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Synchronization in Networks of Identical Linear Systems

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Abstract— **The paper investigates the synchronization of a network of identical linear time-invariant state-space models under a possibly time-varying and directed interconnection structure. The main result is the construction of a dynamic output feedback coupling that achieves synchronization if the decoupled systems have no exponentially unstable mode and if the communication graph is uniformly connected. Stronger conditions are shown to be sufficient – but to some extent, also necessary – to ensure synchronization with the diffusive static output coupling often considered in the literature.**

I. INTRODUCTION

In these last years, consensus, coordination and synchronization problems have been popular subjects in systems and control, motivated by many applications in physics, biology, and engineering. These problems arise in multi-agent systems with the collective objective of reaching agreement about some variables of interest.

In the *consensus* literature, the emphasis is on the communication constraints rather than on the individual dynamics: the agents exchange information according to a communication graph that is not necessarily complete, nor even symmetric or time-invariant, but, in the absence of communication, the agreement variables usually have no dynamics. It is the exchange of information only that determines the time-evolution of the variables, aiming at asymptotic synchronization to a common value. The convergence of such consensus algorithms has attracted a lot of interest in the recent years and it only requires a weak form of connectivity for the communication graph [1], [2], [3], [4], [5].

In the *synchronization* literature, the emphasis is on the individual dynamics rather than on the communication limitations: the communication graph is often assumed to be complete (or all-to-all), but in the absence of communication, the time-evolution of the agents variables can be oscillatory or even chaotic. The system dynamics can be modified through the information exchange, and, as in the consensus problem, the goal of the interconnection is to reach synchronization to a common solution of the individual dynamics [6], [7], [8], [9].

Coordination problems encountered in the engineering world, can often be rephrased as consensus or synchronization problems in which both the individual dynamics and the limited communication aspects play an important role. Designing interconnection control laws that can ensure synchronization of relevant variables is therefore a control problem that has attracted quite some attention in the recent years [10], [11], [12], [13], [14], [15].

The present paper deals with a fairly general solution of the synchronization problem in the linear case. Assuming N identical individual agents dynamics each described by the linear state-space model (A, B, C), the main result is the construction of a dynamic output feedback controller that ensures exponential synchronization to a solution of the linear system $\dot{x} = Ax$ under the following assumptions: (i) A has no exponentially unstable mode, (ii) (A, B) is stabilizable and (A, C) is observable, and (iii) the (possibly time-varying and directed) communication graph is *uniformly connected*.

Uniform connectedness is a very mild condition. It allows the communication from one system to another to be indirect, involving intermediate systems. Also, the required communication does not need to occur instantaneously but may be spread over time.

The result can be interpreted as a generalization of classical consensus algorithms studied in the recent years corresponding to the particular case $A = 0$. The generalization includes the non-trivial examples of synchronizing harmonic oscillators or chains of integrators. The dynamic controller structure proposed in this paper differs from the static diffusive coupling often considered in the synchronization literature, which requires more stringent assumptions on the communication graph. The paper also provides sufficient conditions for synchronization by static diffusive coupling and illustrates, on simple examples, that synchronization may fail under diffusive coupling when the stronger assumptions on the communication graph are not satisfied.

The paper is organized as follows. In Section II the notation used throughout the paper is summarized and some preliminary results are reviewed. In Section III the synchronization problem is introduced and defined. In Section IV the linear case is studied when state coupling among the systems is allowed while, in Section V, the output coupling case is considered. Finally, in Section VI, two-dimensional examples are reported to illustrate the role of the proposed dynamic controller in situations where static diffusive coupling fails to achieve synchronization.

II. PRELIMINARIES

A. Notation and Terminology

Throughout the paper we will use the following notation. Given N vectors $x_1, x_2, \ldots, x_N$ we indicate with x the

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stacking of the vectors, i.e. $x = [x_1^T, x_2^T, \ldots, x_N^T]^T$. We denote with I_N the diagonal matrix of dimension N and we define $1_N \triangleq [1, 1, \ldots, 1]^T \in \mathbb{R}^N$. Given two matrices A and B we denote their Kronecker product with $A \otimes B$, for the definition and basic properties see e.g. [16]. For notational convenience, we use the convention $\tilde{A}_N = I_N \otimes A$ and $\hat{A}_N = A \otimes I_N$.

