Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Almost Invariant Subspaces: -High Gain Feedback? or -Singularly Perturbed Feedback?

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***Abstract*— We show that *almost controllability* can be also obtained by means of singularly perturbed state feedbacks which are approximations of Proportional and Derivative (PD) state feedbacks.**

I. Introduction

With the seminal papers of Brunovsky [3] and Morse [11] began the structural study of linear systems. They made it possible to tackle control problems from a very formal point of view, and to understand how systems structures play a deep role in the solvability of such control problems.

In particular [11] is one of the key papers about structure and geometric approach. More precisely, some important structural properties can be interpreted in terms of the (A, B)*–Invariant* and *Controllability Subspaces*, which are related with the maps of the state space representations of the systems. In a very simplistic way, these subspaces tell us which are the parts of the system, which can be made unobservable (made invariant inside of the kernel of the output map) by state feedback, and for some part with assignable dynamics. This was the starting point for a systematic study of the structure of linear systems. In the important works of Wonham [17] and Marro [2] the principal results of the geometric approach are summarized.

A second milestone occurred with Willems' introduction of the *Almost* (A, B)*–Invariant* and *Almost Controllability Subspaces*, which are related with the maps of the state space representations of the systems [14], [15], [16]. These subspaces are useful when non exact solutions are looked for. *Almost invariance* and *almost controllability* have been connected with the use of high gain state feedback, as approximations of distributional state feedbacks.

The aim of this paper is to show that *almost controllability* can be also obtained by means of singularly perturbed state feedbacks [9] which are approximations of PD state feedbacks [10]. For this, in Section II is recalled the class of systems which we are going to deal with; we use the behavioral approach [12], since it clarifies the action of the involved control laws. In Sections III and IV are presented some basic properties of *almost controllability subspace*. In Section V we give another interpretation of an *almost controllability subspace* in terms of PD state feedbacks. And thus, in Section VI we interpret the *almost controllability subspace* in terms of a singularly perturbed state feedback. In Section VII a simple example is used to illustrate the basic ideas. And in Section VIII we conclude.

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II. Systems

1) Input/State System : An *input/state system*, $\Sigma_{i/s} = \left(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_{fed},\, \mathfrak{B}^{exp}_{[A,B]}\right)$, is a dynamical system defined by the state space representation [12]:

$$\mathrm{d}x_f/\mathrm{d}t = Ax_f + Bu \qquad (1)$$

where $u \in \mathcal{U} \approx \mathbb{R}^m$ is the input variable and $x_f \in \mathcal{X}_{fed} \approx \mathbb{R}^{n_f}$ is the state variable; in this paper it is assumed that the input map B is monic. From the Kronecker theory [6], the associated pencil, $[\lambda \mathrm{I} - A]$, $\lambda \in \mathbb{C}$, only contains finite elementary divisors (integral actions), *fed*. The exponential behavior, $\mathfrak{B}^{exp}_{[A,B]}$, is:[1]

$$\mathfrak{B}^{exp}_{[A,B]} = \Big\{ (u, x_f) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_{fed}) \Big| \; \exists \; x_0 \in \mathcal{X}_{fed} \\ \text{s.t. } x_f(t) = \mathrm{e}^{At}x_0 + \textstyle\int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau \Big\}$$

Sometimes the input variable is decomposed into two components, $u \in \mathcal{U}$ and $q \in \mathcal{Q} \approx \mathbb{R}^\nu$; the first one is free to be used as a controller signal (called controller input or simply input) and the second one is behaving at will (called disturbance input or simply disturbance). In this case we write $\Sigma_{i/s} = \left(\mathbb{R}^+, [\mathcal{U} \times \mathcal{Q}] \times \mathcal{X}_{fed},\, \mathfrak{B}^{exp}_{[A,[B\ S]]}\right)$, being the state space representation: $\mathrm{d}x_f/\mathrm{d}t = Ax_f + Bu + Sq$.

It is also usual to add to the state space representation (1) an output variable, $y \in \mathbb{R}^p$, by means of an output equation: $y = Cx_f + Du$. In this case, we get an *input/state/output system*, $\Sigma_{i/s/o} = \left(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_{fed} \times \mathcal{Y},\, \mathfrak{B}^{exp}_{[A,B,C,D]}\right)$. The exponential behavior is:

$$\mathfrak{B}^{exp}_{[A,B,C,D]} = \Big\{ (u, x_f, y) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_{fed} \times \mathcal{Y}) \Big| \\ \exists \; (u, x_f) \in \mathfrak{B}^{exp}_{[A,B]} \text{ s.t. } y(t) = Cx_f(t) + Du(t) \Big\}$$

[1] $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m)$ stands for the locally integrable functions $v : \mathbb{R}^+ \to \mathcal{W}$.

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The smooth exponential behaviors are defined as:[2] $\widetilde{\mathfrak{B}}^{exp}_{[A,B]} = \mathfrak{B}^{exp}_{[A,B]} \cap \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_{fed})$ and $\widetilde{\mathfrak{B}}^{exp}_{[A,B,C,D]} = \mathfrak{B}^{exp}_{[A,B,C,D]} \cap \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_{fed} \times \mathcal{Y})$. If $D=0$, we write $\mathfrak{B}^{exp}_{[A,B,C]}$ and $\widetilde{\mathfrak{B}}^{exp}_{[A,B,C]}$.

2) Input/State Distributional System : In [13] is considered the *input/state distributional system* (see also [7], [8]; for the singular systems case see [4]):[3] $\widehat{\Sigma}_{i/s} = \left(\mathfrak{T}_{est}, \mathcal{U} \times \mathcal{X}, \widehat{\mathfrak{B}}^{exp}_{[A,B]}\right)$, $\mathcal{U} \approx \mathbb{R}^m$ and $\mathcal{X} \approx \mathbb{R}^{n_f}$, with state space representation (1), where the space of admissible inputs is:[4]

$$\widehat{\mathfrak{U}} = \Big\{ \hat{u} \in \mathfrak{D}_{ist}(\mathfrak{T}_{est}, \mathcal{U}) |\ \hat{u} = u^- + \hat{u}^+,\ u^- \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathcal{U}), \\ \mathrm{supp}\ u^- \subset \mathbb{R}^-, \text{ and } \hat{u}^+ \in \mathfrak{D}_{ist+}(\mathfrak{T}_{est}, \mathcal{U}) \Big\}$$

The exponential distributional behavior, $\widehat{\mathfrak{B}}^{exp}_{[A,B]}$, is:

$$\widehat{\mathfrak{B}}^{exp}_{[A,B]} = \Big\{ (\hat{u}, \hat{x}_f) \in \mathfrak{D}_{ist}(\mathfrak{T}_{est}, \mathcal{U} \times \mathcal{X}_{fed}) \big|\ \exists\ \hat{u} \in \widehat{\mathfrak{U}}\ \& \\ x_0 \in \mathcal{X}_{fed} \text{ s.t. } \hat{x}_f = x^- + \hat{x}^+, \\ x^- = \mathrm{e}^{At} 1_{\mathbb{R}^-}(t) x_0 - \int_t^0 \mathrm{e}^{A(t-\tau)} B(\tau) u^- \mathrm{d}\tau, \\ \hat{x}^+ = \mathrm{e}^{At} 1_{\mathbb{R}^+}(t) x_0 + \int_0^t \mathrm{e}^{A(t-\tau)} B(\tau) \hat{u}^+ \mathrm{d}\tau \Big\}$$

