Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Fixed Order Multivariable Controller Synthesis: A New Algorithm

L.H. Keel and S.P. Bhattacharyya

Abstract— **We consider the synthesis of a fixed order or fixed structure multivariable feedback controller C, parametrized by a design vector x, for a plant P, containing a vector p of uncertain parameters. The characteristic polynomials of such systems contain coefficients which depend polynomially on x and p. Using results on sign definite decomposition we develop a 4-polynomial stability test that gives a sufficient condition for stability of the family of closed loop systems that result when x and p vary over prescribed boxes. This test is reminiscent of Kharitonov's Theorem, even though the family of polynomials considered here is certainly not restricted to be interval or even convex. Moreover this result is tight in the sense that the test does reduce to Kharitonov's Theorem for the special case of interval polynomials. Using this criterion recursively and modularly we design an algorithm to determine sets of controllers that stabilize the family of uncertain plants. Examples and future research directions are provided.**

I. INTRODUCTION

The problem of designing fixed and low order controllers is of current importance in control theory and practice (see, for example [1]). The fixed order control problem does not fit the state feedback observer paradigm that is so successful when the controller order is unconstrained. Indeed, the design of a simple controller is a much more difficult problem than that of designing a high order complex controller. Recent results on this problem includes [2], [3], [4], where a generalization of the KYP lemma designed to be valid over prescribed frequency ranges was developed to deal with fixed order controller synthesis. A relaxation approach to the design of fixed order controllers was advocated in [5]. In [6], the design of H_∞ controllers of fixed order was studied. In [7], the use of quantifier elimination (QE) techniques to deal with the fixed order controller design problem was proposed. In [8], Neimark's D-Decomposition technique [9], [10] was revisited and applied to design fixed order controllers. There has been a number of papers addressing the fixed order controller design problem using LMI techniques [11]. A robust stability problem with multilinear dependencies, which is applicable to analysis and synthesis problems was studied in [12], using the Mapping Theorem.

In this paper, we develop a new algorithm for multivariable fixed order control synthesis problems based on concepts from sign definite decomposition (see [13]). By this means we are able to study the problem of robust stability of the feedback system under polynomial parameter dependencies. A remarkable 4-polynomial test is developed for robust stability of such families. This can be applied recursively to synthesize families of controllers. We illustrate this with examples and indicate future directions of research on this problem.

This work was supported in part by NSF Grant HRD-0531490

Center of Excellence in Information Systems, Tennessee State University, Nashville, TN 37209

Department of Electrical and Computer Engineering, Texas A&M University, College Station, TX 77843

II. PRELIMINARIES

We first introduce some basic results on sign-definite decomposition. These follow [13] where more details are available. The reader should also consult [14], [15] for results on robust positivity.

A. Robust Positivity

The following problem arises in controller synthesis and robust stability problems: Given a box of parameters, determine if a set of polynomial functions of these parameters is positive over this box. For example, consider the characteristic polynomial of a control system containing controller parameters and plant parameters. The Routh table yields a set of polynomial functions of these parameters which must be sign invariant (positive or negative) over the box of design (controller) and uncertain (plant) parameters. Motivated by this, we formulate the following robust positivity problem:

Let

$$\mathbf{x} = (x_1, x_2, \cdots, x_l) \tag{1}$$

be a real vector, $f(\mathbf{x})$ a real polynomial function of $\mathbf{x}$, and consider the problem of determining if $f(\mathbf{x})$ is positive for all $\mathbf{x} \in \mathcal{B}$, where $\mathcal{B}$ is the box:

$$\mathcal{B} = \left\{ \mathbf{x} : x_i^- \leq x_i \leq x_i^+, \quad i = 1, 2, \cdots, l \right\}. \tag{2}$$

A related problem is: In case $f(\mathbf{x})$ is not robustly positive over $\mathcal{B}$, determine subsets $\mathcal{B}^+$ of $\mathcal{B}$ over which it is positive.

Without loss of generality, we can assume that $\mathcal{B}$ lies in the first orthant with $x_i \geq 0$ for $i = 1, 2, \cdots, l$. Indeed if $\hat{\mathcal{B}}$ is an arbitrary box

$$\hat{\mathcal{B}} = \left\{ \hat{\mathbf{x}} : \hat{x}_i^- \leq \hat{x}_i \leq \hat{x}_i^+, \quad i = 1, 2, \cdots, l \right\}. \tag{3}$$

we can introduce the change of coordinates

$$\hat{x}_i = a_i x_i + b_i \tag{4}$$

with

$$a_i = \frac{\hat{x}_i^+ - \hat{x}_i^-}{x_i^+ - x_i^-}, \quad i = 1, 2, \cdots, l \tag{5}$$

$$b_i = \frac{x_i^+ \hat{x}_i^- - x_i^- \hat{x}_i^+}{x_i^+ - x_i^-}, \quad i = 1, 2, \cdots, l \tag{6}$$

to transform the box $\hat{\mathcal{B}}$ in (3) to $\mathcal{B}$ (2). By choosing x_i^- in the first orthant ($x_i^- \geq 0, i = 1, 2, \cdots, i$) the box $\hat{\mathcal{B}}$ is relocated to the first orthant.