B. Communication Graphs

Given a set of interconnected systems the communication topology is encoded through a *communication graph*. The convention is that system j receives information from system i if and only if there is a directed link from node j to node i in the communication graph. Let $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), A_d(t))$ be a time-varying weighted digraph (directed graph) where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is the set of nodes, $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $A_d(t)$ is a weighted adjacency matrix with nonnegative elements $a_{kj}(t)$. In the following we assume that $A_d(t)$ is piece-wise continuous and bounded and $a_{kj}(t) \in \{0\} \cup [\eta, \gamma], \forall k, j,$ for some finite scalars $0 < \eta \le \gamma$ and for all $t \ge 0$. Furthermore $\{v_k, v_j\} \in \mathcal{E}(t)$ if and only if $a_{kj}(t) \ge \eta$. The set of neighbors of node v_k at time t is denoted by $\mathcal{N}_k(t) \triangleq \{v_j \in \mathcal{V} : a_{kj}(t) \ge \eta\}$. A path is a sequence of vertices such that for each of its vertices v_k the next vertex in the sequence is a neighbor of v_k. Assume that there are no self-cycles i.e. $a_{kk}(t) = 0$, $k = 1, \ldots, N$ and for any t.

The graph Laplacian $L(t)$ associated to the graph $\mathcal{G}(t)$ is defined as

$$L_{kj}(t) = \begin{cases} \sum_i a_{ki}(t), & j = k \\ -a_{kj}(t), & j \neq k. \end{cases}$$

The in-degree (respectively out-degree) of node v_k is defined as $d_k^{in} = \sum_{j=1}^N a_{kj}$ (respectively $d_k^{out} = \sum_{j=1}^N a_{jk}$). The digraph G is said to be *balanced* if the in-degree and the out-degree of each node are equal, that is,

$$\sum_j a_{kj} = \sum_j a_{jk}, \qquad k = 1, \ldots, N.$$

Balanced graphs have the particular property that the symmetric part of their Laplacian is nonnegative: $L + L^T \ge 0$ [17]. We recall some definitions that characterize the concept of connectivity for time-varying graphs.

Definition 1: The digraph $\mathcal{G}(t)$ is *connected* at time t if there exists a node v_k such that all the other nodes of the graph are connected to v_k via a path that follows the direction of the edges of the digraph.

Definition 2: Consider a graph $\mathcal{G}(t)$. A node v_k is said to be connected to node v_j ($v_j \neq v_k$) in the interval $I = [t_a, t_b]$ if there is a path from v_k to v_j which respects the orientation of the edges for the directed graph $(\mathcal{V}, \cup_{t \in I} \mathcal{E}(t), \int_I A_d(\tau) d\tau)$.

Definition 3: $\mathcal{G}(t)$ is said to be uniformly connected if there exists a time horizon $T > 0$ and an index k such that for all t all the nodes v_j ($j \neq k$) are connected to node v_k across $[t, t + T]$.

C. Convergence of consensus algorithms

Consider the continuous dynamics

$$\dot{x}_k = \sum_{j=1}^N a_{kj}(t)(x_j - x_k), \qquad k = 1, \ldots, N. \qquad (1)$$

Using the Laplacian definition, (1) can be equivalently expressed as

$$\dot{x} = -\hat{L}_n(t)\, x, \qquad (2)$$

Algorithm (2) has been widely studied in the literature and asymptotic convergence to a consensus value holds under mild assumptions on the communication topology. The following theorem summarizes the main result in [2].

Theorem 1: Let x_k, $k = 1, 2, \ldots, N$, belong to a finite-dimensional Euclidean space W. Let $\mathcal{G}(t)$ be a uniformly connected digraph and $L(t)$ the corresponding Laplacian matrix bounded and piecewise continuous in time. Then the equilibrium sets of consensus states of (1) and (2) are uniformly exponentially stable. Furthermore the solutions of (1) and (2) asymptotically converge to a consensus value $1_N \otimes \beta$ for some $\beta \in W$. □