3) Regular Input/Descriptor System : A *regular input/descriptor system*, $\Sigma_{i/d} = \left(\mathbb{R}^+, \mathcal{U} \times [\mathcal{X}_{fed} \times \mathcal{X}_\infty], \widetilde{\mathfrak{B}}^{exp}_{[A,B]} \oplus \widetilde{\mathfrak{B}}^{pol}_{[N,\Gamma]}\right)$, is a dynamical system defined by the descriptor space representation (expressed in its Weierstrass form) [6]:

$$\mathrm{d}x_f/\mathrm{d}t = Ax_f + Bu \quad ; \quad N\mathrm{d}x_\infty/\mathrm{d}t = x_\infty - \Gamma u \\ x = \begin{bmatrix} x_f^T & x_\infty^T \end{bmatrix}^T \tag{2}$$

where $u \in \mathcal{U} \approx \mathbb{R}^m$ is the input variable and $x \in \mathcal{X}_d = \mathcal{X}_{fed} \oplus \mathcal{X}_{ied} \approx \mathbb{R}^{n_f + n_\infty}$ is the descriptor variable. Its associated pencil [6], $[\lambda N - \mathrm{I}]$, $\lambda \in \mathbb{C}$, only contains infinite elementary divisors (derivative actions), *ied*. The polynomial behavior, $\widetilde{\mathfrak{B}}^{pol}_{[N,\Gamma]}$, is:

$$\widetilde{\mathfrak{B}}^{pol}_{[N,\Gamma]} = \Big\{ (u, x_\infty) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_{ied}) \big|\ x_\infty(t) = \Gamma u(t) + \\ \sum_{j=1}^{n_\infty - 1} N^j \Gamma \frac{\mathrm{d}^j}{\mathrm{d}t^j} u(t) \Big\}$$

In the general case, the *input/descriptor systems* are systems with behavioral equation $\mathbb{E}\mathrm{d}x/\mathrm{d}t = \mathbb{A}x + \mathbb{B}u$, where its associated pencil, $[\lambda\mathbb{E} - \mathbb{A}]$, can be singular, even rectangular, having four types of structural invariants [6]: (*i*) *fed*, (*ii*) *ied*, (*iii*) row minimal indices (variable internal structure), *rmi*, and (*iv*) column minimal indices (algebraic restrictions on the descriptor variable), *cmi*. In this general case, the behavior can be specified by using the differential inclusion theory, as *e.g.* in [5].

If we add to (2) an output variable, $y \in \mathbb{R}^p$, by means of the output equation: $y = \begin{bmatrix} C & \Theta \end{bmatrix} x$, we get a $\Sigma_{i/d/o} = \left(\mathbb{R}^+, \mathcal{U} \times [\mathcal{X}_{fed} \times \mathcal{X}_\infty] \times \mathcal{Y}, \widetilde{\mathfrak{B}}^{exp}_{[A,B,C]} \oplus \widetilde{\mathfrak{B}}^{pol}_{[N,\Gamma,\Theta]}\right)$. The polynomial behavior is:

$$\widetilde{\mathfrak{B}}^{pol}_{[N,\Gamma,\Theta]} = \Big\{ (u, x_\infty, y_\infty) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_{ied} \times \mathcal{Y}) \big| \\ \exists\ (u, x_\infty) \in \widetilde{\mathfrak{B}}^{pol}_{[N,\Gamma]} \text{ s.t. } y_\infty(t) = \Theta x_f(t) \Big\}$$

[2] $\mathcal{C}^\infty(\mathbb{R}^+, \mathcal{W})$ is the space of infinitely differentiable functions $v : \mathbb{R}^+ \to \mathcal{W}$.

[3] $\mathfrak{T}_{est}$ is the space of test functions.

[4] $\mathfrak{D}_{ist}$ is the space of distributions.

III. Almost Controllability Subspaces

Let us write the definition and some geometric characterizations of the almost controllability subspaces:

Definition 1 ([14]): A subspace $\mathcal{R}_a \subset \mathcal{X}_{fed}$ is said to be an almost controllability subspace if $\forall\ x_0,\ x_1 \in \mathcal{R}_a$, $\exists\ T > 0$ such that $\forall\ \rho > 0\ \exists\ x_f \in \mathfrak{B}^{exp}_{[A,B]}$ with the properties that $x_f(0) = x_0$, $x_f(T) = x_1$ and $\sup_{t \in \mathbb{R}^+} \inf_{x' \in \mathcal{R}_a} \|x_f(t) - x'\| \leq \rho$.

Let $\mathcal{K}$ be a subspace of $\mathcal{X}_{fed}$, then the subspace $\mathcal{S}^\infty_\mathcal{K}$ is the limit of the non decreasing *almost controllability subspace algorithm*:

$$\mathcal{S}^0 = \{0\}\ ;\ \ \mathcal{S}^{\mu+1} = \mathcal{K} \cap (A\mathcal{S}^\mu + \mathrm{Im}\ B),\ \mu \in \mathbb{Z}^{*+} \qquad \text{[ACSA]}$$

Corollary 2 ([14], Corollary 1.23 of [13]): A subspace $\mathcal{R}_a$ of $\mathcal{X}_{fed}$ is an almost controllability subspace if and only if there is a mapping $F : \mathcal{X}_{fed} \to \mathcal{U}$ and a chain $\{\mathcal{B}_i\}_{i=1}^k$ in $\mathrm{Im}\ B$ such that $\mathcal{R}_a = \mathcal{B}_1 + A_F \mathcal{B}_2 + \cdots + A_F^{k-1} \mathcal{B}_k$. Moreover, there exist a $k \in \mathbb{Z}^{*+} \cup \{0\}$, $k \leq \dim \mathcal{R}_a$, a chain $\{\mathcal{B}_i\}_{i=1}^k$ in $\mathrm{Im}\ B$ and a mapping $F^* : \mathcal{X}_{fed} \to \mathcal{U}$ such that

$$\mathcal{R}_a = \mathcal{B}_1 \oplus A_{F^*} \mathcal{B}_2 \oplus \cdots \oplus A_{F^*}^{k-1} \mathcal{B}_k \tag{3}$$

$$\mathcal{B}_1 = \mathcal{R}_a \cap \mathrm{Im}\ B \tag{4}$$

$$\dim \mathcal{B}_i = \dim A_F^{i-1} \mathcal{B}_i \\ = \dim \mathcal{S}^i - \dim \mathcal{S}^{i-1}\ ,\ \ i \in \{1, \ldots, k\} \tag{5}$$

where the $\mathcal{S}^i$ are the steps of [ACSA] with $\mathcal{K} = \mathcal{R}_a$.

Theorem 3 ([14], Theorem 1.24 of [13]): Let $\mathcal{K}$ be a subspace of $\mathcal{X}_{fed}$ and $\mathcal{R}^*_{a,\mathcal{K}}$ be the supremal almost controllability subspace contained in $\mathcal{K}$. Then:

$$\mathcal{R}^*_{a,\mathcal{K}} = \Big\{ x_0 \in \mathcal{K} \big|\ \forall\ \rho > 0\ \exists\ x_f \in \mathfrak{B}^{exp}_{[A,B]},\ x_f(0) = x_0, \\ \text{such that } x_f(T) = 0 \text{ and } \mathrm{d}_\infty(x_f, \mathcal{K}) \leq \rho \Big\}$$

Moreover, $\mathcal{R}^*_{a,\mathcal{K}} = \mathcal{S}^\infty_\mathcal{K}$.

