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On Globally Optimal Real-Time Encoding and Decoding Strategies in Multi-Terminal Communication Systems

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Abstract— **We consider a communication system consisting of two encoders communicating with a single receiver over a noiseless channel. The two encoders make distinct partial observations of a discrete-time Markov source. Each encoder must encode its observations into a sequence of discrete variables. The sequence is transmitted over a noiseless channel to a receiver which attempts to reproduce the output of the Markov source. The system must operate in real-time, that is, the encoding at each encoder and decoding at the receiver must be performed without any delay. The goal is to find globally optimal real-time encoding and decoding strategies to minimize an expected distortion metric over a finite time horizon. We determine qualitative properties of optimal real-time encoding and decoding strategies. Using these properties, we develop a sequential decomposition of the problem of finding globally optimal real-time encoding and decoding strategies. Such a sequential decomposition reduces the complexity of the global optimization problem.**

I. Introduction

A multi-terminal communication system with two encoders communicating with a single receiver over a noiseless channel is considered. The two encoders make distinct partial observations of a discrete-time Markov source. Each encoder must encode its observations into a sequence of discrete variables. This sequence is transmitted over a noiseless channel to a receiver which attempts to reproduce the output of the Markov source. The system must operate in real-time, that is, the encoding at each encoder and decoding at the receiver must be performed without any delay. Both encoders and the receiver have perfect recall, i.e, they remember all of their past observations and actions. The goal is to find globally optimal encoding and decoding strategies to minimize an expected distortion metric over a finite time horizon. The problem is motivated by applications such as sensor networks, transportation networks and networked control systems where the communication system is a part of a larger system that requires strict bounds on delays in information transmission.

The key features of the problem are : a) The real-time constraint on information transmission; and b) The presence of multiple encoders with different but correlated information.

The real-time constraint on information-transmission distinguishes our problem from the information-theoretic problem of distributed source coding ([11-13],[14 and references therein]). Information-theoretic approaches deal with encoding and decoding of long sequences that are asymptotically typical. Encoding long sequences introduces delays and this feature is distinctly different from our real-time constraint.

Point-to-point communication systems under the real-time constraint have been investigated in [1], [2], [10], [6], [5]. The structure of real-time encoders and decoders for the broadcast system under the real-time constraint on information transmission and for a real-time variation of the Wyner-Ziv problem was investigated in [10]. In this paper, we consider a multi-terminal communication system; furthermore, our model is different from the broadcast system and the real-time variation of the Wyner-Ziv problem investigated in [10].

The main feature of a multi-terminal problem that distinguishes it from a point to point communication problem is the presence of coupling among the encoders, (that is, each encoder must take into account what other encoders are doing). This coupling arises because of the following reasons - 1) The encoder's observations are correlated with each other. 2) Even if the encoders' observations were independent, the encoding problems remain coupled because the receiver wants to minimize a non-separable distortion metric. That is, the distortion metric cannot be decomposed into separate functions that depend only on one encoder's observations and actions. The nature of optimal strategies strongly depends on the nature and extent of the coupling among the encoders. In this paper, we consider two encoders, a general distortion metric and a simple model of correlation between the two encoders' observations (described in Section II).

The main contributions of this paper are : 1) The determination of structural properties of optimal real-time encoding and decoding strategies, and 2) A sequential decomposition of the problem of finding globally optimal encoding and decoding strategies for the model under consideration. Such a decomposition reduces the complexity of finding globally optimal real-time encoding and decoding strategies.

The rest of this paper is organized as follows. In Section II, we formulate the problem for a specific source model. In Section III, we present results on the structure of optimal real-time encoders and decoders. In Section IV, we present a method for sequentially determining globally optimal real-time encoding and decoding strategies. We conclude in Section V.

Notation: Throughout this paper, we denote random variables by capital letters. We use superscripts to refer to sequences of random variables. Thus, V^t refers to $V_1, V_2, ...V_t$. In case of two superscripts, the first refers to the encoder number for which the variable is being considered and the second refers to the sequence. Thus $X^{1,t}$ indicates the sequence

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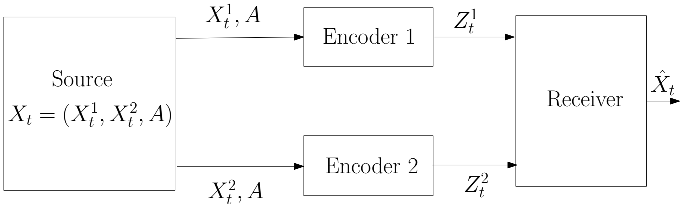

Fig. 1. Multi-Terminal Communication System with two Encoders and one Receiver

$X_1^1, X_2^1, ...X_t^1$.

II. Problem Formulation

A. The Model

We consider a finite state discrete time Markov source. We assume that the source can be in one of finitely many "modes". The random variable $A \in \mathcal{A}$ denotes the mode of the source. In any given mode, the source produces two Markov chains, X_t^1 and X_t^2, which are conditionally independent given A. We assume that the mode A does not change during the time horizon under consideration. Thus, the state of the source at a time t is given as:

$$X_t := (X_t^1, X_t^2, A) \tag{1}$$

where $X_t^i \in \mathcal{X}^i$, i=1,2 and $A \in \mathcal{A}$. $\mathcal{X}^i, i = 1,2$ and $\mathcal{A}$ are finite spaces. The state space of the source is $\mathcal{X} = \mathcal{X}^1 \times \mathcal{X}^2 \times \mathcal{A}$
The evolution of the Markov source is given in terms of the statistics of the initial state and the transition probability as:

$$\begin{aligned} Pr(X_1^1, X_1^2, A) =& Pr(X_1^1, X_1^2/A).Pr(A) \\ =& Pr(X_1^1/A).Pr(X_1^2/A).Pr(A) \end{aligned} \tag{2}$$

$$\begin{aligned} &Pr(X_{t+1}^1, X_{t+1}^2, A^{'}/X_t^1, X_t^2, A) \\ &\quad = Pr(X_{t+1}^1/X_t^1, A).Pr(X_{t+1}^2/X_t^2, A).\delta(A^{'}, A) \end{aligned} \tag{3}$$

