0 + \bar{\Gamma} e^{\bar{S}t} r_2(w_0) w_0 \tag{10}$$

where $\bar{S} = \text{blk diag}(S_k), k = 1, ..., m-1$, and $S_k = \begin{pmatrix} 0 & (k+1)\omega_0 \\ -(k+1)\omega_0 & 0 \end{pmatrix}$. The functions $r_0 : \mathbb{R}^2 \to \mathbb{R}$ and

$r_2 : \mathbb{R}^2 \to \mathbb{R}^{2(m-1)\times 2(m-1)}$ are respectively related to the mean value of $y_{ss}(w(t))$ and to the harmonics of frequency multiple than the fundamental. From Proposition 1, it follows that the smooth matrix-valued function $r_1 : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ can be given the following structure

$$r_1(w_0) = \begin{pmatrix} a_{1,0} + m(w_0) & n(w_0) \\ -n(w_0) & a_{1,0} + m(w_0) \end{pmatrix} \tag{11}$$

where, in particular, $m(w_0) = \sum_{\substack{d=1\\ \text{odd}}}^{m} \sum_{\substack{p=2\\ \text{even}}}^{m} n_{d,p}(w_0)$. The next assumption, which can be regarded as a nonlinear version of the one formulated in [5], ensures that the fundamental harmonic of $y_{ss}(w(t))$ carries enough information so as to allow an asymptotic reconstruction of the disturbance signal.

Assumption 3.1: The coefficients $a_{d,p}$ are such that $a_{1,0} > 0$ and $m(w_0) \geq 0$ for all $w_0 \in \mathbb{R}^2$. As a result, $r_1(w_0)$ is positive definite for all $w_0 \in \mathbb{R}^2$.

IV. CONTROLLER DESIGN

Thanks to Assumption 3.1, the contribution of the fundamental harmonic to $y_{ss}(w(t))$ (given by the second term in the right-hand side of equation (10)) should be the one to used be the controller to provide cancelation of the disturbance, while the remaining terms of the expansion (10) are undesired. Consequently, it is convenient to process the output of the system (1) by a linear filter in order to reject the constant contribution $r_0(w_0)$ and to exert a sufficiently large attenuation on the harmonics of frequencies higher than ω.

Since the filter should reproduce exactly the term in (10) associated with the first harmonic, a simple internal model of the exosystem (2) should be embedded in the filter, which is chosen as the third-order, relative degree-one system

$$F_\lambda(s) = \frac{s(s+z_0)}{s(s+z_0) + (s+\lambda)(s^2+\omega_0^2)} \tag{12}$$

where $z_0 > 0$ is fixed, and $\lambda \geq 1$ is a design parameter. It can be verified that the filter is stable for any $z_0 > 0$ and $\lambda \geq 1$. The zero at the origin secures rejection of constant signals at the input; in addition, it is possible to verify that (12) embeds an internal model for the fundamental tone at frequency ω_0. Recall that we are interesting in attenuating the harmonics of $y_{ss}(w(t))$ at $\omega = k\,\omega_0$, $k \in \{2, 3, \dots, m\}$. The frequency response of the filter, for $\omega \neq 0$, reads as

$$F_\lambda(j\omega) = \frac{j\omega z_0 - \omega^2}{j\omega[z_0 + \omega_0^2 - \omega^2] - \omega^2[1 + \lambda - \lambda\frac{\omega_0^2}{\omega^2}]}$$

showing that $|F_\lambda(j\omega_0)| = 1$ and that $\lim_{\lambda\to\infty} |F_\lambda(jk\,\omega)| = 0$, $k \in \{2, 3, \dots, m\}$. As a result, once z_0 is fixed, it is possible to achieve an arbitrary degree of attenuation at any given frequency $k\,\omega_0$ in the considering range by choosing λ large enough. For convenience, we will denote by γ_λ the response of the filter at the first frequency of interest, that is, $\gamma_\lambda = |F_\lambda(j2\omega_0)|$, with the understanding that $|F_\lambda(jk\,\omega_0)| < |F_\lambda(j(k+1)\omega_0)|$, $k = 2, 3, \dots, m-1$. The filter (12) can be realized in observer canonical form as

$$\begin{aligned} \dot{x}_f &= A_F(\lambda)x_f + B_F u_f \\ y_f &= C_F x_f \end{aligned} \tag{13}$$

with $x_f \in \mathbb{R}^3$. Since system (13) is linear, the cascade system (1)-(13) obtained by setting $u_f = y$ inherits the properties of the plant model, as far as existence and uniqueness of the steady-state solution are concerned. In particular, let the cascade be written as

$$\begin{aligned} \dot{x}^a &= f^a(x, u+d, \lambda)\,, \quad x^a(0) = x_0^a \\ y^a &= C^a x^a \end{aligned} \tag{14}$$