The following Lemma gives a nice space decomposition, in terms of a suitable feedback, which will enable us to get some important structural conclusions:

Lemma 4 ((Lemma 1.15 of [13])): Let $\mathcal{K}$ be a subspace of $\mathcal{X}_{fed}$. There are subspaces $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ of $\mathcal{X}_{fed}$ and $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ of $\mathcal{U}$, a linear mapping $F^* : \mathcal{X}_{fed} \to \mathcal{U}$, an integer $k \leq \dim \mathcal{K}$ and integers r_i, such that:

1) $\mathcal{S}^\infty_\mathcal{K} = \mathcal{X}_1 \oplus \mathcal{X}_2$,

2) $\mathcal{X}_{fed} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$,

3) $A_{F^*} \mathcal{X}_1 \subset \mathcal{X}_1 \oplus \mathcal{X}_2$,

4) $B\mathcal{U}_i \subset \mathcal{X}_i$, $i \in \{1, 2, 3\}$

5) When applying the state feedback $u = F^* x_f + u^*$ to (1), then under the decompositions $\mathcal{X}_{fed} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ and $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{U}_3$, the state space representation is:

$$\mathrm{d}x_f/\mathrm{d}t = A_{F^*} x_f + Bu^* \tag{6}$$

$$A_{F^*} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix},\ B = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix} \tag{7}$$

where:[5]

[5] These geometric properties directly follow from the matricial expressions of Trentelman. For example, for item a): $\mathrm{Ker}\ [A_{21}\ B_2]^T = \{0\}$ implies $\mathcal{X}_2 = \mathrm{Im}\ A_{21} + \mathrm{Im}\ B_2$ and $\dim \mathcal{X}_2 = \mathrm{rank}\ A_{21} + \mathrm{rank}\ B_2$ implies $\mathcal{X}_2 = \mathrm{Im}\ A_{21} \oplus \mathrm{Im}\ B_2$.

(a) $\mathcal{X}_2 = \text{Im } A_{21} \oplus \text{Im } B_2$,

(b) Let $\overline{A}_{21} = P_{A_{21}} A_{21}$, where $P_{A_{21}}$ is the natural projection on $\text{Im } A_{21}$ along $\text{Im } B_2$, then $\mathcal{X}_1 = A_{11}^{-1}\text{Im } B_1 \oplus \text{Ker } \overline{A}_{21}$ and $\text{Im } \overline{A}_{21} \approx \text{Im } B_1$.

(c) The associated pencil, $\left[\begin{array}{c|c} \lambda \mathrm{I} - A_{11} & -B_1 \\ \hline \overline{A}_{21} & \end{array}\right]$, $\lambda \in \mathbb{C}$, only contains *ied*, namely the standard controllable triple $(\overline{A}_{21}, A_{11}, B_1)$ is prime.

Morse [11] introduced the prime systems, which roughly speaking are controllable and observable systems, represented by a (C, A, B) state space form. Moreover, his Theorem 3.1 shows that there exists a state feedback F such that:[6] $(A + BF) \sim BDM\{A_1, \ldots, A_m\}$, $B \sim BDM\{B_1, \ldots, B_m\}$, and $C \sim BDM\{C_1, \ldots, C_m\}$; where:

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & \cdots & 0 & 1 \\ 0 & \cdot & \cdot & \cdots & 0 \end{bmatrix},\ B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ C_i^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{8}$$

IV. Distributional Input and High Gain Feedback

Let $\mathcal{K}$ be a subspace of $\mathcal{X}_{fed}$ and let $x_0 \in \mathcal{S}_{\mathcal{K}}^{\infty}$ be an initial condition for (6), namely $x_f(0) = V_{\mathcal{S}_{\mathcal{K}}^{\infty}} x_0$, where $V_{\mathcal{S}_{\mathcal{K}}^{\infty}} : \mathcal{S}_{\mathcal{K}}^{\infty} \to \mathcal{X}_{fed}$ is the insertion map.

In [15] the supremal almost controllability subspace contained in $\mathcal{K}$, $\mathcal{S}_{\mathcal{K}}^{\infty}$, is related with the ability of bringing instantaneously any initial condition, $x_0 \in \mathcal{S}_{\mathcal{K}}^{\infty}$, to zero by means of a suitable distributional input $\hat{u}^* \in \widehat{\mathfrak{U}}$. Indeed, in [13] is proposed the distributional control law (recall Corollary 2):[7] $\hat{u}^* = -\alpha_1 \delta B u_1 - \sum_{i=1}^{k-1} \alpha_{i+1} \delta^{(i)} B u_{i+1} \in \widehat{\mathfrak{U}}$; where the $\alpha_i \in \mathbb{R}$ and the $u_i \in B^{-1}\mathcal{B}_i$ are such that $V_{\mathcal{S}_{\mathcal{K}}^{\infty}} x_0 = \sum_{i=1}^{k} \alpha_i A_{F^*}^{i-1} B u_i$. Leading to $(\hat{u}^*, \hat{x}_f) = (\hat{u}^*,\ \mathrm{e}^{A_{F^*} t} 1_{\mathbb{R}^-}(t) V_{\mathcal{S}_{\mathcal{K}}^{\infty}} x_0) \in \widehat{\mathfrak{B}}^{exp}_{[A_{F^*}, B]}$.

In [13], it is shown how to approximate, in the *generalized limit* sense, the distributional control law, $\hat{u}^* \in \widehat{\mathfrak{U}}$, by a sequence of smooth inputs, $u_n^*(t) = -\alpha_1 \varphi_n(t) B u_1 - \sum_{i=1}^{k-1} \alpha_{i+1} \mathrm{d}^i \varphi_n(t)/\mathrm{d}t^i B u_{i+1} \in \widetilde{\mathfrak{U}}$, $n \in \mathbb{Z}^{*+}$; where the φ_n are non negative $\mathcal{C}^{\infty}$ functions of unit area, with $\text{supp } \varphi_n \subset [0, 1/n]$ and $\varphi_n(0) = 0$ and $\mathrm{d}^i \varphi_n(0)/\mathrm{d}t^i = 0$, $i \in [|1, k|]$. Leading to $(u_n^*, x_{f,n}^*) = \Big(u_n^*,\ \mathrm{e}^{A_{F^*} t} V_{\mathcal{S}_{\mathcal{K}}^{\infty}} x_0 - \big(\int_0^t \mathrm{e}^{A_{F^*}(t-\tau)} \varphi_n(\tau) \mathrm{d}\tau\big) V_{\mathcal{S}_{\mathcal{K}}^{\infty}} x_0 - \sum_{j=1}^{k} \big(\varphi_n(t) \alpha_j A_{F^*}^{j-1} B u_j + \sum_{i=j+1}^{k} \mathrm{d}^{i-j} \varphi_n(t)/\mathrm{d}t^{i-j} \alpha_i A_{F^*}^{j-1} B u_j\big)\Big) \in \widetilde{\mathfrak{B}}^{exp}_{[A_{F^*}, B]}$, $n \in \mathbb{Z}^{*+}$; with the property that for every $\rho \in \mathbb{R}^{*+}$ there exist $T \in \mathbb{R}^{*+}$ and $N \in \mathbb{Z}^{*+}$, such that $x_{f,n}^*(T) = 0$ and $\mathrm{d}_{\infty}(x_{f,n}, \mathcal{S}_{\mathcal{K}}^{\infty}) \leq \rho$, for every $n \geq N$.