At each time t, the first encoder observes X_t^1 and A, and the second encoder observes X_t^2 and A. Thus, each encoder gets a perfect observation of the underlying source mode and it observes one of the two conditionally independent Markov chains. The two encoders produce Z_t^1 and Z_t^2 that belong to finite alphabets $\mathcal{Z}^1$ and $\mathcal{Z}^2$, respectively. Both encoders must encode in real time hence the encoded symbols at time t are functions of observations available till time t only. Thus,

$$Z_t^i = f_t^i(X^{i,t}, A) \tag{4}$$

for i=1,2 where $X^{i,t} = X_1^i, X_2^i, ..., X_t^i$. The encoders' outputs at time t, (Z_t^1, Z_t^2), are transmitted to a receiver over a noiseless channel. A perfect memory receiver must produce estimates $\hat{X}_t$ of the state of the source X_t in real time, i.e,

$$\hat{X}_t = g_t(Z^{1,t}, Z^{2,t}) \tag{5}$$

where $Z^{i,t} = Z_1^i, Z_2^i, ..., Z_t^i, i = 1, 2$.
A non-negative distortion function $\rho_t(X_t, \hat{X}_t)$ measures the instantaneous distortion between the source and the estimate at time t. The overall performance of the system is the expected total distortion over a finite time-horizon, T.

B. The Optimization Problem

Given the source statistics, the encoding alphabets, the time horizon T, and the distortion function $\rho_t(X_t, \hat{X}_t)$, the objective is to find globally optimal encoding and decoding functions $f^{1,T}, f^{2,T}, g^T$ so as to minimise

$$J(f^{1,T}, f^{2,T}, g^T) = E[\sum_{t=1}^{T} \rho_t(X_t, \hat{X}_t)], \tag{6}$$

where we use the notation $f^{i,t}$ for $f_1^i, f_2^i, ..., f_t^i$ and g^t for $g^1, ..., g^t$.
Remark: Since the state space of the source, the encoding alphabets and the time horizon are all finite, the number of possible real-time encoding and decoding strategies $(f^{1,T}, f^{2,T}, g^T)$ is finite. Therefore, an optimal strategy $(\tilde{f}^{1,T}, \tilde{f}^{2,T}, \tilde{g}^T)$ always exists.

C. Features of the Problem

The problem formulated in this paper is a dynamic team problem. Dynamic team problems are difficult because they are, in general, non-convex functional optimization problems. We would like to develop a methodology that reduces the complexity of determining an optimal solution to our problem. For that matter, we wish to obtain a sequential decomposition of the optimization problem. The fundamental difficulty in obtaining such a decomposition is the discovery of an information state appropriate for performance evaluation [9]. This difficulty is a fundamental conceptual issue for any decentralized optimization problem. We wish to identify an information state that is not only appropriate for performance evaluation but also has a time-invariant domain, that is, the space in which this state lies does not keep increasing with time. Such an information state would be applicable to both finite as well as infinite time horizon problems.

Identifying structural properties of optimal real-time encoding and decoding strategies could lead to the discovery of information states with a time-invariant domain. Therefore, we proceed as follows. In Section III, we present structural properties of optimal strategies. We use these structural results in Section IV to identify an information state for the global optimization problem of Section II.B that is appropriate for performance evaluation and has a time-invariant domain. We show how such an information state leads to a sequential decomposition of the global optimization problem.

III. Structural Properties of an Optimal Design

As shown in Appendix I, for any arbitrary but fixed encoding rules, the decoder can be assumed to have the following structure without any loss of optimality:

$$\hat{X}_t = \tau(\psi_t) \tag{7}$$

where

$$\psi_t = Pr(X_t/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t}) \tag{8}$$

and

$$\tau(\psi) = \arg\min_{a \in \mathcal{X}} \sum_{x \in \mathcal{X}} \psi(x) \rho_t(x, a) \tag{9}$$

Note that for a fixed $z^{1,t}, z^{2,t}$, the $Pr(X_t/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t})$ depends only on the encoding rules used.

We now prove a structural result for the encoders in two steps. In the first step (Theorem 1), we establish a result similar to those proven in Theorem 1 in [1] and Theorem 1 in [2]. A drawback of this result is that the domain of the optimal real-time encoding rule is changing (increasing) with time. We wish to have (if possible) optimal real-time encoding rules whose domain does not change with time. This consideration motivates the structural result we obtain in Theorem 2, where we show the existence of optimal real-time encoding rules whose domain is time-invariant. The result of the second step is similar to that of [2]. However, the presence of two encoders with different but correlated information does not permit us to use the methodology adopted in [2] to achieve the second structural result.

A. First structural Result

Theorem 1 : There is no loss of optimality if one restricts attention to encoders of the form :

$$Z^i_t = f^i_t(X^i_t, A, Z^{i,t-1}) \tag{10}$$

for i=1,2.