where $x^a = \mathrm{col}(x, x_f) \in \mathbb{R}^{n_a}$ denotes the combined state, and $n_a = n+3$. From [7, p.21 and 28] it follows that the ISC and LEC properties of the plant model are preserved for the cascade, therefore a unique globally attractive steady-state trajectory $x^a_{ss}(w, \lambda) = \mathrm{col}(x_{ss}(w), x_{f_{ss}}(w, \lambda))$ is defined when $u = 0$, where $x_{f_{ss}}(w, \lambda)$ is a polynomial in w with λ-dependent coefficients. In particular, letting $\tilde{x}^a = x^a - x^a_{ss}(w, \lambda)$ and $w(t) = \exp(St)w_0$, the transient dynamics of system (14), when $u = 0$, can be written as

$$\dot{\tilde{x}}^a = \tilde{f}^a(t, \tilde{x}^a, w_0, \lambda)\,, \quad \tilde{x}^a(0) = x^a(0) - x^a_{ss}(w_0) \tag{15}$$

where $\tilde{f}^a(t, \tilde{x}^a, w_0, \lambda) = f^a(\tilde{x}^a + x^a_{ss}(w(t), \lambda), \Gamma w(t), \lambda) - f^a(x^a_{ss}(w(t), \lambda), \Gamma w(t), \lambda)$. The following result, whose proof needs to be omitted for lack of space, holds for system (15) due to Assumption 2.1 and the fact that the $\tilde{x}_f$-dynamics is exponentially stable:

Proposition 2: Fix $\lambda > 0$. For any given compact set $\mathcal{K}_w \subset \mathbb{R}^2$, there exists a smooth T-periodic function $V^a : [0, T) \times \mathbb{R}^{n_a} \times \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying for any $t \in [0, T)$, $\tilde{x}^a \in \mathbb{R}^{n_a}$ and $w_0 \in \mathcal{K}_w$

$$\bar{\alpha}_1(\|\tilde{x}^a\|) \leq V^a(t, \tilde{x}^a, w_0, \lambda) \leq \bar{\alpha}_2(\|\tilde{x}^a\|)$$
$$\frac{\partial V^a}{\partial t} + \frac{\partial V^a}{\partial \tilde{x}^a}\tilde{f}(t, \tilde{x}^a, w_0, \lambda) \leq -\bar{a}_3\|\tilde{x}^a\|^2$$
$$\left\|\frac{\partial V^a}{\partial \tilde{x}^a}\right\| \leq \bar{\alpha}_4(\|\tilde{x}^a\|).$$

for some class-$\mathcal{K}_\infty$ functions $\bar{\alpha}_1(\cdot)$ and $\bar{\alpha}_2(\cdot)$, some positive constant $\bar{a}_3$, and some class-$\mathcal{K}$ function $\bar{\alpha}_4(\cdot)$. Moreover, there exist positive constants $\bar{r}$, $\bar{a}_i$, such that $\bar{a}_1 s^2 \leq \bar{\alpha}_1(s)$, $\bar{\alpha}_2(s) \leq \bar{a}_2 s^2$ and $\bar{\alpha}_4(s) \leq \bar{a}_4 s$, for all $s \in [0, \bar{r}]$.

Next, we analyze the output of the cascaded system (14). As a result of the properties of the filter (13), the steady-state output response, $y^a_{ss}(w, \lambda) = C^a x^a_{ss}(w, \lambda)$, reads as

$$y^a_{ss}(e^{St}w_0, \lambda) = \Gamma e^{St} r_1(w_0)w_0 + \gamma_\lambda \bar{\Gamma}_2(\lambda)e^{\bar{S}t} r_2(w_0)w_0 \tag{16}$$

where $\bar{\Gamma}_2(\lambda) = \frac{1}{\gamma_\lambda} C_F \bar{\Pi}(\lambda)$ and $\bar{\Pi}(\lambda)$ is the unique solution of the Sylvester equation $\bar{\Pi}(\lambda)\bar{S} = A_F(\lambda)\bar{S} + B_F\bar{\Gamma}$ which defines the steady-state response of the filter (13) when forced by the higher-order harmonics of $y_{ss}(w(t))$. A comparison of (10) with the filtered steady-state response (16) shows that in $y^a_{ss}(w(t), \lambda)$ the constant term has been rejected, the term related to the fundamental harmonic has been perfectly reproduced, and the high-frequency terms have been attenuated by a factor γ_λ, which can be rendered arbitrarily small by increasing the parameter λ, since $\|\bar{\Gamma}_2(\lambda)\| = \mathcal{O}(\gamma_\lambda)$ as $\lambda \to \infty$. The term $\bar{\Gamma}_2(\lambda)$ also accounts for the phase shift exerted by the filter on the higher harmonics of the output $y_{\rm ss}(w(t))$.