Furthermore, following [15], the next Theorem shows that the distributional control law is also approximated by a sequence of high gain state feedbacks:

Theorem 5 (Theorem 2.35 of [13]): Let $\mathcal{R}_a$ be an almost controllability subspace and $F^* : \mathcal{X}_{fed} \to \mathcal{U}$ be a mapping satisfying (3)–(5). Let $x_0 \in \mathcal{R}_a$ be an initial condition for the state space representation (6) and let $\{\mathcal{L}_n\}$, $n \geq N$, $N \in \mathbb{Z}^+$, be a sequence of subspaces[8] generated by the sequences of vectors $\{x_{1,j}(n, \bar{u}_j), \ldots, x_{k,j}(n, \bar{u}_j)\}$, where $x_{1,j}(n, \bar{u}_j) = \big(\mathrm{I} - (1/n) A_{F^*}\big)^{-1} B \bar{u}_j$, $x_{i+1,j}(n, \bar{u}_j) = \big(\mathrm{I} - (1/n) A_{F^*}\big)^{-1} A_{F^*} x_{i,j}(n, \bar{u}_j)$, $i \in [|1, k|]$, and $B\bar{u}_j \in \{\mathcal{B}_i\}_{i=1}^{k}$; such that the map $\big(\mathrm{I} - (1/n) A_{F^*}\big)$ is invertible and $x_0 \in \mathcal{L}_n$, for all $n \geq N$. Let a sequence of friends mappings of the $\mathcal{L}_n$, $F_n : \mathcal{L}_n \to \mathcal{U}$, such that $F_n x_{i,j}(n, \bar{u}_j) = -n^i \bar{u}_j$. Then for all $\rho > 0$ there exists a $N \in \mathbb{Z}^{*+}$ such that $\mathrm{d}_{\infty}(x_f, \mathcal{R}_a) \leq \rho$, $x_f \in \mathfrak{B}^{exp}_{[A_{F^*} + BF_n,\ B]}$, for all $n \geq N$.

V. Smooth Input and PD Feedback

In the proof of Lemma 4, Trentelman [13] comments that given $x_{f,2} \in \mathcal{C}^{\infty}(\mathbb{R}^+, \mathcal{X}_2)$ and $u_3 \in \mathcal{C}^{\infty}(\mathbb{R}^+, \mathcal{U}_3)$ the dynamic constraints, $\mathrm{d}x_{f,2}/\mathrm{d}t = A_{21} x_{f,1} + A_{22} x_{f,2} + A_{23} x_{f,3} + B_2 u_2$ and $\mathrm{d}x_{f,1}/\mathrm{d}t = A_{11} x_{f,1} + A_{12} x_{f,2} + A_{13} x_{f,3} + B_1 u_1$, yield unique solutions $x_{f,1} \in \mathcal{C}^{\infty}(\mathbb{R}^+, \mathcal{X}_1)$, $u_1 \in \mathcal{C}^{\infty}(\mathbb{R}^+, \mathcal{U}_1)$, and $u_2 \in \mathcal{C}^{\infty}(\mathbb{R}^+, \mathcal{U}_2)$. So this comment suggests us to decompose the system into three particular subsystems: *a)* one whose aim of the manifest behavior, (u_1, y_1), is to lead the state trajectory to zero, *b)* another whose manifest behavior, (u_2, y_2), obeys for getting the state trajectory to zero, and *c)* a subsystem initially at rest, which is perturbed and in a pre-specified finite time comes back to zero.

A. Decomposition into Subsystems

Let us decompose (6)–(7) in three subsystems:

a) Master subsystem: $\Sigma_{i/d/o} = \Big(\mathbb{R}^+, [\mathcal{U}_1 \times (\mathcal{X}_2 \times \mathcal{X}_3)] \times [\mathcal{X}_1 \times \{0\}] \times \text{Im } \overline{A}_{21},\ \widetilde{\mathfrak{B}}^{exp}_{[A_{11}, [B_1\ [A_{12}\ A_{13}]], \overline{A}_{21}]} \oplus \{0\}\Big)$,

$$\begin{aligned} \mathrm{d}x_{f,1}/\mathrm{d}t &= A_{11} x_{f,1} + B_1 u_1 + \begin{bmatrix} A_{12} & A_{13} \end{bmatrix} \begin{bmatrix} x_{f,2} \\ x_{f,3} \end{bmatrix} \\ x_1 &= x_{f,1} \quad ; \quad y_1 = \overline{A}_{21} x_1 \end{aligned} \tag{9}$$

where: *(a)* u_1 is the controller input variable, *(b)* y_1 is a virtual output variable, and *(c)* $\begin{bmatrix} x_{f,2}^T & x_{f,3}^T \end{bmatrix}^T$ is considered as a measurable disturbance input variable. Note that, this subsystem is controllable, observable, and with no invariant zeros.

b) Slave subsystem: $\Sigma_{i/d/o} = \Big(\mathbb{R}^+, [(\text{Im } \overline{A}_{21}) \times \mathcal{U}_2 \times \mathcal{X}_3] \times [\mathcal{X}_2 \times \{0\}] \times \text{Im } A_{32},\ \widetilde{\mathfrak{B}}^{exp}_{[A_{22}, [W_2\ A_{23}], A_{32}]} \oplus \{0\}\Big)$,

$$\begin{aligned} \mathrm{d}x_{f,2}/\mathrm{d}t &= A_{22} x_{f,2} + W_2 \begin{bmatrix} y_1 \\ u_2 \end{bmatrix} + A_{23} x_{f,3} \\ x_2 &= x_{f,2} \quad ; \quad y_2 = A_{32} x_2 \end{aligned} \tag{10}$$

where: *(a)* $W_2 = [V_{A_{21}}\ B_2]$, $V_{A_{21}} : \text{Im } \overline{A}_{21} \to \mathcal{X}_2$ is the insertion map, *(b)* $[\ y_1^T \quad u_2^T\ ]^T$ is a virtual controller input variable, *(c)* y_2 is a virtual output variable, and *(d)* $x_{f,3}$ is considered as a measurable disturbance input variable. Note that W_2 is an isomorphism.

[6] BDM denotes block diagonal matrix.

[7] The *generalized derivatives*, $\psi^{(i)}$, $i \in \mathcal{Z}^{*+}$, of $\psi \in \mathfrak{D}_{ist}$ are defined by: $< \varphi, \psi^{(i)} >= (-1)^i < \mathrm{d}^i \varphi/\mathrm{d}t^i, \psi >$ for all $\varphi \in \mathfrak{T}_{est}$. The *Dirac delta* distribution, δ, is defined as $< \varphi, \delta >= \varphi(0)$, $\varphi \in \mathfrak{T}_{est}$. The successive *generalized derivatives* of δ are $< \varphi, \delta^{(k)} >= (-1)^k \mathrm{d}^k \varphi(0)/\mathrm{d}t^k$, $\varphi \in \mathfrak{T}_{est}$.