Proof: Consider an arbitrary decoding rule $\hat{X}_t = g_t(Z^{1,t}, Z^{2,t})$ and an arbitrary encoding rule $Z^2_t = f^2_t(X^{2,t}, A)$ for the second encoder. We will show that the first encoder can use encoding rules of the form $Z^1_t = f^1_t(X^1_t, A, Z^{1,t-1})$ without any loss of optimality.

Define $V_1 := (X^1_1, A)$
and $V_t := (X^1_t, A, Z^{1,t-1})$, for $t = 2, 3..T$.
Then V_t is a conditionally Markov process given the Z^1_ts since

$$\begin{aligned} Pr(V_{t+1}/V^t, Z^{1,t}) &= Pr(X^1_{t+1}, A, Z^{1,t}/X^{1,t}, A, Z^{1,t}) \\ &= Pr(X^1_{t+1}, A, Z^{1,t}/X^1_t, A, Z^{1,t}) \quad (11) \\ &= Pr(X^1_{t+1}, A, Z^{1,t}/V_t, Z^1_t) \\ &= Pr(V_{t+1}/V_t, Z^1_t) \quad (12) \end{aligned}$$

where the equality in (11) holds because of the Markovian nature of X^1_t when conditioned on A.

As seen by the first encoder, the cost function of this system (with second encoder's and decoder's rules fixed) can be written as:

$$\begin{aligned} J(f^{1,T}, f^{2,T}, g^T) &= E[\sum_{t=1}^{T} \rho_t(X_t, \hat{X}_t)] = \sum_{t=1}^{T} E[\rho_t(X_t, \hat{X}_t)] \\ &= \sum_{t=1}^{T} E[E[\rho_t(X_t, \hat{X}_t)/X^{1,t}, A, Z^{1,t}]] \end{aligned} \tag{13}$$

$$= \sum_{t=1}^{T} E[E[\rho_t((X^1_t, X^2_t, A), g_t(Z^{1,t}, Z^{2,t}))/X^{1,t}, A, Z^{1,t}]] \tag{14}$$

where (13) follows from the smoothing property of conditional expectation and (14) by direct substitution. In the inner expectation of (14), the only random quantities are X^2_t and $Z^{2,t}$ since the rest appear in the conditioning variables. Since the second encoder's rule has been fixed, $Z^{2,t}$ itself is a function of $X^{2,t}$ and A. Thus the only randomness in the inner expectation is due to $X^{2,t}$ which conditioned on A is independent of the first encoders private observations $X^{1,t}$ and actions $Z^{1,t}$. Therefore, the above expectation can be written as:

$$J(f^{1,T}, f^{2,T}, g^T) = \sum_{t=1}^{T} E[E[\rho_t((X^1_t, X^2_t, A), g_t(Z^{1,t}, Z^{2,t}))/X^1_t, A, Z^{1,t}]] \tag{15}$$

$$= \sum_{t=1}^{T} E[\hat{\rho}_t(X^1_t, A, Z^{1,t})]] \tag{16}$$

$$= \sum_{t=1}^{T} E[\hat{\rho}_t(V_t, Z^1_t)]] \tag{17}$$

In (16), we have expressed the inner conditional expectation as a function of the conditioning random variables.

Hence, the optimal encoding problem from the first encoder's point of view is to find the optimal control actions Z^1_t for the controlled Markov chain V_t when the cost function is of the form in (17). It is a well known result of Markov decision theory (see [3], Chapter 6) that there is an optimal control law of the form :

$$Z^1_t = f^1_t(V_t) \tag{18}$$

or equivalently,

$$Z^1_t = f^1_t(X^1_t, A, Z^{1,t-1}) \tag{19}$$

We can repeat the same argument for the second encoder to establish the same structural result for the second encoder. ■

B. Second structural Result

As mentioned before, the structural result of equation (10) suffers from the drawback that the domain of the encoding rules f^i_t, $(\mathcal{X}^i \times \mathcal{A} \times \mathcal{Z}^{i,t-1}), i = 1, 2$, keeps increasing with time. We prove a second structural result for the encoders that is free from this drawback.

In the proof of Theorem 1, we fixed the second encoder and the decoder to arbitrary rules and considered the problem of selecting optimal encoding rules of the first encoder. This simpler problem is a classical centralized decision making problem allowing us to use results from Markov decision theory to obtain the first structural result.

To further refine this result, we will keep the second encoder fixed to an arbitrary rule (of the from in (10)) and consider the problem of finding an optimal encoding rule of the first encoder and an optimal decoding rule for the decoder. Because of the first structural result on the encoder and the structural result of the decoder, we will only consider encoding rules of the form $Z^i_t = f^i_t(X^i_t, A, Z^{i,t-1})$ and a decoding rule that uses the receiver's belief on the source at time t, (ψ_t), to make the estimate $\hat{X}_t$. Note that the receiver's belief on source depends on the received messages *and* the encoding functions used (equation (8)).

We are now looking for optimal strategies for the first encoder and the decoder. This is a decentralized team problem since the two decision makers - the first encoder and the decoder- make decisions based on different information. We analyze this problem as follows:

Step 1: We convert the above decentralized team problem into a centralized stochastic control problem.

Step 2: We show that the centralized stochastic control problem is a Partially Observed Markov Decision Process (POMDP).

Step 3: We identify an information state for the resulting POMDP and use it to deduce the second structural result.

Below, we elaborate on these steps.