Following [5], the control signal u is generated by periodic nonlinear system

$$\begin{aligned}\dot\eta &= \epsilon e^{-St}\Gamma^T y^a\,, \quad \eta(0)=\eta_0\\ u &= -\Gamma e^{St}\eta\end{aligned} \tag{17}$$

where $\eta\in\mathbb{R}^2$, and $\epsilon>0$ is tunable gain parameter. By changing variables as $\theta = w_0-\eta$, $z = x^a - x^a_{\rm ss}(e^{St}\theta)$ ad using (16), it follows that the closed-loop system can be written as

$$\begin{aligned}\dot z &= \tilde f^a(t,z,\theta,\lambda) - \tfrac{\partial}{\partial\theta}x^a_{\rm ss}(e^{St}\theta,\lambda)\dot\theta\\ \dot\theta &= -\epsilon f_\theta(t,\theta) - \epsilon\gamma_\lambda g_1(t,\theta,\lambda) - \epsilon g_2(t,z)\end{aligned} \tag{18}$$

where $f_\theta(t,\theta) \triangleq e^{-St}\Gamma^T\Gamma e^{St}r_1(\theta)\theta$, $g_1(t,\theta,\lambda) \triangleq e^{-St}\Gamma^T\bar\Gamma_2(\lambda)e^{\bar St}r_2(\theta)\theta$, and $g_2(t,z)\triangleq e^{-St}\Gamma^T C^a z$. Note that all functions above are smooth, T-periodic in the first argument, and such that $f_\theta(t,0)\equiv 0$, $g_1(t,0,\lambda)\equiv 0$, and $g_2(t,0)\equiv 0$.

The main result of the paper is the following:

Theorem 3: System (18) is semi-gobally uniformly asymptotically (locally exponentially) stable in the parameters (λ,ϵ). Specifically, for any closed ball $\bar{\mathcal{B}}_{R_1}\subset\mathbb{R}^{n_a}$ and $\bar{\mathcal{B}}_{r_1}\subset\mathbb{R}^2$ there exist $\lambda^*\geq 1$ and $\epsilon^*>0$ such that for all $\lambda\geq\lambda^*$ and all $\epsilon\in(0,\epsilon^*]$ the origin of system (18) is uniformly asymptotically (locally exponentially) stable, with domain of attraction which includes $\bar{\mathcal{B}}_{R_1}\times\bar{\mathcal{B}}_{r_1}$.

The proof follows from an application of averaging theory and the semi-global stabilization lemmas of Teel and Praly [8]. The technical machinery needed for the proof will be developed in the next section.

V. Stability Analysis

To prove semi-global stability of the closed-loop system (18), we begin by looking at the properties of the lower subsystem. Since $f_\theta(t,\theta)$ is smooth and T-periodic in t, from [9, p. 404] it follows that the average

$$f_{\rm av}(\theta) = \frac{1}{T}\int_0^T f_\theta(\tau,\theta)d\tau$$

is well defined. Consider the standard near-identity transformation

$$\theta \,=\, \vartheta + \epsilon\int_0^t [f_\theta(\tau,\vartheta) - f_{\rm av}(\vartheta)]d\tau \,\triangleq\, \theta(t,\vartheta,\epsilon)\,. \tag{19}$$

The following result holds by continuity of the map (19) with respect to all its arguments, and the fact that $\theta(0,\vartheta,\epsilon) = \theta(T,\vartheta,\epsilon)$ and $\theta(t,\vartheta,0)=\vartheta$.

Lemma 4: For any three numbers $0<r_1<r_2<r_3$ there exists $\bar\epsilon>0$ such that for any $t\in[0,T)$ and any $\epsilon\in(0,\bar\epsilon]$

1) The map $\theta=\theta(t,\vartheta,\epsilon)$ is a diffemorphism over an open neighborhood of the closed-ball $\bar{\mathcal{B}}_{r_3} = \{\vartheta : |\vartheta|\leq r_3\}$.
2) The image of the set $\bar{\mathcal{B}}_{r_2} = \{\vartheta : |\vartheta|\leq r_2\}$ under the map $\theta=\theta(t,\vartheta,\epsilon)$ includes the set $\bar{\mathcal{B}}_{r_1} = \{\theta : |\theta|\leq r_1\}$.
3) The image of the set $\bar{\mathcal{B}}_{r_2} = \{\vartheta : |\vartheta|\leq r_2\}$ under the map $\theta=\theta(t,\vartheta,\epsilon)$ is included in the set $\bar{\mathcal{B}}_{r_4} = \{\theta : |\theta|\leq r_4\}$, for some finite number $r_4>0$.

Assume that $\bar\epsilon>0$ has been fixed in correspondence of some *arbitrary* $r_i>0$, $i=1,2,3$, as discussed above. Following the same arguments as in [9, pp. 404-405], it is seen that the lower subsystem in equation (18) can be written in the new variable as

$$\dot\vartheta = -\epsilon f_{\rm av}(\vartheta) + \epsilon^2 q_1(t,\vartheta,\epsilon) + \epsilon\gamma_\lambda q_2(t,\vartheta,\epsilon,\lambda) + \epsilon q_3(t,\vartheta,z,\epsilon) \tag{20}$$

where $f_{\rm av}(\vartheta) = \frac{1}{2}r_1(\vartheta)\vartheta$ and the functions $q_1(t,\vartheta,\epsilon)$, $q_2(t,\vartheta,\epsilon)$ and $q_3(t,z,\vartheta,\epsilon)$, which are well-defined in $(t,\vartheta,z,\epsilon,\lambda)\in[0,T)\times\bar{\mathcal{B}}_{r_3}\times\mathbb{R}^{n_a}\times[0,\bar\epsilon]\times[1,\infty)$, satisfy $q_1(t,0,\epsilon)\equiv 0$, $q_2(t,0,\epsilon,\lambda)\equiv 0$ and $q_3(t,\vartheta,0,\epsilon)\equiv 0$.