III. The Synchronization Problem

Consider N identical dynamical systems

$$\begin{aligned} \dot{x}_k &= f(x_k, u_k) \\ y_k &= h(x_k), \qquad k = 1, \ldots, N \end{aligned} \qquad (3)$$

where $x_k \in \mathbb{R}^n$ is the state of the system, $u_k \in \mathbb{R}^m$ is the control and $y_k \in \mathbb{R}^p$ is the output. We assume that the coupling among the systems involves only the output differences $y_k - y_j$ and the controller state differences $\xi_k - \xi_j$. Following the terminology and notation of Section II, two systems are coupled at time t if there exists an edge connecting them in the associated (time-varying) communication graph $\mathcal{G}(t)$ at time t. We will call a control law *dynamic* if it depends on an internal (controller) state, otherwise it is called *static*. For the systems to be synchronized, the control action (that will depend on the coupling) must vanish asymptotically and must force the solutions of the closed-loop systems to asymptotically converge to a common solution of the individual systems. This leads to the formulation of the following problem:

Synchronization Problem: Given N identical systems described by the model (3) and a communication graph $\mathcal{G}(t)$, find a (distributed) control law such that the solutions of (3) asymptotically synchronize to a solution of the open-loop system $\dot{x}_0 = f(t, x_0, 0)$. □

In the present paper we focus the attention on synchronization of linear time-invariant systems. Generalizations will be the subject of future work.

IV. Synchronization of linear systems with state feedback

Consider N identical linear systems, each described by the linear model

$$\dot{x}_k = Ax_k + Bu_k, \qquad k = 1, 2, \ldots, N, \qquad (4)$$

where $x_k \in \mathbb{R}^n$ is the state of the system and $u_k \in \mathbb{R}^m$ is the control vector. For notational convenience it is possible to rewrite (4) in compact form as

$$\dot{x} = \tilde{A}_N x + \tilde{B}_N u. \quad (5)$$

Theorem 1 can be interpreted as a synchronization result for linear systems with $A = 0$ and $B = I$. A straightforward generalization is as follows.

Theorem 2: Consider the linear systems (4). Let B be a $n \times n$ nonsingular matrix and assume that all the eigenvalues of A belong to the imaginary axis. Assume that the communication graph $G(t)$ is uniformly connected and the corresponding Laplacian matrix $L(t)$ piecewise continuous and bounded. Then, under the control law

$$u_k = B^{-1} \sum_{j=1}^{N} a_{kj}(t)(x_j - x_k), \quad k = 1, 2, \ldots, N, \quad (6)$$

all solutions of (4) exponentially synchronize to a solution of the system $\dot{x}_0 = A x_0$. □

Proof: Consider the closed-loop system

$$\dot{x}_k = A x_k + \sum_{j=1}^{N} a_{kj}(t)(x_j - x_k).$$

The change of variable

$$z_k(t) = e^{-A(t-t_0)} x_k(t), \quad k = 1, 2, \ldots, N \quad (7)$$

$t \geq t_0$, leads to

$$\begin{aligned} \dot{z}_k &= -A e^{-A(t-t_0)} x_k + e^{-A(t-t_0)} A x_k \\ &\quad + e^{-A(t-t_0)} \textstyle\sum_{j=1}^{N} a_{kj}(t)(x_j - x_k) \\ &= \textstyle\sum_{j=1}^{N} a_{kj}(t)(z_j - z_k) \end{aligned}$$

or, in compact form,

$$\dot{z} = -\hat{L}_n(t)\, z. \quad (8)$$

From Theorem 1 the solutions $z_k(t)$, $k = 1, 2, \ldots, N$ exponentially converge to a common value $x_0 \in \mathbb{R}^n$ as $t \to \infty$, that is, there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that for all t_0,

$$||z_k(t) - x_0|| \leq \delta_1 e^{-\delta_2 (t-t_0)} \, ||z_k(t_0) - x_0|| \,, \forall t > t_0. \quad (9)$$