Step 1: We observe that the first encoder and the decoder have some information that they both know in common. At time t, they both know $Z^{1,t-1}$. We now formulate the original decentralized problem as a centralized problem from the perspective of an agent that knows just the common information $Z^{1,t-1}$. We call this fictitious agent the "pre-encoder". The system can now be described as follows : Based on $Z^{1,t-1}$, the pre-encoder selects a pre-encoding function $w_t^1 : \mathcal{X}^1 \times \mathcal{A} \longrightarrow \mathcal{Z}^1$. Once w_t^1 is selected, the encoder observes X_t^1 and A and uses w_t^1 to find

$$Z_t^1 = w_t^1(X_t^1, A) \tag{20}$$

The decoder receives Z_t^1 and Z_t^2 and updates its belief on the source (ψ_t) using the fixed second encoding rule f_t^2 and the pre-encoding rule w_t^1. (See Appendix II for the receiver's belief). Once the decoder forms ψ_t, it selects the estimate $\hat{X}_t$ according to the function τ in equation (9). The system incurs a distortion cost $\rho_t(X_t, \hat{X}_t)$. The pre-encoder's information changes to $Z^{1,t}$.

Viewed in this way, the original decentralized problem is now a centralized with the pre-encoder as the only decision maker. Once w_t^1 has been selected, the encoder and the decoder simply carry out fixed transformations.

Step 2: We proceed to analyse the pre-encoder's optimization problem. For that matter, we define:

$$R_1 := (X_1^1, A) \tag{21}$$

$$R_t := (X_t^1, A, \xi_{t-1}^1) \tag{22}$$

where

$$\xi_{t-1}^1 = Pr(X_{t-1}/Z^{1,t-1}, w^{1,t-1}) \tag{23}$$

for t=2,3..T, and proceed as follows. We first obtain functional relations among different random variables of interest in *Claims 1* and *2* below. These relations are used to prove that R_t is a controlled Markov chain and the instantaneous cost of the system depends on R_t and w_t^1 (*Lemmas 1* and *2* below).

Claim 1: $\xi_t^1 = F_t(Z_t^1, w_t^1, \xi_{t-1}^1) = \hat{F}_t(R_t, w_t^1)$, t=2,3,..,T, where F_t, $\hat{F}_t$ are deterministic functions.

Claim 2: $\psi_t = H_t(X^{2,t}, R_t, w_t^1)$,t=1,2,..,T, where H_t are deterministic functions.

For the proofs of Claim 1 and Claim 2, we refer the reader to [15].

Lemma 1 : R_t is a controlled Markov chain with w_t^1 as control action.

Proof:

$$Pr(R_{t+1}/R^t, w^{1,t}) \tag{24}$$

$$= Pr(X_{t+1}^1, A, \xi_t^1 / X^{1,t}, A, \xi^{1,t-1}, w^{1,t}) \tag{25}$$

$$= Pr(X_{t+1}^1, A / X^{1,t}, A, \xi^{1,t}, w^{1,t}). \\ Pr(\xi_t^1 / X^{1,t}, A, \xi^{1,t-1}, w^{1,t}) \tag{26}$$

$$= P(X_{t+1}^1, A / X^{1,t}, A, \xi^{1,t}, w^{1,t}, R_t) \\ .Pr(\xi_t^1 / X^{1,t}, A, \xi^{1,t-1}, w^{1,t}, R_t) \tag{27}$$

$$= Pr(X_{t+1}^1, A/\xi_t^1, w_t^1, R_t).Pr(\xi_t^1/w_t^1, R_t) \tag{28}$$

$$= Pr(X_{t+1}^1, A, \xi_t^1/w_t^1, R_t) \tag{29}$$

$$= Pr(R_{t+1}/R_t, w_t^1) \tag{30}$$

The equality in (27) follows because the variables in conditioning determine R_t exactly, so its inclusion in the conditioning does not alter the probability. In the first term of (28), because of the Markovian nature of source, one only needs X_t^1 and A in the conditioning (X_t^1 and A are present in R_t) while in the second term of (28) one only needs $w_t^1$and R_t in the conditioning because of *Claim 1*. Equation (30) proves the lemma. Thus R_t is a conditional Markov chain given w_t^1. ■

Lemma 2 : For a fixed $f^{2,t}$ of the form in (10) and a decoder of the form in (7), the cost function can be written as :

$$J(f^{1,T}, f^{2,T}, g^T) = \sum_{t=1}^{T} E[\rho_t^*(R_t, w_t^1)] \tag{31}$$

where ρ_t^* is a deterministic function.

Proof: The cost function can be written as :

$$J(f^{1,T}, f^{2,T}, g^T) = E[\sum_{t=1}^{T} \rho_t(X_t, \hat{X}_t)]$$

$$= \sum_{t=1}^{T} E[\rho_t(X_t, \hat{X}_t)]$$

$$= \sum_{t=1}^{T} E[\rho_t(X_t, \tau(\psi_t)]] \tag{32}$$

$$= \sum_{t=1}^{T} E[\rho_t(X_t^1, X_t^2, A, \tau(H_t(X^{2,t}, R_t, w_t^1)]] \tag{33}$$

$$= \sum_{t=1}^{T} E[\hat{\rho}_t(X^{2,t}, R_t, w_t^1)] \tag{34}$$

where equality in (33) follows because of *Claim 2*, and $\hat{\rho}_t$ in (34) is simply a different representation of the composite function in (33). The expectation in (34) can be evaluated as:

$$\sum_{x^{2,t}\in(\mathcal{X}^2)^t} Pr(x^{2,t}/A).\hat{\rho}_t(x^{2,t}, R_t, w_t^1) \quad (35)$$