[8] The $\mathcal{L}_n$ are (A, B)–*invariant subspaces* which tend to $\mathcal{R}_a$.

c) Almost decoupled subsystem: $\Sigma_{i/d/o} = \Big(\mathbb{R}^+, [\mathcal{U}_3, \mathcal{X}_2] \times [\mathcal{X}_3 \times \{0\}] \times A_{33}^{-1}\text{Im } B_3, \widetilde{\mathfrak{B}}^{exp}_{[A_{33},[B_3\ A_{22}],P_3]} \oplus \{0\}\Big)$,

$$\begin{array}{c} \mathrm{d}x_{f,3}/\mathrm{d}t = A_{33}x_{f,3} + B_3u_3 + A_{32}x_{f,2} \\ x_3 = x_{f,3} \quad ; \quad y_3 = P_3x_3 \end{array} \tag{11}$$

where: *(a)* u_3 is the controller input variable, *(b)* y_3 is a virtual output variable, and *(c)* $x_{f,2}$, is considered as a measurable disturbance input variable. As we consider that the initial conditions for (6) are contained in $\mathcal{S}^\infty_\mathcal{K} = \mathcal{X}_1 \oplus \mathcal{X}_2$, namely $x_{f,3}(0) \equiv 0$, we can assume without lost of generality that the pair (A_{33}, B_3) is controllable. Thus, if we select as output map, a natural projection $P_3 : \mathcal{X}_3 \to A_{33}^{-1}\text{Im } B_3$, then the standard controllable triple (P_3, A_{33}, B_3) is prime. So this subsystem is also controllable, observable, and with no invariant zeros.

B. Invertible PD–Feedback

Lemma 6: Let a system, $\Sigma_{i/d/o} = \Big(\mathbb{R}^+, [\mathcal{U} \times \mathcal{Q}] \times [\mathcal{X}_{fed} \times \{0\}] \times \mathcal{Y}, \ \widetilde{\mathfrak{B}}^{exp}_{[A,[B\ S],C]} \oplus \{0\}\Big)$, where (C, A, B) is a prime standard controllable triple. There then exists a PD–feedback, $u = F_D\mathrm{d}x_f/\mathrm{d}t + F_Px_f + F_dq + g$, $g \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{G})$, which inverts the system, namely $\Sigma_{i/d/o} = \Big(\mathbb{R}^+, [\mathcal{G} \times \mathcal{Q}] \times [\{0\} \times \mathcal{X}_{ied}] \times \mathcal{Y}, \{0\} \oplus \widetilde{\mathfrak{B}}^{pol}_{[N,[\Gamma\ \overline{S}],\Theta]}\Big)$. Furthermore, $\widetilde{\mathfrak{B}}^{pol}_{[N,[\Gamma\ \overline{S}],\Theta]} = \Big\{((g,q), x_\infty, y) \in \mathcal{C}^\infty(\mathbb{R}^+, [\mathcal{G} \times \mathcal{Q}] \times \mathcal{X}_{ied} \times \mathcal{Y}) \big| \ \exists\ ((g,q), x_\infty) \in \widetilde{\mathfrak{B}}^{pol}_{[N,[\Gamma\ \overline{S}]]} \text{ s.t. } y(t) = g(t)\Big\}$.

Proof: Since the standard controllable triple (C, A, B) is prime, there then exist bases for $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{X}_{fed}$, and a linear map $F^* : \mathcal{X}_{fed} \to \mathcal{U}$ such that [11]:

$$\begin{array}{c} \mathrm{d}x_f/\mathrm{d}t = A_{F^*}x_f + Bu + Sq, \ x = x_f, \ y = Cx \\ A_{F^*} = BDM\{A_1, \ldots, A_m\}, \ B = BDM\{B_1, \ldots, B_m\} \\ C = BDM\{C_1, \ldots, C_m\}, \ S = \left[\begin{array}{ccc} S_1^T & \cdots & S_m^T \end{array}\right]^T \end{array} \tag{12}$$

where A_i, B_i, and C_i are as (8). Then, with the PD–feedback $u = B^T\mathrm{d}x_f/\mathrm{d}t - Cx_f + g - B^TSq$, $g(0) = y(0)$, we get $N\mathrm{d}x_\infty/\mathrm{d}t = x_\infty - \Gamma g + \overline{S}q$, $x = x_\infty$, $y = \Theta x$, $N = A^T_{F^*}$, $\Gamma = C^T$, and $\Theta = C$; *i.e.* $(g, x_\infty) \in \widetilde{\mathfrak{B}}^{pol}_{[N,\Gamma]}$. Finally, for $i \in \{1, \ldots, m\}$ and $t > 0$, we have:

$$\begin{array}{l} y_i(t) = C_i\mathrm{e}^{A_it}x_{f,i}(0) + C_i\int_0^t \mathrm{e}^{A_i(t-\tau)}\Big(B_i\big(B_i^T\frac{\mathrm{d}}{\mathrm{d}\tau}x_{f,i}(\tau) - \\ \qquad\qquad y_i(\tau) + g_i(\tau) - B_i^TSq(\tau)\big) + S_iq(\tau)\Big)\mathrm{d}\tau \\ \mathrm{d}^{\kappa_i}y_i(t)/\mathrm{d}t^{\kappa_i} = B_i^T\mathrm{d}x_{f,i}(t)/\mathrm{d}t - y_i(t) + g_i(t) - B_i^TSq(t) \\ \quad +C_i\Big(\sum_{i=1}^{\kappa_i-1} A_i^{i-1}S_i\mathrm{d}^{\kappa_i-i}q(t)/\mathrm{d}t^{\kappa_i-i} + A_i^{\kappa_i-1}Sq(t)\Big) \\ \quad = \mathrm{d}^{\kappa_i}y_i(t)/\mathrm{d}t^{\kappa_i} - y_i(t) + g_i(t) \end{array}$$

□

C. Almost Decoupling PD–Feedback

Based on Lemma 6, let us propose the PD control law:

$$\begin{array}{l} u_1 = \big(B_1^T\mathrm{d}/\mathrm{d}t - \overline{A}_{21}\big)x_{f,1} + h_1 - B_1^T\big(A_{12}x_{f,2} + A_{13}x_{f,3}\big) \\ W_2\left[\begin{array}{c} h_1 \\ u_2 \end{array}\right] = \big(\mathrm{d}/\mathrm{d}t - A_{22} - \mathrm{I}\big)x_{f,2} - A_{23}x_{f,3} + h_2 \\ u_3 = \big(B_3^T\mathrm{d}/\mathrm{d}t - P_3\big)x_{f,3} - B_3^TA_{32}x_{f,2} \\ h_2(t) = \left\{\begin{array}{l} x_{f,2}(0)\mathrm{e}^{-(t/T)^2/\left(1-(t/T)^2\right)}, \ 0 \le t < T \\ 0, \ t \ge T \end{array}\right. \end{array} \tag{13}$$

$h_2 \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{X}_2)$ is taken from the Section 2.4 of [12]; this function satisfies $h_2(0) = x_{f,2}(0)$ and $h_2(t) = 0$ for all $t \ge T$.

Let us note that the closed loop system satisfies:

1) The *master subsystem* satisfies $x_{f,1} = (\Gamma_1h_1 + \overline{S}_{12}x_{f,2} + \overline{S}_{13}x_{f,3}) + \sum_{i=1}^{n_1-1} N_1^i\mathrm{d}^i/\mathrm{d}t^i(\Gamma_1h_1 + \overline{S}_{12}x_{f,2} + \overline{S}_{13}x_{f,3})$ and $y_1 = h_1$. Then the *slave subsystem* satisfies $x_{f,2} \equiv h_2$, thus $x_{f,2}(t) = 0 \ \forall\ t \ge T$.

2) The *almost decoupled subsystem* satisfies $y_3 = 0$ and $x_{f,3} = \overline{S}_3x_{f,2} + \sum_{i=1}^{n_3-1} N_3^i\mathrm{d}^i/\mathrm{d}t^i\overline{S}_3x_{f,2}$. Then $x_{f,3}(t) = 0$ for $t = 0$ and $\forall\ t \ge T$.