In the original coordinates, this means

$$\begin{aligned} \left|\left|x_k(t) - e^{A(t-t_0)} x_0\right|\right| &\leq \delta_1 \left|\left|e^{A(t-t_0}\right|\right| \\ &\quad \times e^{-\delta_2(t-t_0)} \, ||x_k(t_0) - x_0|| \,, \end{aligned} \quad (10)$$

for every $t > t_0$. Because all the eigenvalues of the matrix A lie on the imaginary axis, there exists a constant $\delta_3 > 0$ such that

$$\left|\left|x_k(t) - e^{A(t-t_0)} x_0\right|\right| \leq \delta_1 e^{-\delta_3(t-t_0)} \, ||x_k(t_0) - x_0|| \,, \quad (11)$$

for every $t > t_0$, which proves that all solutions exponentially synchronize to a solution of the open loop system. ■

Remark 1: The result is of course unchanged if A also possesses eigenvalues with a negative real part. Exponentially stable modes synchronize to zero, even in the absence of coupling. In contrast, the situation of systems with some eigenvalues with a positive real part can be addressed in a similar way but it requires that the graph connectivity is sufficiently strong to dominate the instability of the system. This is clear from the last part of the proof of Theorem 2 where the exponential synchronization in the z coordinates must dominate the divergence of the unstable modes of A. □

The assumption of a *square* (nonsingular) matrix B in Theorem 2 can be weakened to a stabilizability assumption on the pair (A, B). For an arbitrary stabilizing feedback matrix K, consider the (dynamic) control law

$$\begin{aligned} \dot{\xi} &= \left(\tilde{A}_N + \tilde{B}_N \tilde{K}_N\right) \xi + \hat{L}_n(t)(x - \xi), \\ u &= \tilde{K}_N \xi \end{aligned} \quad (12)$$

which leads to the closed-loop system

$$\dot{x} = \tilde{A}_N x + \tilde{B}_N \tilde{K}_N \xi \quad (13a)$$

$$\dot{\xi} = \left(\tilde{A}_N + \tilde{B}_N \tilde{K}_N\right) \xi + \hat{L}_n(t)(x - \xi). \quad (13b)$$

Theorem 3: Consider the system (4). Assume that all the eigenvalues of A belong to the closed left-half complex plane. Assume that the pair (A, B) stabilizable and let K a stabilizing matrix such that $A + BK$ is Hurwitz. Assume that the graph is uniformly connected and the Laplacian is piecewise continuous and bounded. Then the solutions of (13) exponentially synchronize to a solution of the open loop system $\dot{x}_0 = A x_0$. □

Proof: With the the change of variable $s_k = x_k - \xi_k$ we can rewrite (13b) as

$$\dot{s} = \tilde{A}_N s - \hat{L}_n(t) s,$$

and the closed-loop dynamics write

$$\dot{x} = \left(\tilde{A}_N + \tilde{B}_N \tilde{K}_N\right) x + \tilde{B}_N \tilde{K}_N s \quad (14a)$$

$$\dot{s} = \tilde{A}_N s - \hat{L}_n(t) s. \quad (14b)$$

Observe that the two systems (14a) and (14b) are decoupled. Since the hypotheses of Theorem 2 are satisfied for the subsystem (14b), its solutions exponentially synchronize to a solution of $\dot{s}_0 = A s_0$. The subsystem (13b) is therefore an exponentially stable system driven by an input $\hat{L}_n(t) s(t)$ that exponentially converges to zero. As a consequence, its solution $\xi(t)$ exponentially converges to zero, which implies that the solutions of (13a) exponentially synchronize to a solution of $\dot{x}_0 = A x_0$. ■

Remark 2: In the present paper we focus on *time-invariant* linear systems in *continuous time*. However, the results here presented, are easily extendable to discrete-time systems and periodic systems. For the interested reader these generalizations are discussed in [18]. □

V. Synchronization of Linear Systems with Output Feedback

Consider a group of N identical linear systems described by the linear model

$$\begin{aligned} \dot{x}_k &= A x_k + B u_k, \\ y_k &= C x_k \end{aligned} \qquad k = 1, 2, \ldots, N, \quad (15)$$

where $x_k \in \mathbb{R}^n$ is the state of the system, $u_k \in \mathbb{R}^m$ is the control vector, and $y_k \in \mathbb{R}^p$ is the output. For notational convenience it is possible to rewrite (15) in compact form as

$$\begin{array}{rcl} \dot{x} & = & \tilde{A}_N x + \tilde{B}_N u \\ y & = & \tilde{C}_N y. \end{array} \qquad (16)$$