$$= \rho_t^*(R_t, w_t^1) \quad (36)$$

where we have used the conditionally independent nature of the source to get the first term in the summation in (35), and (36) follows because $Pr(X^{2,t}/A)$ is a known statistic. Thus we can write (34) as:

$$= \sum_{t=1}^{T} E[\rho_t^*(R_t, w_t^1)] \quad (37)$$

which proves the lemma. ∎

The optimization problem from the pre-encoder's perspective can now be seen as follows. There is an underlying controlled Markov chain R_t for which the pre-encoder has to find the optimal control actions w_t^1 (Lemma 1). The expected cost of an action (w_t^1) at time t is $E[\rho_t^*(R_t, w_t^1)]$. At time t, the Markov chain is in state R_t, the pre-encoder takes an action w_t^1 and makes an observation Z_t^1 which depends only on the state R_t and the action w_t^1. The state then changes to R_{t+1} with the transition statistic depending only on R_t and w_t^1. This is a typical partially observed Markov decision problem.

Step 3: From Markov decision theory (see [3], [7]), we know that $\pi_t = Pr(R_t/Z^{1,t-1}, w^{1,t-1})$ is an information state appropriate for performance evaluation and there is an optimal policy for the pre-encoder of the form:

$$w_t^1 = G_t(\pi_t) \quad (38)$$

We now argue that ξ_{t-1}^1 is an equivalent information state. To show that, we need to show that a) π_t can be obtained from ξ_{t-1}^1. b) ξ_t^1 can be obtained from the ξ_{t-1}^1, the action (w_t^1) at time t and the observation (Z_t^1) at time t ; and c) ξ_{t-1}^1 is a function of the pre-encoder's previous observations ($Z^{1,t-1}$) and actions ($w^{1,t-1}$),

By (23), ξ_{t-1}^1 is a function of the pre-encoder's previous observations ($Z^{1,t-1}$) and actions ($w^{1,t-1}$), and *Claim 1* establishes the required update, that is, $\xi_t^1 = F(Z_t^1, w_t^1, \xi_{t-1}^1)$.

Consider $\pi_t = Pr(R_t/Z^{1,t-1}, w^{1,t-1})$

$$= Pr((X_t^1, A, \xi_{t-1}^1)/Z^{1,t-1}, w^{1,t-1}) \quad (39)$$

Given $Z^{1,t-1}, w^{1,t-1}$, ξ_{t-1}^1 is known exactly, hence (39) can be written as :

$$\pi_t = Pr((X_t^1, A)/Z^{1,t-1}, w^{1,t-1}) \quad (40)$$

$$= \sum_{x\in\mathcal{X}} Pr((X_t^1, A)/X_{t-1} = x). \\ Pr(X_{t-1} = x/\xi_{t-1}^1, Z^{1,t-1}, w^{1,t-1}) \quad (41)$$

$$= \sum_{x\in\mathcal{X}} Pr((X_t^1, A)/X_{t-1} = x).\xi_{t-1}^1(x) \quad (42)$$

where (41) uses the Markov property of the source. Observe that the first term in (42) is a known source statistic and the second term depends only on ξ_{t-1}^1 . Thus, π_t is a deterministic function of ξ_{t-1}^1.

Hence, ξ_{t-1}^1 is an equivalent information state. Therefore, there exists an optimal control law of the form :

$$w_t^1 = G_t^1(\xi_{t-1}^1). \quad (43)$$

∎

We can now state the desired structural result for the encoders.

Theorem 2: With a decoder of the form $\hat{X}_t = \tau(\psi_t)$, there is no loss in optimality in restricting attention to encoders of the form:

$$Z_t^i = f_t^i(X_t^i, A, \xi_{t-1}^i) \quad (44)$$

where $\psi_t = Pr(X_t/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t}))$ and $\xi_{t-1}^i = Pr(X_{t-1}/Z^{i,t-1}, f^{i,t-1}), i = 1, 2$.

Proof: By Theorem 1, one can restrict attention to encoders of the form:

$$Z_t^i = f_t^i(X_t^i, A, Z^{i,t-1}) \quad (45)$$

Consider a fixed encoding rule of the second encoder of the form in (45). Then, by Step 3 above (equation 43), there exists an optimal selection rule of the first pre-encoder of the form :

$$w_t^1 = G_t^1(\xi_{t-1}^1)$$

With this selection rule, the encoded symbol at time t is given as :

$$Z_t^1 = w_t^1(X_t^1, A)$$

$$= G_t^1(\xi_{t-1}^1)(X_t^1, A) \quad (46)$$

$$= f_t^1(X_t^1, A, \xi_{t-1}^1) \quad (47)$$

where (47) is simply another representation of (46). Thus, there exists an optimal encoder of the form in (44) for the first encoder. Consequently, one can restrict attention to encoders of the form in (44) for the first encoder. Now observe that any encoder of the form in (44) is also of the form $Z_t^i = f_t^i(X_t^i, A, Z^{i,t-1})$. Hence with the first encoder as in (44), we can repeat the same argument for the second encoder.

Therefore, by only considering encoders of the form $Z_t^i = f_t^i(X_t^i, A, \xi_{t-1}^i)$, we do not lose optimality. ∎

C. Discussion

It is worthwhile to compare our results with those obtained in [2] for a communication system with a single encoder. The results in [2] are also true for a system with a noiseless channel and no feedback. The key result in [2] is a structural result on the encoder (Theorem 1 of [2]). With the help of this result, the authors have been able to formulate the problem of finding optimal real-time encoding and decoding rules as a centralized optimization problem with receiver as the only control agent. For this centralized problem they present an optimal solution by a dynamic program. This optimal solution has the structural property that the authors proposed in their Theorem 2 of [2].