Proposition 5: The origin $\vartheta=0$ of system

$$\dot\vartheta = -\epsilon f_{\rm av}(\vartheta) + \epsilon^2 q_1(t,\vartheta,\epsilon) + \epsilon\gamma_\lambda q_2(t,\vartheta,\epsilon,\lambda) \tag{21}$$

is exponentially stabilizable in the parameters ϵ and λ, with domain of attraction that includes the closed invariant set $\bar{\mathcal{B}}_{r_3}$.

Proof: Let $\Omega^\vartheta_a = \{\vartheta\in\mathbb{R}^2 : \mathcal{U}(\vartheta)\leq a\}$ denote the level sets of the Lyapunov function candidate $\mathcal{U}(\vartheta) = \vartheta^{\rm T}\vartheta$, and fix $a=r_3^2$ so that $\Omega^\vartheta_a = \bar{\mathcal{B}}_{r_3}$. By virtue of the fact that $q_1(t,\vartheta,\epsilon)$ and $q_2(t,\vartheta,\epsilon,\lambda)$ are continuously differentiable with respect to ϑ in an open neighborhood of $\bar{\mathcal{B}}_{r_3}$, and the fact that they vanish at $\vartheta=0$, it is possible to write $q_1(t,\vartheta,\epsilon) = \bar q_1(t,\vartheta,\epsilon)\vartheta$ and $q_2(t,\vartheta,\epsilon,\lambda) = \bar q_2(t,\vartheta,\epsilon,\lambda)\vartheta$, where $\bar q_1(t,\vartheta,\epsilon)$ and $\bar q_2(t,\vartheta,\epsilon,\lambda)$ are continuous with respect to their arguments and *bounded* in $(t,\vartheta,\epsilon,\lambda)\in[0,T)\times\Omega^\vartheta_a\times[0,\bar\epsilon]\times[1,\infty)$. As a result, the derivative of $\mathcal{U}$ along the trajectories of (21) can be estimated as

$$\begin{aligned}\dot{\mathcal{U}}(\vartheta) \leq -\epsilon\,\|\vartheta\|^2\big[a_{1,0} &- 2\epsilon\,\|\bar q_1(t,\vartheta,\epsilon)\|\\ &- 2\gamma_\lambda\,\|\bar q_2(t,\vartheta,\epsilon,\lambda)\|\,\big]\end{aligned} \tag{22}$$

for all $t\in[0,T)$ and all $\vartheta\in\Omega^\vartheta_a$. Recalling that $\lim_{\lambda\to\infty}\gamma_\lambda = 0$, it follows that there exist numbers $\bar\epsilon^*\in(0,\bar\epsilon]$, $\lambda^*\geq 1$ and $\bar a>0$ such that, for all $\epsilon\in(0,\bar\epsilon^*]$ and all $\lambda\geq\lambda^*$: $\dot{\mathcal{U}}(\vartheta)\leq -\epsilon\,\bar a\,\|\vartheta\|^2$ for all $t\in[0,T)$ and all $\vartheta\in\Omega^\vartheta_a$; from which the result follows. ■

The result of Proposition 5 is used as follows: Let the set $\bar{\mathcal{B}}_{r_1}$ for $\theta(0)$ be given as in Theorem 3, and determine a ball $\bar{\mathcal{B}}_{r_2}$ for $\vartheta(0)$ on the basis of Lemma 4. Proposition 5 implies that, when $z=0$, system (21) enjoys the *uniform Lyapunov property* as defined in [8], and that $\bar{\mathcal{B}}_{r_2}$ is properly contained in the open invariant subset of the domain of attraction given by $\{\vartheta\in\mathbb{R}^2 : \mathcal{U}(\vartheta)<r_3^2\}$. To facilitate the use of the results of [8, Lemma 2.2] in our proof, we let r_2 and r_3 in Lemma 4 and Proposition 5 be defined, without loss of generality, as $r_2=\sqrt{\mu}$ and $r_3=\sqrt{\mu+1}$, with $\mu\geq 1$. Consequently, we determine $\bar\epsilon^*$ and λ^* as in the proof of Proposition 5, and we *fix* $\lambda\geq\lambda^*$ *once and for all*. As a result of this assignment, from now on *we will omit the explicit dependence on* λ *of all terms in our equations*.