The state feedback controller of Theorem 3 is easily extended to an output feedback controller if we assume observability of the pair (A, C). Pick an observer matrix H such that $A + HC$ is Hurwitz and consider the output feedback controller

$$\begin{array}{rcl} \dot{\xi} & = & \left(\tilde{A}_N + \tilde{B}_N \tilde{K}_N\right)\xi + \hat{L}_n(t)(\hat{x} - \xi) \\ \dot{\hat{x}} & = & \tilde{A}_N \hat{x} + \tilde{B}_N u + \tilde{H}_N(\hat{y} - y) \\ u & = & \tilde{K}_N \xi \\ \hat{y} & = & \tilde{C}_N \hat{x}, \end{array} \qquad (17)$$

where observability is assumed and H is a suitable observer matrix. The convergence analysis is similar to the one for Theorem 2 and is mainly based on the observation that the estimation error is decoupled from the consensus dynamics.

Theorem 4: Assume that the open-loop system (4) is stabilizable and observable and that all the eigenvalues of A belong to the closed left-half complex plane. Assume that the communication graph is uniformly connected and the Laplacian is piecewise continuous and bounded. Then for any gain matrices K and H such that $A+BK$ and $A+HC$ are Hurwitz, the solutions of (5) with the dynamic controller (17) exponentially synchronize to a solution of $\dot{x}_0 = Ax_0$. □

Proof: Define $s_k = \hat{x}_k - \xi_k$ and $e_k = x_k - \hat{x}_k$, and rewrite the closed loop system as

$$\begin{array}{rcl} \dot{x} & = & \left(\tilde{A}_N + \tilde{B}_N \tilde{K}_N\right)x + \tilde{B}_N \tilde{K}_N (e + s) \\ \dot{s} & = & \tilde{A}_N s - \hat{L}_n s \\ \dot{e} & = & \left(\tilde{A}_N + \tilde{H}_N \tilde{C}_N\right) e. \end{array}$$

This system is the cascade of the closed-loop system analyzed in the proof of Theorem 3 with an exponentially stable estimation error dynamics, which proves the result. ■

Theorem 4 provides a general synchronization result for linear systems but the solution requires a dynamic controller. For the sake of comparison, we provide a set of sufficient conditions to prove synchronization under a simple static output feedback (diffusive) interconnection. These sufficient conditions assume a passivity property for the system (A, B, C), that is, the existence of a symmetric positive definite matrix $P > 0$ that verifies

$$PA + A^T P \leq 0, \quad B^T P = C. \qquad (18)$$

Passity conditions have been considered previously in [19] (where it is assumed that the communication topology is bidirectional and strongly connected) and in [8] (where synchronization is studied for a class of (nonlinear) oscillators assuming that the communication topology is time-invariant and balanced). Assumptions A1 and A2 below lead to a time-varying extension of the results in [8] and [19] in the special case of linear systems.

Theorem 5: Consider systems (15) equipped with the static output feedback control laws

$$u_k = \sum_{j=1}^{N} a_{kj}(t)(y_j - y_k).$$

Let the graph Laplacian matrix $L(t)$ be piecewise continuous and bounded. Then exponential synchronization to a solution of $\dot{x}_0 = Ax_0$ is achieved under either one of the following assumptions: A1. The system (A, B, C) is passive and observable, the communication graph is connected and balanced at each time;
A2. The system (A, B, C) is passive and observable, the communication graph is symmetric, i.e. the Laplacian matrix can be factorized as $L = DD^T(t)$, and the pair $(\tilde{A}_N, \hat{D}_p^T \tilde{C}_N)$ is uniformly observable. □

Proof: Supppose first that assumption A1 holds and consider the matrix P solution of (18).
Consider the Lyapunov function

$$V(x) = \frac{1}{2}(\hat{\Pi}_n x)^T \tilde{P}_N (\hat{\Pi}_n x), \qquad (19)$$

the derivative along the solutions of the closed loop system is

$$\dot{V}(x) = \frac{1}{2}\dot{x}^T \hat{\Pi}_n \tilde{P}_N \hat{\Pi}_n \tilde{A}_N x + \frac{1}{2} x^T \hat{\Pi}_n \tilde{P}_N \hat{\Pi}_n \tilde{A}_N \dot{x}. \quad (20)$$