3) Now, in view that $h_2(t) = 0$, $x_{f,2}(t) = 0$, and $x_{f,3}(t) = 0$ $\forall\ t \ge T$, then it also holds $x_{f,1}(t) = 0 \ \forall\ t \ge T$.

4) Finally, since we are dealing with $\mathcal{C}^\infty$ functions, then for any $\rho > 0$, there exists a sufficiently small $T > 0$, such that $\|x_{f,3}\|_\infty \le \rho$.

We have proved in this way the following result:

Theorem 7: Let $x_0 \in \mathcal{S}^\infty_\mathcal{K}$ be an initial condition for (6). For any $\rho > 0$ there exist a PD–feedback, $u^* = F_D\mathrm{d}x_f/\mathrm{d}t + F_Px_f + h$, $h \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U})$, and a finite time, $T > 0$, such that the trajectory $(u^*, x_f) \in \widetilde{\mathfrak{B}}^{pol}_{[A_{F^*},B]}$ satisfies $\mathrm{d}_\infty(x_f, \mathcal{S}^\infty_\mathcal{K}) \le \rho$ and $x_f(t) = 0$ for all $t \ge T$.

VI. Smooth Input and Singularly Perturbed Feedback

A. Singularly Perturbed Coupling Filter

Lemma 8: Let a prime system, $\Sigma_{i/s/o} = \Big(\mathbb{R}^+, [\mathcal{U} \times \mathcal{Q}] \times \mathcal{X}_{fed} \times \mathcal{Y}, \widetilde{\mathfrak{B}}^{exp}_{[A_{F^*},[B\ S],C]}\Big)$, described by (12); where q and the $\mathrm{d}^iq/\mathrm{d}t^i$, $i \in \{1, \ldots, \overline{\kappa}\}$ with $\overline{\kappa} = \max\{\kappa_1, \ldots, \kappa_m\}$, are bounded.

1) There then exists a singularly perturbed control law, $u = F_\varepsilon x_f + F_dq + K_\varepsilon\bar{g}$, $\bar{g} = \bar{x}_f + g$, $\bar{x}_f(t) = -\ \varepsilon\int_0^t \mathrm{e}^{-\beta(t-\tau)}K_\varepsilon y(\tau)\mathrm{d}\tau$, $g \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{G})$, $\bar{x}_f \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$, such that for any trajectory, $((\bar{x}_f + g, q), x_f, y) \in \widetilde{\mathfrak{B}}^{exp}_{[(A_{F^*}+BF_\varepsilon),[BG_\varepsilon\ (S+BF_\varepsilon)],C]}$, holds:[9]

$$y(t) = g(t) + \mathcal{O}(\sqrt{\varepsilon}), \quad \text{for all } t \ge t^* \tag{14}$$

for $\beta = \mathcal{O}(1/\varepsilon)$ and where $t^* = \mathcal{O}(\varepsilon\ln(1/\sqrt{\varepsilon}))$.

2) Moreover, If ε and β are chosen as in Theorem 10 of [10], then the gain margins of the characteristic functions of the Hurwitz stable closed loop system, $\ell_i(\jmath\omega)$, $i \in \{1, \ldots, m\}$, are lower bounded:

$$\textit{Gain Margin}(\ell_i(\jmath\omega)) \ge \mathcal{O}((1/\varepsilon)^{\underline{\kappa}+2}) \tag{15}$$

where $\underline{\kappa} = \min\{\kappa_1, \ldots, \kappa_m\}$.

3) Furthermore, let the set of trajectories of the Hurwitz stable closed loop system $\big\{((\bar{x}_f + g, q), x_f, y_\varepsilon) \mid \varepsilon = 1/\eta, \ \eta \in \mathbb{Z}^{*+}\big\}$ and let $((g,q), x_\infty, y)$ the trajectory of the behavior, $\widetilde{\mathfrak{B}}^{pol}_{[N,[\Gamma\ \overline{S}],\Theta]}$, obtained with the invertible PD–feedback of Lemma 6, if ε and β are chosen as in Corollary 11 of [10], then:

$$\lim_{\varepsilon\to 0}(g, y_\varepsilon) = (g, y) \text{ in the sense of } \mathcal{L}_1^{\text{loc}}(\mathbb{R}^{*+}, \mathbb{R}^m) \tag{16}$$

[9] $\mathcal{O}(\varphi(\varepsilon))$ means: $\exists\ \varepsilon^* > 0\ \&\ K > 0$ s.t. $|f(\varepsilon)| \le K\varphi(\varepsilon)\ \forall\ \varepsilon \in (0,\ \varepsilon^*)$ & $\varphi(\varepsilon) > 0$; $g + \mathcal{O}(\varphi(\varepsilon))$ means: $g + f(\varepsilon)$ with $f(\varepsilon) = \mathcal{O}(\varphi(\varepsilon))$.

Proof: Let us consider the following singularly perturbed control law:

$$\begin{array}{c} u(t) = K_\varepsilon^{-1}\big(\overline{F}_\varepsilon x_f(t) + \bar{x}_f(t) + g(t)\big) - B^T S q(t),\ g(0) = y(0) \\ \mathrm{d}\bar{x}_f/\mathrm{d}t = -\beta \bar{x}_f - \varepsilon K_\varepsilon y \\ K_\varepsilon = BDM\{\varepsilon^{\kappa_1}, \varepsilon^{\kappa_2}, \ldots, \varepsilon^{\kappa_m}\} \\ \overline{F}_\varepsilon = BDM\{\mathfrak{a}_{1,\varepsilon}, \mathfrak{a}_{2,\varepsilon}, \ldots, \mathfrak{a}_{m,\varepsilon}\} \\ \mathfrak{a}_{i,\varepsilon} = \left[\begin{array}{cccc} -\mathfrak{b}^B_{i,\kappa_i} & -\varepsilon \mathfrak{b}^B_{i,\kappa_i-1} & \cdots & -\varepsilon^{\kappa_i-1}\mathfrak{b}^B_{i,1} \end{array}\right] \end{array} \tag{17}$$

where the coefficients $\mathfrak{b}^B_{i,j}$ are those of the Butterworth polynomials $\Delta_{B,i}(\mathrm{s})$:

$$\Delta_{B,i}(\mathrm{s}) = \left(\mathrm{s}^{\kappa_i} + \mathfrak{b}^B_{i,1}\mathrm{s}^{\kappa_i-1} + \cdots + \mathfrak{b}^B_{i,\kappa_i-1}\mathrm{s} + \mathfrak{b}^B_{i,\kappa_i}\right) = \left\{\begin{array}{l} \prod_{j=1}^{\frac{\kappa_i}{2}}\left((\mathrm{s}+\sin\theta_{j,\kappa_i})^2 + \cos^2\theta_{j,\kappa_i}\right),\ \text{for } \kappa_i \text{ even} \\ (\mathrm{s}+1)\prod_{j=1}^{\frac{(\kappa_i-1)}{2}}\left((\mathrm{s}+\sin\theta_{j,\kappa_i})^2 + \cos^2\theta_{j,\kappa_i}\right),\ \text{for } \kappa_i \text{ odd} \end{array}\right.$$