Our first structural result in Theorem 1 is analogous to Theorem 1 in [2]. However, in spite of this result, we cannot formulate the problem of finding optimal encoding and decoding rules as a centralized optimization problem because of the following reason.

An essential feature of any centralized problem is that all decisions at time t must be made on the basis of the *same information*. In Theorem 2 of [2], the two decisions to be made at time t (a pre-encoding function at the encoder and the source estimate at the decoder) are both based on the same information which is the encoded symbols sent till time (t-1). This is crucial for the centralized formulation proposed in [2]. Now note that in the problem we consider in this paper, the two pre-encoding functions (w_t^1, w_t^2) and the source estimate are based on *different information*. In particular, w_t^1 is selected on the basis of $Z^{1,t-1}$ and w_t^2 on the basis of $Z^{2,t-1}$. This fact of making different decisions based on different information is unavoidable in our problem, and it gives our problem its decentralized nature. It must be emphasized here that the fact that the receiver knows the information of both encoders ($Z^{1,t-1}$ and $Z^{2,t-1}$) does not alter the decentralized nature of the problem. Even though the receiver can choose w_t^1 and w_t^2, it must do so on the basis of different information - $Z^{1,t-1}$ and $Z^{2,t-1}$, respectively. The mere fact that these two decisions could be made at the same location (the receiver) does not remove their informational separation, and even from the receiver's perspective, the problem is still equivalent to one with two separate agents making decisions based on separate information.

The fact that our problem cannot be viewed as a centralized optimization problem has two important implications :

a) Firstly, we had to introduce an imaginary pre-encoder that essentially represents the common information between the receiver and one encoder. This enabled us to identify a structural result similar to Theorem 2 in [2].

b) More importantly, the decentralized nature of the problem makes the task of finding globally optimal real-time encoding and decoding functions considerably more difficult than in [2]. The main difficulty is the identification of an information state that is sufficient for performance evaluation [9]. This difficulty is a fundamental conceptual issue for any decentralized optimization problem. Since there are multiple agents (the encoders and the decoder) taking actions based on different information, the usual information states from Markov decision theory [3] are not appropriate for our problem.

In the next section, we present an information state that is sufficient for performance evaluation and has a time-invariant domain. We then present the resulting sequential decomposition of the global optimization problem.

IV. Global Optimization

The structural results presented in Section III allow us to restrict the space in which one must look for optimal encoding and decoding rules. Now, we want to find globally optimal strategies. Note that for any choice of encoding strategies, the decoder's structural result of (7) and (8) gives us the optimal decoder. Hence, we are looking for globally optimal encoding strategies $f^{1,T}, f^{2,T}$ of the form in equation (44) that along with the corresponding optimal decoder of (7) give the best performance.

We propose a sequential decomposition of the problem since it reduces the complexity of the optimization problem. For that matter, we need to determine an information state appropriate for performance evaluation. Motivated by the approach in [4] and [5], we consider the problem from the point of a fictitious designer who has to select the strategies f_t^1 and $f_t^2, t = 1, 2..T$, without having access to any observations. An information state appropriate for performance evaluation for this designer should satisfy conditions of sequential update and sufficiency for cost evaluation. Specifically, if θ_t is an information state, then we want :

$$\theta_{t+1} = T_t(\theta_t, f_t^1, f_t^2) \tag{48}$$

and

$$E[\rho_t(X_t, \hat{X}_t)] = C_t(\theta_t, f_t^1, f_t^2) \tag{49}$$

If we can find an information state that satisfies (48) and (49), then we can transform the original optimization problem, formulated in II A and II B, into an equivalent deterministic functional optimization problem, namely,

Select $f^{1,T}, f^{2,T}$ to minimize

$$\sum_{t=1}^{T} C_t(\theta_t, f_t^1, f_t^2) \tag{50}$$

subject to ,for t=1,..,T-1,

$$\theta_{t+1} = T_t(\theta_t, f_t^1, f_t^2) \tag{51}$$

For this problem the optimal strategies $f^{1,T}, f^{2,T}$ can be determined as follows.

Theorem 4: For the functional optimization problem described by the equations (50) and (51), the optimal encoding functions f_t^1, f_t^2 are given by the following optimality equations:

$$V_{T+1}(\theta) = 0 \tag{52}$$

$$V_t(\theta) = \inf_{f_t^1, f_t^2} [C_t(\theta, f_t^1, f_t^2) + V_{t+1}(T_t(\theta_t, f_t^1, f_t^2))] \tag{53}$$

for t =1,2..,T, where $f_t^i \in \mathcal{F}_t^i$ and $\mathcal{F}_t^i$ is the set of functions of the form $Z_t^i = f_t^i(X_t^i, A, \xi_{t-1}^i)$

Proof: For the system in (50) and (51), a standard dynamic programming argument results in the above optimality equations. The optimal actions (f_t^1, f_t^2) for the information state θ at time t minimize the instantaneous cost $C_t(\theta, f_t^1, f_t^2)$ and the future cost to go $V_{t+1}(T_t(\theta_t, f_t^1, f_t^2))$. This is a standard result (see [8], Chapter 2). ■

To proceed further we start with the following Claim.

Claim 3: $\psi_t = \hat{B}_t(X_t, \psi_{t-1}, \xi_{t-1}^1, \xi_{t-1}^2, f_t^1, f_t^2)$, where $\hat{B}_t$ are deterministic transformations.