By using the commutation property of Kronecker products (see e.g. [16]) and the passivity relation (18) we obtain

$$\begin{array}{rcl} \dot{V}(x) & = & \frac{1}{2} x^T \hat{\Pi}_n (\tilde{P}_N \tilde{A}_N + \tilde{A}_N^T \tilde{P}_N) \hat{\Pi}_n x \\ & & - x^T \tilde{C}_N^T \Pi_p \hat{L}_p^{\text{sym}}(t) \hat{\Pi}_p y \\ & \leq & -y^T \hat{\Pi}_p \hat{L}_p^{\text{sym}}(t) \hat{\Pi}_p y. \end{array} \qquad (21)$$

Because the graph is balanced, the matrix $L^{\text{sym}}(t) \triangleq (L(t) + L^T(t))/2$ is positive semi-definite for each t and

$$(\hat{\Pi}_p y)^T \hat{L}_p^{\text{sym}}(t) \hat{\Pi}_p y \geq \lambda_2^* \left|\left|\hat{\Pi}_p y\right|\right|^2,$$

where $\lambda_2^* = \inf_t \lambda_2(t)$, and $\lambda_2(t)$ is the algebraic connectivity of the graph at time t. Note that $\lambda_2^* > 0$ because the graph is connected at each time t and the values of the adjacency matrix related to the connected components are assumed to be bounded away from zero (see Section II). This allows to rewrite (21) as

$$\dot{V}(x) \leq -\lambda_2^* \left|\left|\hat{\Pi}_p y\right|\right|^2, \quad \lambda_2^* > 0. \qquad (22)$$

Integrating (22) over the interval $[t_0, t_0 + T]$ where $T > 0$ is arbitrary, we obtain

$$\begin{array}{rcl} \int_{t_0}^{t_0+T} \dot{V} dt & \leq & -\lambda_2^* \int_{t_0}^{t_0+T} \left|\left|\hat{\Pi}_p y\right|\right|^2 dt \\ & \leq & -\gamma \lambda_2^* \left|\left|\hat{\Pi}_n x(t_0)\right|\right|^2, \quad \gamma > 0, \end{array} \qquad (23)$$

for all $x(t_0)$, where the last inequality follows from the observability condition of the pair (A, C). We conclude from a standard Lyapunov argument that the solutions exponentially synchronize.

Assume that assumption A2 holds. First observe that from the symmetry of the communication graph the Laplacian matrix can be factorized as $L(t) = DD^T(t)$. Uniform observability of the pair $(\tilde{A}_N, \hat{D}_p^T \tilde{C}_N)$ means that for all $t_0 > 0$ there exist positive constants T and α (independent from t_0) such that

$$\int_{t_0}^{t_0+T} \tilde{\Phi}_N(t,t_0)^T \tilde{C}_N^T \hat{D}_p \hat{D}_p^T(\tau) \tilde{C}_N \tilde{\Phi}_N(t,t_0) dt \geq \alpha I_{nN}, \quad (24)$$

where $\Phi(t,\tau)$ is the transition matrix. This implies that the system

$$\begin{aligned} \dot{x} &= \tilde{A}_N x \\ z &= \hat{D}_p^T(t) \tilde{C}_N x, \end{aligned} \quad (25)$$

is uniformly observable. Applying output injection to system (25) we obtain

$$\begin{aligned} \dot{x} &= \tilde{A}_N x - K(t) \tilde{D}_p^T \tilde{C}_N x \\ z &= \hat{D}_p^T(t) \tilde{C}_N x. \end{aligned} \quad (26)$$

Choose $K(t) \triangleq \tilde{P}_N^{-1} \tilde{C}_N^T \hat{D}_p^T(t)$ and observe that, since $L(t)$ is bounded, $K(t)$ belongs to $L_2(t, t+T)$. Then output injection preserves observability (see [20] and references therein) and the system

$$\begin{aligned} \dot{x} &= \tilde{A} x - \tilde{B}_N \hat{D}_p^T \hat{D}_p(t) \tilde{C}_N x \\ z &= \hat{D}_p^T(t) \tilde{C}_N x \end{aligned} \quad (27)$$

is still uniformly observable (here we have also used the passivity condition $\tilde{C}_N = \tilde{B}_N^T \tilde{P}_N$). Therefore for all $t_0 > 0$ there exist positive constants T and β (independent from t_0) such that for every $x(0) \neq 0$

$$\int_{t_0}^{t_0+T} ||z||^2 \, dt = \int_{t_0}^{t_0+T} y(t)^T \hat{D}_p \hat{D}_p^T(t) y(t) dt \geq \beta. \quad (28)$$