We then get (after a change of basis in $\mathcal{X}_f$):

$$\begin{array}{rl} \mathrm{d}\bar{x}_f/\mathrm{d}t & = -\beta\bar{x}_f - \varepsilon K_\varepsilon C_o z_f\ , \\ \varepsilon \mathrm{d}z_f/\mathrm{d}t & = B_o x_f + A_o z_f + B_o(g + \bar{q}) \\ & y = C_o z_f,\ \bar{q} = \mathcal{O}(\varepsilon) \end{array} \tag{18}$$

where the matrices are the one shown in the equations (29), (30), (4) and (5) of [10], with $n = m$. Then: 1) (14) follows from Theorem 8 of [10], 2) (15) follows from Theorem 10 of [10], and 3) (16) follows from Corollary 11 of [10]. □

B. Almost Decoupling Singularly Perturbed–Feedback

Based on the singularly perturbed control law (17) of Lemma 8, let us propose the following Singularly Perturbed control law:

$$\begin{array}{l} u_1 = K_{1,\varepsilon}^{-1}\big(\overline{F}_{1,\varepsilon}x_{f,1} + \bar{x}_{f,1} + h_1\big) - B_1^T\big(A_{12}x_{f,2} + A_{13}x_{f,3}\big) \\ \mathrm{d}\bar{x}_{f,1}/\mathrm{d}t = -\beta\bar{x}_{f,1} - \varepsilon K_{1,\varepsilon}\overline{A}_{21}x_{f,1} \\ W_2\left[\begin{array}{c} h_1 \\ u_2 \end{array}\right] = -\Big(A_{22} + \frac{1}{\varepsilon}\mathrm{I}\Big)x_{f,2} - A_{23}x_{f,3} + \frac{1}{\varepsilon}(h_2 + \bar{x}_{f,2}) \\ \mathrm{d}\bar{x}_{f,2}/\mathrm{d}t = -\beta\bar{x}_{f,2} - \varepsilon^2 x_{f,2} \\ u_3 = K_{3,\varepsilon}^{-1}\big(\overline{F}_{3,\varepsilon}x_{f,3} + \bar{x}_{f,3}\big) - B_3^T A_{32}x_{f,2} \\ \mathrm{d}\bar{x}_{f,3}/\mathrm{d}t = -\beta\bar{x}_{f,3} - \varepsilon K_{3,\varepsilon}P_3 x_{f,3} \\ h_2(t) = \left\{\begin{array}{l} x_{f,2}(0)\mathrm{e}^{-(t/T)^2/\left(1-(t/T)^2\right)},\ 0 \le t < T \\ 0,\ t \ge T \end{array}\right. \end{array} \tag{19}$$

with $T >> \varepsilon\ln(1/\sqrt{\varepsilon})$.

Let us note that the closed loop system satisfies:

1) In view of (15) the closed loop system is Hurwitz stable and all its latent variables are bounded.

2) The *master subsystem* satisfies $x_{f,1} = (\Gamma_1 h_1 + \overline{S}_{12}x_{f,2} + \overline{S}_{13}x_{f,3}) + \sum_{i=1}^{n_1-1} N_1^i \frac{\mathrm{d}^i}{\mathrm{d}t^i}(\Gamma_1 h_1 + \overline{S}_{12}x_{f,2} + \overline{S}_{13}x_{f,3}) + \mathcal{O}(\sqrt{\varepsilon})$ and $y_1 = h_1 + \mathcal{O}(\sqrt{\varepsilon})$, for all $t \ge t^*$. Then the *slave subsystem* satisfies $x_{f,2} = h_2 + \mathcal{O}(\sqrt{\varepsilon})$, for all $t \ge t^*$, thus $x_{f,2}(t) = \mathcal{O}(\sqrt{\varepsilon})$, for all $t \ge T$.

3) The *almost decoupled subsystem* satisfies $y_3 = \mathcal{O}(\sqrt{\varepsilon})$ and $x_{f,3} = \overline{S}_3 x_{f,2} + \sum_{i=1}^{n_3-1} N_3^i \frac{\mathrm{d}^i}{\mathrm{d}t^i}\overline{S}_3 x_{f,2} + \mathcal{O}(\sqrt{\varepsilon})$, for all $t \ge t^*$. Then $x_{f,3}(t) = \mathcal{O}(\sqrt{\varepsilon})$ for $t = 0$ and for all $t \ge T$.

4) Now, in view that $h_2(t) = \mathcal{O}(\sqrt{\varepsilon})$, $x_{f,2}(t) = \mathcal{O}(\sqrt{\varepsilon})$, and $x_{f,3}(t) = \mathcal{O}(\sqrt{\varepsilon})$, for all $t \ge T$, then also holds $x_{f,1}(t) = \mathcal{O}(\sqrt{\varepsilon})$ for all $t \ge T$.

5) Finally, since we are dealing with $\mathcal{C}^\infty$ functions, then for any $\rho > 0$, there exists a sufficiently small $T > 0$, such that $\|x_{f,3}\|_\infty \le \rho$.

We have proved in this way the following result:

Theorem 9: Let $x_0 \in \mathcal{S}^\infty_{\mathcal{K}}$ be an initial condition for (6). For any $\rho > 0$ there exist a Singularly Perturbed–feedback, $u = (F_\varepsilon + F_d)x_f + K_\varepsilon h - \varepsilon\int_0^t \mathrm{e}^{-\beta(t-\tau)}y(\tau)\mathrm{d}\tau$, $h \in \mathcal{C}^\infty(\mathbb{R}^+ \to \mathcal{U})$, and a finite time, $T > 0$, such that the trajectory $(u^*, x_f) \in \widetilde{\mathfrak{B}}^{pol}_{[A_{F^*}, B]}$ satisfies $\mathrm{d}_\infty(x_f, \mathcal{S}^\infty_{\mathcal{K}}) \le \rho$ and $x_f(t) = \mathcal{O}(\sqrt{\varepsilon})$ for all $t \ge T$.

VII. Illustrative Example

Let us consider (6) and (7), with: $A_{11} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$, $A_{12} = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right]$, $A_{13} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array}\right]$, $A_{21} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$, $A_{22} = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$, $A_{23} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array}\right]$, $A_{32} = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right]$, $A_{33} = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$, $P_3 = \left[\begin{array}{cc} 1 & 0 \end{array}\right]$, $B_1 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right]$, $B_2 = \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \end{array}\right]$, and $B_3 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$. Note that: $\overline{A}_{21} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$, $V_{A_{21}} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right]$, and $W_2 = \mathrm{I}_3$.