Proof: See Appendix III.

We now present an information state for the designer. For that matter, we define:

$$\theta_t = Pr(X_{t-1}, \xi^1_{t-1}, \xi^2_{t-1}, \psi_{t-1}) \tag{54}$$

for t=2,3..T, and $\theta_1 = Pr(X_0)$, where X_0 is a arbitrary fixed initial state of the Markov source before the start of time.

We show that θ_t satisfies (48) and (49). Thus, from the arguments above, one can use the result of Theorem 4 to sequentially determine the optimal encoding functions $f^{1,T}, f^{2,T}$.

Lemma 3 : $\theta_t = Pr(X_{t-1}, \xi^1_{t-1}, \xi^2_{t-1}, \psi_{t-1})$ satisfies equations (48) and (49).

Proof: By definition,

$$\theta_{t+1} = Pr(X_t, \xi^1_t, \xi^2_t, \psi_t)$$

Using *Claim 1* and *Claim 3*, we can write :

$$(\xi^1_t, \xi^2_t, \psi_t) = Q^{f^1_t, f^2_t}(X_t, \xi^1_{t-1}, \xi^2_{t-1}, \psi_{t-1}) \tag{55}$$

where the transformation $Q^{f^1_t, f^2_t}$ is derived from the transformations $\hat{F}_t$ and $\hat{T}_t$ of *Claim 1* and *Claim 3*.($Q^{f^1_t, f^2_t}(X_t, \xi^1_{t-1}, \xi^2_{t-1}, \psi_{t-1})$ gives $\hat{F}_t(X^1_t, A, f^1_t)$, $\hat{F}_t(X^2_t, A, f^2_t)$ and $\hat{T}_t(X_t, \psi_{t-1}, \xi^1_{t-1}, \xi^2_{t-1}, f^1_t, f^2_t)$). Hence,

$$\begin{aligned}\theta_{t+1} &= Pr(X_t, \xi^1_t, \xi^2_t, \psi_t) \\ &= Pr(X_t, Q^{f^1_t, f^2_t}(X_t, \xi^1_{t-1}, \xi^2_{t-1}, \psi_{t-1})) \quad (56)\end{aligned}$$

$$= \tilde{T}_t(Pr(X_t, \xi^1_{t-1}, \xi^2_{t-1}, \psi_{t-1}), f^1_t, f^2_t) \tag{57}$$

where (57) simply states that the probability of a function of random variables can be obtained from the joint probability of the random variables and the function[1]. Therefore,

$$\begin{aligned}\theta_{t+1} = \tilde{T}_t(&\sum_{x \in \mathcal{X}} Pr(X_t/X_{t-1} = x). \\ &Pr(X_{t-1} = x, \xi^1_{t-1}, \xi^2_{t-1}, \psi_{t-1}), f^1_t, f^2_t) \quad (58)\end{aligned}$$

$$= \tilde{T}_t(\sum_{x \in \mathcal{X}} Pr(X_t/X_{t-1} = x).\theta_t(x, \xi^1_{t-1}, \xi^2_{t-1}, \psi_{t-1}) , f^1_t, f^2_t) \tag{59}$$

Since $Pr(X_t/X_{t-1} = x)$ is given by the known statistical description of the source, (59) implies

$$\theta_{t+1} = T_t(\theta_t, f^1_t, f^2_t) \tag{60}$$

Now, consider

$$E[\rho_t(X_t, \hat{X}_t)] = E[\rho_t(X_t, \tau(\psi_t))] \tag{61}$$

The expectation in (61) is a function of the joint distribution of X_t and ψ_t which is a marginal of $Pr(X_t, \xi^1_t, \xi^2_t, \psi_t)$. Hence,

$$\begin{aligned}E[\rho_t(X_t, \hat{X}_t)] &= \tilde{C}_t(Pr(X_t, \psi_t)) && (62)\\ &= \hat{C}_t(Pr(X_t, \xi^1_t, \xi^2_t, \psi_t)) && (63)\\ &= \hat{C}_t(\theta_{t+1}) && (64)\\ &= C_t(\theta_t, f^1_t, f^2_t) && (65)\end{aligned}$$

[1]For the specific form of $\tilde{T}_t$, see [15]

where the equality in (65) is a consequence of (61).

(For the specific form of the functions T_t and C_t, we refer the reader to [15].) ■

V. CONCLUSION

We have discovered the structure of optimal real-time encoders and decoders for the multi-terminal communication system considered in this paper. The structure of the Markov source, the nature of encoders' observations and the noiseless nature of the channel are critical in obtaining the results of Section III for the following reasons. In general, to determine its encoding rule at any time t, each encoder must form a belief about the information of the other encoder and the receiver's information. The conditional independence of X^1_t and X^2_t on A allows each encoder to use only the value of the random variable A to form a belief about the other encoder's information. The noiseless nature of the communication channel allows each encoder at any time t to use the value of the random variable A and its previous transmissions (up to time t-1) to form its belief on the receiver's information. These considerations lead to the structural results of Theorem 1 and Theorem 2. The structural result of Theorem 2 plays an important role in identifying an information state that has a time-invariant domain and is appropriate for performance evaluation. Such an information state is appropriate for obtaining a sequential decomposition of the finite horizon global optimization problem (considered in this paper) as well as of the corresponding infinite horizon problem.