Consider the Lyapunov function (19). Integrating its time derivative over the interval $[t_0, t_0+T]$ where $T > 0$ is arbitrary we obtain

$$\begin{aligned} \int_{t_0}^{t_0+T} \dot{V} dt &\leq -\int_{t_0}^{t_0+T} \left|\left| \hat{\Pi}_p \hat{D}_p \hat{D}_p^T(t) y \right|\right|^2 dt \\ &\leq -\sigma \left|\left| \hat{\Pi}_n x(t_0) \right|\right|^2, \quad \sigma > 0. \end{aligned}$$

We conclude from standard Lyapunov results that the solutions asymptotically synchronize. ■

VI. Examples

The conditions of Theorem 5 are only *sufficient* conditions for exponential synchronization under diffusive coupling. We provide two simple examples to illustrate that these conditions are not far from being necessary when considering time-varying and directed graphs and that the *internal model* of the dynamic controller (12) plays an important role in such situations.

Example 1: Synchronization of harmonic oscillators Consider a group of N harmonic oscillators

$$\begin{aligned} \dot{x}_{1k} &= x_{2k} \\ \dot{x}_{2k} &= -x_{1k} + u_k, \end{aligned} \quad (29)$$

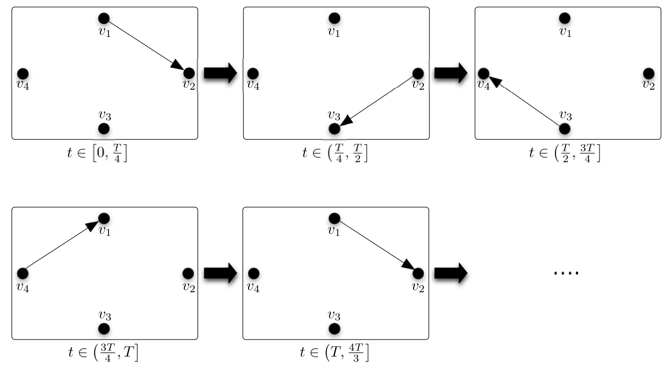

Fig. 1. The time-varying communication topology used in Example 1 and Example 2.

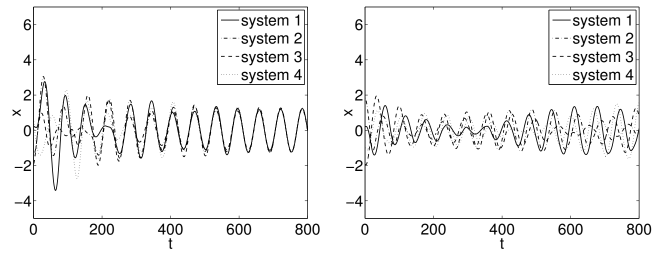

Fig. 2. First component of the solutions of the closed loop harmonic oscillators by using the dynamic control law (to the left) and the static control law (31) (to the right). The dynamic control ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection.

for $k = 1, 2, \ldots, N$, which corresponds to system (4) with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The assumptions of Theorem 2 are satisfied: A is Lyapunov stable and (A, B) is stabilizable. Choosing the stabilizing gain $K = (0 \; -1)$, the dynamic control law (12) yields the closed-loop system

$$\begin{aligned} \dot{x}_{1k} &= x_{2k} \\ \dot{x}_{2k} &= -x_{1k} - \xi_{2k} \\ \dot{\xi}_{1k} &= \xi_{2k} + \textstyle\sum_{j=1}^N a_{kj}(t)(\xi_{1j} - \xi_{1k} + x_{1k} - x_{1j}) \\ \dot{\xi}_{2k} &= -\xi_{1k} - \xi_{2k} + \textstyle\sum_{j=1}^N a_{kj}(t)(\xi_{2j} - \xi_{2k} + x_{2k} - x_{2j}). \end{aligned} \quad (30)$$