A. Almost Decoupling PD–Feedback

The control law (13) is:

$$\begin{array}{rl} h_1 = & \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]\frac{\mathrm{d}x_{f,2}}{\mathrm{d}t} - \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right]x_{f,2} \\ & - \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]x_{f,3} + \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]h_2 \\ u_1 = & \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]\frac{\mathrm{d}x_{f,1}}{\mathrm{d}t} - \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]x_{f,1} + h_1 \\ & - \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right]x_{f,2} - \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]x_{f,3} \\ u_2 = & \left[\begin{array}{ccc} 0 & 0 & 1 \end{array}\right]\frac{\mathrm{d}x_{f,2}}{\mathrm{d}t}x_{f,2} - \left[\begin{array}{ccc} 0 & 0 & 1 \end{array}\right]x_{f,2} \\ & - \left[\begin{array}{cc} 1 & 1 \end{array}\right]x_{f,3} + \left[\begin{array}{ccc} 0 & 0 & 1 \end{array}\right]h_2 \\ u_3 = & \left[\begin{array}{cc} 0 & 1 \end{array}\right]\frac{\mathrm{d}x_{f,3}}{\mathrm{d}t} - \left[\begin{array}{cc} 1 & 0 \end{array}\right]x_{f,3} - \left[\begin{array}{ccc} 1 & 1 & 1 \end{array}\right]x_{f,2} \end{array}$$

The closed loop system is:

$$\begin{array}{l} x_{f,1} = \left[\begin{array}{ccc} (\mathrm{d}/\mathrm{d}t + 1) & 0 & 1 \\ 1 & (\mathrm{d}/\mathrm{d}t + 1) & 0 \\ \mathrm{d}/\mathrm{d}t & (\mathrm{d}^2/\mathrm{d}t^2 + \mathrm{d}/\mathrm{d}t) & 0 \end{array}\right]h_2 \\ x_{f,2} = h_2 \\ x_{f,3} = -\left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}\right]h_2 \end{array}$$

B. Almost Decoupling Singularly Perturbed–Feedback

The matrices of the control law (19) are: $K_{1,\varepsilon} = BDM\{\varepsilon, \varepsilon^2\}$, $\overline{F}_{1,\varepsilon} = BDM\{\mathfrak{a}_{1,\varepsilon}, \mathfrak{a}_{2,\varepsilon}\}$, $\mathfrak{a}_{1,\varepsilon} = [-1]$, $\mathfrak{a}_{2,\varepsilon} = [-1\ \ -\varepsilon\sqrt{2}]$, $K_{3,\varepsilon} = [\varepsilon^2]$, $\overline{F}_{3,\varepsilon} = [-1\ \ -\varepsilon\sqrt{2}]$.

The closed loop system is:

$$\begin{array}{rl} \mathrm{d}\bar{x}_{f,1}/\mathrm{d}t & = -\beta\bar{x}_{f,1} - \varepsilon K_{1,\varepsilon}y_1 \\ \varepsilon\mathrm{d}z_{f,1}/\mathrm{d}t & = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & -\sqrt{2}/2 & 1 \\ 0 & -1/2 & -\sqrt{2}/2 \end{array}\right]z_{f,1} + B_1(\bar{x}_{f,1} + h_1) \\ & +\varepsilon\left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{array}\right]\left[\begin{array}{c} x_{f,2} \\ \hline x_{f,3} \end{array}\right] \\ y_1 & = \overline{A}_{21}z_{f,1} \end{array}$$

$$\begin{aligned}
\mathrm{d}\bar{x}_{f,2}/\mathrm{dt} &= -\beta\bar{x}_{f,2} - \varepsilon^2 x_{f,2} \\
\varepsilon \mathrm{d}z_{f,2}/\mathrm{dt} &= \begin{bmatrix} 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \\ 0 & 0 & -1 \end{bmatrix} z_{f,2} + \varepsilon \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} x_{f,3} \\
&\quad + \varepsilon V_{A_{21}} y_1 + B_2 B_2^T (h_2 + \bar{x}_{f,2}) \\
\varepsilon h_1 &= \begin{bmatrix} -1 & -\varepsilon & 0 \\ 0 & -1 & -\varepsilon \end{bmatrix} z_{f,2} - \varepsilon \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_{f,3} \\
&\quad + (h_2 + \bar{x}_{f,2}) \\
y_2 &= A_{32} z_{f,2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\bar{x}_{f,3}/\mathrm{dt} &= -\beta\bar{x}_{f,3} - \varepsilon K_{3,\varepsilon} y_3 \\
\varepsilon \mathrm{d}z_{f,3}/\mathrm{dt} &= \begin{bmatrix} -\sqrt{2}/2 & 1 \\ -1/2 & \sqrt{2}/2 \end{bmatrix} z_{f,3} + B_3 \bar{x}_{f,3} \\
&\quad + \varepsilon \begin{bmatrix} 1 & 1 & 1 \\ \sqrt{2}/2 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} x_{f,2} \\
y_3 &= P_3 z_{f,3}
\end{aligned}$$

where $z_{f,1} = (BDM\{1, T_\varepsilon\}) x_{f,1}$, $z_{f,2} = x_{f,2}$, and $z_{f,3} = T_\varepsilon x_{f,3}$, with $T_\varepsilon = \begin{bmatrix} 1 & 0 \\ \sqrt{2}/2 & \varepsilon \end{bmatrix}$. In Fig. 1, we show some MATLAB® simulations with a *relative tolerance* of 1×10^{-5}, a *variable step* and an *ODE 45 (Domain–Prince)*; the other parameters are set to *auto*. The initial conditions are set as: $z_{f,1}(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, $z_{f,2}(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, $z_{f,3}(0) = 0$. The parameter's controller were chosen as: $\varepsilon = 0.01$, $\beta = 100$, and $T = 1$.

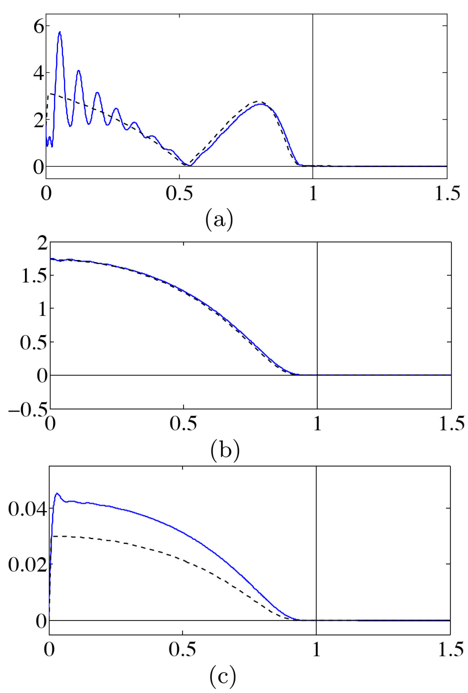

Fig. 1. The dashed lines corresponds to the *PD–feedback* and the solid lines corresponds to the *singularly perturbed–feedback*. (a) $\|z_{f,1}\|$, (b) $\|z_{f,2}\|$, and (c) $\|z_{f,3}\|$.

VIII. Conclusion

In this paper the *almost rejection of initial conditions* is studied by a PD state feedback law (see Theorem 7) $u = F_D \mathrm{d}x_f/\mathrm{dt} + (F_P + F^*)x_f$. It is shown that with the tools and results from Willems [15] and Trentelman [13] it is possible to solve this problem by means of a PD law. This can be performed with a finite map, F_d, and a map, F_ε, parametrized in the precision positive coefficient ε, namely (see Theorem 9) $u = (F_\varepsilon + F_d + F^*)x_f - \varepsilon \int_0^t \mathrm{e}^{-\beta(t-\tau)} y(\tau)\mathrm{d}\tau$. The β integral term, characterizing a slow subsystem, is introduced for remaining in the singularly perturbed framework of Kokotović [9]; also the positive coefficient β guarantees a certain stability margin. Thus, when ε tends to zero the singularly perturbed state feedback tends to the PD state feedback in the sense $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^{*+}, \mathcal{U})$ (see Corollary 11 of [10]). Let us note that Trentelman [13] has shown that the use of high gain state feedback to solve the *almost disturbance decoupling problem* may cause certain state variables in the closed loop system to become unacceptably large.

The synthesis procedure, introduced in this paper, simplifies in a great manner the design task and makes the application of these so important subspaces introduced by Willems [14] more feasible. Let us note that our results are also complementary to those of Armentano [1].

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