Once λ has been fixed, we move on to considering the semi-global stabilization (in the parameter ϵ) of the origin of the closed-loop system (18), with respect to the domain

$\mho = \{z \in \mathbb{R}^{n_a}\} \times \{\vartheta : \mathcal{U}(\vartheta) < \mu+1\}$. In the new coordinates (z, ϑ), the upper subsystem in equation (18) is written as

$$\dot{z} = F(t, z, \vartheta, \epsilon) + \epsilon l_1(t, z, \vartheta, \epsilon) + \epsilon l_2(t, \vartheta, \epsilon) \qquad (23)$$

where $F(t, z, \vartheta, \epsilon) \triangleq \tilde{f}^a(t, z, \theta(t, \vartheta, \epsilon), \lambda)$, and

$$\begin{aligned} l_1(t, z, \vartheta, \epsilon) &\triangleq \tfrac{\partial}{\partial \theta} x^a_{\rm ss}(e^{St}\theta, \lambda)_{|\theta = \theta(t,\vartheta,\epsilon)} e^{-St} \Gamma^T C^a z \\ l_2(t, \vartheta, \epsilon) &\triangleq \big[\tfrac{\partial}{\partial \theta} x^a_{\rm ss}(e^{St}\theta, \lambda) e^{-St} \Gamma^T \Gamma e^{St} \\ &\quad \times r_1(\theta)\theta\big]_{|\theta = \theta(t,\vartheta,\epsilon)} + \gamma_\lambda \big[\tfrac{\partial}{\partial \theta} x^a_{\rm ss}(e^{St}\theta, \lambda) \\ &\quad \times e^{-St} \Gamma^T \bar{\Gamma}_2(\lambda) e^{\bar{S}t} r_2(\theta)\theta\big]_{|\theta = \theta(t,\vartheta,\epsilon)} . \end{aligned}$$

Note that the above vector-fields are T-periodic in the first argument, and that $F^a(t, 0, \vartheta, \epsilon) \equiv 0$, $l_1(t, 0, \vartheta, \epsilon) = 0$, and $l_2(t, 0, \epsilon) = 0$. As a Lyapunov function candidate for (23), we choose the function $\mathcal{V}(t, z, \vartheta, \epsilon) \triangleq V^a(t, z, \theta(t, \vartheta, \epsilon), \lambda)$ from which, for a given positive constant b, we define the following parameterized family of level sets

$$\begin{aligned} \Omega^z_{b,t,\vartheta,\epsilon} = \big\{ &z \in \mathbb{R}^{n_a} : \mathcal{V}(t, z, \vartheta, \epsilon) \le b,\ (t, \vartheta, \epsilon) \in [0, T) \\ &\times \Omega^\vartheta_{\mu+1} \times [0, \bar{\epsilon}^*] \big\} . \end{aligned}$$

Due to Lemma 4, there exists $r_4 > 0$ such that $\theta(t, \vartheta, \epsilon) \in \bar{\mathcal{B}}_{r_4}$ for all $(t, \vartheta, \epsilon) \in [0, T) \times \Omega^\vartheta_{\mu+1} \times [0, \bar{\epsilon}^*]$. Consequently, from Proposition 2 it follows that for any $R_1 > 0$ the inclusions $\bar{\mathcal{B}}_{R_1} \subset \Omega^z_{c,t,\vartheta,\epsilon} \subset \bar{\mathcal{B}}_{R_2}$ hold for all $(t, \vartheta, \epsilon) \in [0, T) \times \Omega^\vartheta_{\mu+1} \times [0, \bar{\epsilon}^*]$, where $c = \bar{\alpha}_2(R)$ and $R_2 = \bar{\alpha}_1^{-1}(c)$. Therefore, in the domain $\mho$ (which is unbounded in the z-direction) any given compact set of initial conditions $z(0)$ (specified by Theorem 3) can be included in all elements of the family of parameterized level sets of $\mathcal{V}$ of the form $\Omega^z_{b,t,\vartheta,\epsilon}$. Following [8], consider now the Lyapunov function candidate

$$\mathcal{W}(t, z, \vartheta, \epsilon) = \mathcal{V}(t, z, \vartheta, \epsilon) + \mu \frac{\mathcal{U}(\vartheta)}{\mu + 1 - \mathcal{U}(\vartheta)},$$

which, by construction, is proper in the set $\mho$. Fix a positive constant d and define the parameterized family of sets

$$\begin{aligned} \Omega_{d,t,\epsilon} = \big\{ &(z, \vartheta) \in \mathbb{R}^{n_a} \times \mathbb{R}^2 : \mathcal{W}(t, z, \vartheta, \epsilon) \le d\,, \\ &(t, \epsilon) \in [0, T) \times [0, \bar{\epsilon}^*] \big\} . \end{aligned}$$

Notice that for any $(t, \epsilon) \in [0, T) \times [0, \bar{\epsilon}^*]$ the following inclusions hold (see [8, Lemma 2.2])

$$\Omega^z_{c,t,\vartheta,\epsilon} \times \Omega^\vartheta_\mu \subset \Omega_{c+\mu^2+1,t,\epsilon} \subset \bar{\mathcal{B}}_{R_3} \times \Omega^\vartheta_\sigma \subset \mho \qquad (24)$$

where $R_3 = \bar{\alpha}_1^{-1}(c + \mu^2 + 1)$ and $\sigma = \frac{(\mu+1)(c+\mu^2+1}{c+\mu^2+\mu+1}$. This shows that every fixed compact set of initial conditions for (z, ϑ) can be included in any element of the parameterized family of level sets $\Omega_{d,t,\epsilon}$, and that the union of these sets lies in a compact set. For convenience, we denote this set by $\bar{\mathcal{S}} \triangleq \bar{\mathcal{B}}_{R_3} \times \Omega^\vartheta_\sigma$.