Theorem 3 ensures exponential synchronization of the oscillators to a solution of the harmonic oscillator if the graph is uniformly connected. Fig. 2 illustrates the simulation of a group of 4 oscillators coupled according to the time-varying communication topology shown in Fig. 1 (the period T is set to 7 sec). The dynamic control ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection

$$u_k = \sum_{j=1}^N a_{kj}(t)(x_{2j} - x_{2k}). \quad (31)$$

The system (A, B, C) is nevertheless passive, meaning that stronger assumptions on the communication graph would ensure synchronization with the diffusive coupling (31). We

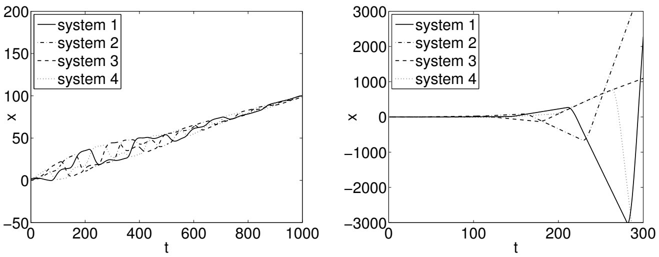

Fig. 3. First component of the solutions of the closed loop double integrators by using the dynamic control law (to the left) and the static control law (34) (to the right). The dynamic control ensures exponential synchronization. In contrast synchronization is not observed with the diffusive interconnection.

mention the recent result [15] that proves (in discrete-time) synchronization of harmonic oscillators with diffusive coupling under the assumption that the graph is time-invariant and connected. The following example illustrates an analog scenario with unstable dynamics.

Example 2: Consensus for double integrators
Consider a group of N double integrators

$$\begin{aligned} \dot{x}_{1k} &= x_{2k} \\ \dot{x}_{2k} &= u_k, \end{aligned} \tag{32}$$

for $k = 1, 2, \ldots, N$, which corresponds to system (4) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The assumptions of Theorem 2 are satisfied: the two eigenvalues of A are at zero and (A, B) is stabilizable. Choosing the stabilizing gain $K = (-1\ -1)$, the dynamic control law (12) yields closed-loop system

$$\begin{aligned} \dot{x}_{1k} &= x_{2k} \\ \dot{x}_{2k} &= -\xi_{1k} - \xi_{2k} \\ \dot{\xi}_{1k} &= \xi_{2k} + \textstyle\sum_{j=1}^{N} a_{kj}(t)(\xi_{1j} - \xi_{1k} + x_{1k} - x_{1j}) \\ \dot{\xi}_{2k} &= -\xi_{1k} - \xi_{2k} + \textstyle\sum_{j=1}^{N} a_{kj}(t)(\xi_{2j} - \xi_{2k} + x_{2k} - x_{2j}). \end{aligned} \tag{33}$$

Theorem 3 ensures exponential synchronization to a solution of the double integrator if the graph is uniformly connected. Fig. 3 illustrates the simulation of a group of 4 double integrators coupled according to the time-varying communication topology shown in Fig. 1 (the period T is set to 2 sec). The dynamic control ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection

$$u_k = \sum_{j=1}^{N} a_{kj}(t)(y_j - y_k),\ y_k = x_{1k} + x_{2k}. \tag{34}$$

The matrix $A - \alpha BC$ is nevertheless stable for every $\alpha > 0$, suggesting that stronger assumptions on the communication graph would ensure synchronization.

VII. CONCLUSION AND FUTURE WORK

In this paper the problem of synchronizing a network of identical linear systems described by the state-space model (A, B, C) under general interconnection topologies has been addressed. A dynamic controller ensuring exponential convergence of the solutions to a synchronized solution of the decoupled systems is provided assuming that (i) A has no exponentially unstable mode, (ii) (A, B) is stabilizable and (A, C) is observable, and (iii) the communication graph is uniformly connected. Stronger conditions are shown to be sufficient (and, to some extent, also necessary) to ensure synchronization with the often considered static diffusive output coupling. The extension of the proposed technique for synchronization of nonlinear systems is the subject of ongoing work.

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