The problem of global optimization is significantly more difficult than the one considered in [2] because of the following reason. The presence of two encoders with different information implies that encoding decisions have to be necessarily based on different information. It is this fact that makes our problem decentralized. Decentralized optimization problems are considerably more challenging than centralized problems and the information states appropriate for them are more complicated than their centralized counterparts. We presented an information state for the global optimization problem and obtained a methodology that allows us to sequentially determine globally optimal real-time encoding and decoding strategies. The results of this paper can be extended to multi-terminal systems consisting of N encoders, communication with one receiver by noiseless channels, general distortion metrics and Markov sources of the form $X_t := (X^1_t, X^2_t, ..., X^N_t, A)$, where A does not change with time and conditioned on A, $X^1_t, X^2_t, ..., X^N_t$ form independent Markov chains.

APPENDIX I
STRUCTURAL RESULT FOR THE DECODER

Observe that with fixed encoding rules, minimizing $J(f^{1,T}, f^{2,T}, g^T) = E[\sum_{t=1}^{T} \rho_t(X_t, \hat{X}_t)]$ is equivalent to minimizing $E[\rho_t(X_t, \hat{X}_t)]$ for each t. This can be minimized by minimizing $E[\rho_t(X_t, \hat{X}_t)/Z^{1,t}, Z^{2,t}]$

for each $Z^{1,t}, Z^{2,t}$. The structural property of the decoder then follows from the definition of ψ_t and τ in (8) and (9).

APPENDIX II
RECEIVER'S BELIEF UPDATE

By definition,

$$\psi_t := Pr(X_t/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t})$$

By Bayes' rule,

$$\psi_t = Pr(X_t, Z_t^1, Z_t^2/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t})/ \sum_{x\in\mathcal{X}} Pr(x, Z_t^1, Z_t^2/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}) \quad (66)$$

We can write the numerator as:

$$\begin{aligned} &Pr(Z_t^1, Z_t^2/X_t, Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}).\\ &\quad Pr(X_t/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t})\\ &= Pr(Z_t^1/X_t, Z^{1,t-1}, w_t^1).Pr(Z_t^2/X_t, Z^{2,t-1}, f_t^2).\\ &\quad \sum_{x\in\mathcal{X}} Pr(X_t/X_{t-1}=x).\\ &\quad Pr(X_{t-1}=x/Z^{1,t-1}, Z^{2,t-1}, f^{1,t-1}, f^{2,t-1}) \quad (67)\end{aligned}$$

The last term on the right side of the above equation is simply ψ_{t-1}. Thus, the receiver can update its belief on the source based on the pre-encoding function (w_t^1) used at time t and the fixed rule of the second encoder (f_t^2).

APPENDIX III
PROOF OF CLAIM 3

By definition,

$$\begin{aligned}\psi_t(y) &= Pr(X_t = y/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t})\\ &= Pr(X_t = y/Z^{1,t}, Z^{2,t}, f^{1,t}, f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2) \quad (68)\end{aligned}$$

We can introduce ξ_{t-1}^1, ξ_{t-1}^2 in the conditioning in (68) since they are functions of the conditioning variables. By Bayes' rule, we have

$$\psi_t(y) = Pr(X_t = y, Z_t^1, Z_t^2/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2)/ \sum_{x\in\mathcal{X}} Pr(X_t = x, Z_t^1, Z_t^2/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}.f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2)$$

We can write the numerator as:

$$\begin{aligned}&Pr(Z_t^1, Z_t^2/X_t = y, Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2).\\ &\quad Pr(X_t = y/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2) \quad (69)\end{aligned}$$

$$\begin{aligned}&= Pr(Z_t^1, Z_t^2/X_t = y, f_t^1, f_t^2, \xi_{t-1}^1, \xi_{t-1}^2).\\ &\quad \sum_{x\in\mathcal{X}} Pr(X_t = y/X_{t-1} = x).\\ &\quad Pr(X_{t-1} = x/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2) \quad (70)\end{aligned}$$

The first term in (70) is because of the structural result of the encoders and the second by the Markov nature of the source. Observe that the first term in (70) is either 1 or 0, the first term in the summation is the source statistic known apriori and the second term in the summation is $\psi_{t-1}(x)$. Therefore, (70) is equal to

$$\begin{aligned}&1_{[Z_t^1 = f_t^1(y^1, a, \xi_{t-1}^1), Z_t^2 = f_t^2(y^2, a, \xi_{t-1}^2)]}.\\ &\quad \sum_{x\in\mathcal{X}} Pr(X_t = y/X_{t-1} = x).\\ &Pr(X_{t-1} = x/Z^{1,t-1}, Z^{2,t-1}, f^{1,t}, f^{2,t}, \xi_{t-1}^1, \xi_{t-1}^2) \quad (71)\end{aligned}$$

where $y = (y^1, y^2, a)$ The same holds true for each term in the summation in the denominator of (66). Since Z_t^i is simply $f_t^i(X_t^i, A, \xi_{t-1}^i)$, the indicator function in the above expression is $1_{[f_t^1(X_t^1, A, \xi_{t-1}^1) = f_t^1(y^1, a, \xi_{t-1}^1), f_t^2(X_t^2, A, \xi_{t-1}^2) = f_t^2(y^2, a, \xi_{t-1}^2)]}$ and we conclude that ψ_t^1 is a function of $X_t, f_t^1, f_t^2, \xi_{t-1}^1, \xi_{t-1}^2, \psi_{t-1}$.That is,

$$\psi_t = \hat{B}_t(X_t, \psi_{t-1}, \xi_{t-1}^1, \xi_{t-1}^2, f_t^1, f_t^2) \quad (72)$$

■

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