Proposition 6: For any number $\rho > 0$ there exist real numbers $0 < \epsilon_1^* \le \bar{\epsilon}^*$, $\kappa > 0$ such that for each $\epsilon \in (0, \epsilon_1^*]$ and $t \in [0, T)$ the Lie derivative of the Lyapunov function candidate $\mathcal{W}$ along the vector field of system (23)–(20) satisfies $\dot{\mathcal{W}}(t, z, \vartheta, \epsilon) \le -\kappa$ for all (z, ϑ) in the set $\mathcal{S}_{t,\epsilon} = \big\{(z, \vartheta) : \rho \le \mathcal{W}(t, z, \vartheta, \epsilon) \le c + \mu^2 + 1\big\}$.

Proof: The Lie derivative of $\mathcal{W}$ along (23)–(20) reads as

$$\begin{aligned} \dot{\mathcal{W}} = &\frac{\partial V^a}{\partial t} + \frac{\partial V^a}{\partial z} F(t, z, \vartheta, \epsilon) + \epsilon \frac{\partial V^a}{\partial z} \big[l_1(t, z, \vartheta, \epsilon) + \\ &l_2(t, \vartheta, \epsilon)\big] + \frac{\partial V^a}{\partial \theta}[f_\theta(t, \vartheta) - f_{\rm av}(\vartheta)] + \big[\frac{\partial V^a}{\partial \theta}\frac{\partial \theta}{\partial \vartheta} \\ &+ \frac{\partial \mathcal{U}}{\partial \vartheta}\frac{\mu(\mu+1)}{(\mu + 1 - \mathcal{U}(\vartheta))^2}\big]\big[-\epsilon f_{\rm av}(\vartheta) + \epsilon^2 q_1(t, \vartheta, \epsilon) \\ &+ \epsilon\gamma q_2(t, \vartheta, \epsilon) + \epsilon q_3(t, \vartheta, z, \epsilon)\big] \end{aligned}$$

and satisfies for all $(z, \vartheta) \in \mho$ and all $(t, \epsilon) \in [0, T) \times (0, \bar{\epsilon}^*]$

$$\begin{aligned} \dot{\mathcal{W}} \le &-\bar{a}_3\|z\|^2 - \epsilon\,\bar{a}\frac{\mu}{\mu+1}\|\vartheta\|^2 + \epsilon\, m_1(t, z, \vartheta, \epsilon) \\ &+ \epsilon\, m_2(t, z, \vartheta, \epsilon) \end{aligned} \qquad (25)$$

where we have used Proposition 5 and we have denoted

$$\begin{aligned} m_1(t, z, \vartheta, \epsilon) &\triangleq \frac{\partial V^a}{\partial z} l_1(t, z, \vartheta, \epsilon) + \frac{\partial V^a}{\partial \theta}\frac{\partial \theta}{\partial \vartheta} q_3(t, z, \vartheta, \epsilon) \\ m_2(t, z, \vartheta, \epsilon) &\triangleq \frac{\partial V^a}{\partial z} l_2(t, \vartheta, \epsilon) + \frac{\partial V^a}{\partial \theta}\frac{\partial \theta}{\partial \vartheta}\big[-f_{\rm av}(\vartheta) + \\ &\quad \epsilon q_1(t, \vartheta, \epsilon) + \gamma q_2(t, \vartheta, \epsilon)\big] + \frac{\partial V^a}{\partial \theta}\big[f_\theta(t, \vartheta) \\ &\quad - f_{av}(\vartheta)\big] + \frac{\partial \mathcal{U}}{\partial \vartheta}\frac{\mu(\mu+1)}{(\mu + 1 - \mathcal{U}(\vartheta))^2} q_3(t, z, \vartheta, \epsilon) \end{aligned}$$

Define the set $\underline{\mathcal{S}} = \{z \in \mathbb{R}^{n_a} : \|z\| < \bar{\alpha}_2^{-1}(\rho/2)\} \times \{\vartheta \in \mathbb{R}^2 : \mathcal{U}(\vartheta) < \sqrt{\rho/2}\}$. Since $\mathcal{U}(\vartheta) < \sqrt{\rho/2}$ implies $\mu\frac{\mathcal{U}(\vartheta)}{\mu+1-\mathcal{U}(\vartheta)} < \rho/2$, it follows that $\underline{\mathcal{S}} \subset \Omega_{\rho,t,\epsilon}$ for all $(t, \epsilon) \in [0, T) \times [0, \bar{\epsilon}^*]$. Recalling (24), it can be concluded that each $\mathcal{S}_{t,\epsilon}$ is contained in the compact set $\mathcal{S} \triangleq \bar{\mathcal{S}} \setminus \underline{\mathcal{S}}$. From this point on, the proof follows by a suitable application of Bacciotti's semi-global stabilization lemma (see [10, Theorem 9.3.1]). Specifically, notice that both $m_1(t, z, \vartheta, \epsilon)$ and $m_2(t, 0, \vartheta, \epsilon)$ vanish at $z = 0$ (see Proposition 2). Therefore, for any $\epsilon \in (0, \bar{\epsilon}^*]$ it follows that $\mathcal{W} < 0$ on the compact set $\mathcal{S}_0 = \{(z, \vartheta) \in \mathcal{S} : z = 0\}$ where $\|\vartheta\|$ is bounded away from zero. By continuity, $\mathcal{W} < 0$ on an open neighborhood $\mathcal{N}$ of $\mathcal{S}_0$. Since $z \ne 0$ on the compact set $\tilde{\mathcal{S}} = \mathcal{S} \setminus \mathcal{N}$, and $m_1(t, z, \vartheta, \epsilon)$ and $m_2(t, z, \vartheta, \epsilon)$ are bounded in $\tilde{\mathcal{S}}$ for all $(t, \epsilon) \in [0, T) \times [0, \bar{\epsilon}^*]$, one obtains that in this set: $\dot{\mathcal{W}} \le -\delta_1 - \epsilon\,\bar{a}\frac{\mu}{\mu+1}\|\vartheta\|^2 + \epsilon\delta_2$, for some finite numbers $\delta_1 > 0$ and $\delta_2 > 0$. Choosing ϵ_1^* such that $0 < \epsilon_1^* < \delta_1/\delta_2$ completes the proof. ∎

Proposition 6 implies that, for any fixed $\epsilon \in (0, \epsilon_1^*]$, all trajectories originating within the set $\mathcal{S}_{0,\epsilon}$ satisfy $\mathcal{W}(t, z(t), \vartheta(t), \epsilon) \le \mathcal{W}(0, z(0), \vartheta(0), \epsilon) - \kappa\, t \le c + \mu^2 + 1 - \kappa t$, and thus are trapped by the set $\Omega_{\rho,t,\epsilon}$ for all $t \ge (c + \mu^2 + 1 - \rho)/\kappa$. Exponential convergence from a suitable family of level sets $\Omega_{\rho,t,\epsilon}$ is established as follows:

Proposition 7: Let $\bar{r} > 0$ be defined as in Proposition 2. Then, there exists $0 < \epsilon_2^* \le \bar{\epsilon}^*$ such that the following inequalities hold in the set $\{(z, \vartheta) \in \mathbb{R}^{n_a} \times \mathbb{R}^2 : \|(z, \vartheta)\| \le \bar{r}\}$ for all $\epsilon \in (0, \epsilon_2^*]$ and all $t \in [0, T)$

$$\begin{aligned} c_1\|(z, \vartheta)\|^2 \le \mathcal{W}(t, z, \vartheta, \epsilon) &\le c_2\|(z, \vartheta)\|^2 \\ \dot{\mathcal{W}}(t, z, \vartheta, \epsilon) &\le -c_3\|(z, \vartheta)|^2 \end{aligned}$$

for some numbers $c_i > 0$, $1 \le i \le 3$.

Proof: The first two inequalities follow directly from Proposition 2 and the definition of $\mathcal{W}$. For the last inequality, notice that, by definition of $m_1(\cdot)$ and $m_2(\cdot)$, it follows that, for all $(z,\vartheta) \in \mho$, $m_1(t,0,\vartheta,\epsilon) \equiv 0$, $\frac{\partial m_1}{\partial z}(t,0,\vartheta,\epsilon) \equiv 0$, $m_2(t,0,\vartheta,\epsilon) \equiv 0$, and $m_2(t,z,0,\epsilon) \equiv 0$. Since $m_1(\cdot)$ and $m_2(\cdot)$ are smooth, there exist numbers $M_1 > 0$, $M_2 > 0$ such that $\|m_1(t,z,\vartheta,\epsilon)\| \le M_1\|z\|^2$ and $\|m_2(t,z,\vartheta,\epsilon)\| \le M_2\|z\|\,\|\vartheta\|$, for all $(t,z,\vartheta,\epsilon) \in [0,T] \times \{(z,\vartheta) : \|(z,\vartheta)\| \le \bar{r}\} \times [0,\bar{\epsilon}^*]$. From (25), and using again Proposition 2, one obtains $\dot{\mathcal{W}} \le -\bar{a}_3\|z\|^2 - \epsilon\bar{a}\frac{\mu}{\mu+1}\|\vartheta\|^2 + \epsilon M_1\|z\|^2 + \epsilon M_2\|z\|\|\vartheta\|$, for all $\|(z,\vartheta)\| \le \bar{r}$, all $\epsilon \in (0,\bar{\epsilon}^*]$ and all $t \in [0,T)$, hence the result follows from a simple application of Young's inequality. ■

The proof of Theorem 3 is completed by choosing ρ in Proposition 6 small enough so that $\Omega_{\rho,t,\epsilon} \subset \{(z,\vartheta) : \|(z,\vartheta)\| \le \bar{r}\}$ for all $(t,\epsilon) \in [0,T) \times [0,\bar{\epsilon}^*]$, and by choosing $\epsilon^* = \min\{\epsilon_1^*, \epsilon_2^*\}$.

VI. Illustrative Example

Consider the following ISC nonlinear system [7]

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + u + d \\ y &= x_1 + x_2 . \end{aligned} \tag{26}$$

When forced by the output of (2), the steady-state of system (26) reads as $x_{\mathrm{ss},1}(w) = b_1(\omega_0)w_1^2 + 2b_2(\omega_0)w_1w_2 + b_3(\omega_0)w_2^2$ and $x_{\mathrm{ss},2}(w) = a_1(\omega_0)w_1 + a_2(\omega_0)w_2$, where $a_1(\omega_0)$ and $a_2(\omega_0)$ are given by $a_1(\omega_0) = \frac{1}{1+\omega_0^2}$ and $a_2(\omega_0) = -\frac{\omega_0}{1+\omega_0^2}$ and the expressions of the polynomials $b_i(\omega_0)$ can be found in [7, p.67]. This shows that the steady-state output $y_{\mathrm{ss}}(\exp(St)w_0)$ satisfies Assumption 3.1. The change of variable $\tilde{x} = x - x_{\mathrm{ss}}(w)$ transforms system (26) into

$$\begin{aligned} \dot{\tilde{x}}_1 &= -\tilde{x}_1 + \tilde{x}_2^2 + \delta(w)\tilde{x}_2 \\ \dot{\tilde{x}}_2 &= -\tilde{x}_2 \end{aligned} \tag{27}$$

where $\delta(w) = 2(a_1(\omega_0)w_1 + a_2(\omega_0)w_2)$. Notice that for all $\omega_0 > 0$, $|a_1(\omega_0)| \le 1$ and $|a_2(\omega_0)| \le 1/2$. Also, since $|w_i(t)| \le \|w_0\|$, $i = 1,2$, it follows that $|\delta(e^{St}w_0)| \le 3\|w_0\|$, for all $t \in [0,T)$. Consider the Lyapunov function

$$V(\tilde{x},w_0) = \frac{1}{2}\tilde{x}_1^2 + \frac{1}{4}\tilde{x}_2^4 + \kappa(w_0)\frac{1}{2}\tilde{x}_2^2 \tag{28}$$

where $\kappa(w_0) \ge 0$ is to be determined. The derivative of V along the trajectories of (27) reads as $\dot{V} = -\tilde{x}_1^2 + \tilde{x}_1\tilde{x}_2^2 + \delta(e^{St}w_0)\tilde{x}_1\tilde{x}_2 - \tilde{x}_2^4 - k\tilde{x}_2^2$. Applying Young's inequality, a simple algebraic manipulation shows that $\dot{V} \le -\frac{1}{4}\tilde{x}_1^2 - \frac{1}{2}\tilde{x}_2^4 + 36\|w_0\|^2\tilde{x}_2^2 - \kappa(w_0)\tilde{x}^2$. By choosing $\kappa(w_0) = 1 + 36\|w_0\|^2$, it is seen that (28) fulfills Assumption 2.1. Since Assumption 3.1 is fulfilled as well, the proposed control strategy can be applied to system (26), where, due to the structure of the steady-state response $y_{\mathrm{ss}}^a(\exp(St)w_0)$, it has not been necessary to add the filter (12) to the output of (26). The initial conditions for (2) are chosen so that $w_0 = (1\ 1)$, those of the plant at $x_0 = (1\ 2)$, while the remaining initial conditions for (17) are chosen at the origin. The frequency of the sinusoidal disturbance has been selected as $\omega_0 = 0.5$ rad/s. The effect of the controller on the plant output is visible in Figure 1, which shows the regulated output for three different values of the controller parameter, namely $\epsilon = 0.5$, $\epsilon = 1$, and $\epsilon = 0.1$. It is worth noting that instability occurs for $\epsilon > 1.2$.

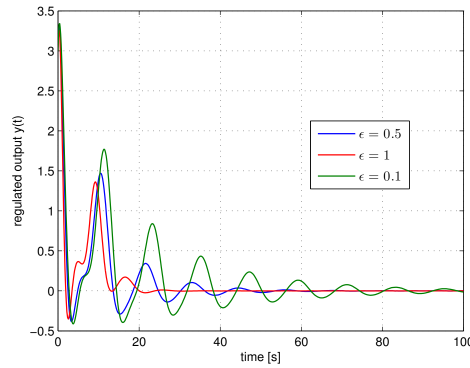

Fig. 1. Regulated output for different values of the parameter ϵ.

VII. Conclusions

In this work, the problem of adaptive feedforward compensation for a class of nonlinear systems has been addressed. It has been shown how, under particular assumptions, the scheme proposed in [5] could be reinterpreted in a nonlinear setting and applied to achieve disturbance rejection of a harmonic disturbance at the input of a stable nonlinear system with a semi-global domain of convergence. The stability analysis was carried out using tools from averaging analysis and semi-global stabilization.

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