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

With $\mathcal{B}$ situated in the first orthant, we can make the sign definite decomposition

$$f(\mathbf{x}) = f^+(\mathbf{x}) - f^-(\mathbf{x}) \tag{7}$$

where $f^+(\mathbf{x})$ and $f^-(\mathbf{x})$ are uniquely determined polynomial functions of $\mathbf{x}$ with positive coefficients.

Identify the following vertices of $\mathcal{B}$

$$\mathbf{x}^- := \left(x_1^-, x_2^-, \cdots, x_l^-\right) \tag{8}$$
$$\mathbf{x}^+ := \left(x_1^+, x_2^+, \cdots, x_l^+\right). \tag{9}$$

Example 1 Consider the function

$$\hat{f}(\hat{\mathbf{x}}) = 2 + 3\hat{x}_1^3 - \hat{x}_1\hat{x}_2 - \hat{x}_1^2$$

and the box

$$\hat{\mathcal{B}} = \{\hat{\mathbf{x}} : \hat{x}_1 \in [-1, 1], \hat{x}_2 \in [-1, 2]\}.$$

Using the transformation

$$\hat{x}_1 = 2x_1 - 1, \quad \hat{x}_2 = 3x_2 - 1.$$

$\hat{\mathcal{B}}$ is transformed into the new box:

$$\mathcal{B} = \{\mathbf{x} : x_1 \in [0, 1], x_2 \in [0, 1]\}$$

and the corresponding function:

$$f(\mathbf{x}) = -3 - 6x_1x_2 + 24x_1 + 3x_2 + 24x_1^3 - 40x_1^2$$

so that

$$f^+(\mathbf{x}) = 24x_1 + 3x_2 + 24x_1^3$$
$$f^-(\mathbf{x}) = 3 + 6x_1x_2 + 40x_1^2$$

and $\mathbf{x}^- = [0, 0]$, $\mathbf{x}^+ = [1, 1]$.

Based on the above sign definite decomposition, we have the following.

Lemma 1 *For all* $\mathbf{x} \in \mathcal{B}$*, the following inequalities hold:*

$$f^+(\mathbf{x}^-) \le f^+(\mathbf{x}) \le f^+(\mathbf{x}^+) \tag{10}$$
$$f^-(\mathbf{x}^-) \le f^-(\mathbf{x}) \le f^-(\mathbf{x}^+). \tag{11}$$

The function $f(\mathbf{x})$ can be represented in the (f^-, f^+) plane by associating $f(\mathbf{x})$ with the point $f^-(\mathbf{x})$, $f^+(\mathbf{x})$ as shown below (Fig. 1).

Consider the rectangle formed by the four points in the (f^-, f^+) plane

$$\mathrm{A} = \left(f^-(\mathbf{x}^-), f^+(\mathbf{x}^-)\right),$$
$$\mathrm{B} = \left(f^-(\mathbf{x}^-), f^+(\mathbf{x}^+)\right),$$
$$\mathrm{C} = \left(f^-(\mathbf{x}^+), f^+(\mathbf{x}^+)\right),$$
$$\mathrm{D} = \left(f^-(\mathbf{x}^+), f^+(\mathbf{x}^-)\right).$$

From Lemma 1, we have the following results.

Lemma 2 *For every* $\mathbf{x} \in \mathcal{B}$, $(f^-(\mathbf{x}), f^+(\mathbf{x}))$ *lies inside* ABCD *(Fig. 2):*

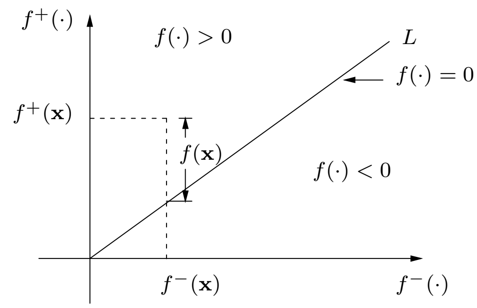

Fig. 1. $f^+(\cdot)$ and $f^-(\cdot)$ representation

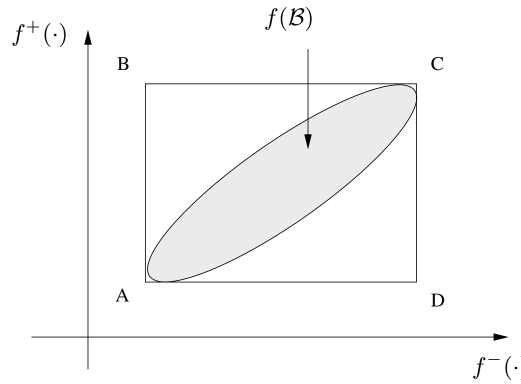

Fig. 2. A rectangle ABCD

Lemma 3 *For all* $\mathbf{x} \in \mathcal{B}$*,*

$$f(\mathbf{x}) \begin{cases} > 0, & \text{if } f^+(\mathbf{x}^-) - f^-(\mathbf{x}^+) > 0, \\ < 0, & \text{if } f^+(\mathbf{x}^+) - f^-(\mathbf{x}^-) < 0, \end{cases}$$

Proof: Follows from Lemmas 1 and 2 and the three possible relationships between the line L in Fig. 1 and the rectangle ABCD as shown in (Fig. 3). ∎

Recursive Algorithm

In Fig. 3.(III), B and D lie on opposite side of L

$$f^+(\mathbf{x}^+) - f^-(\mathbf{x}^-) > 0$$

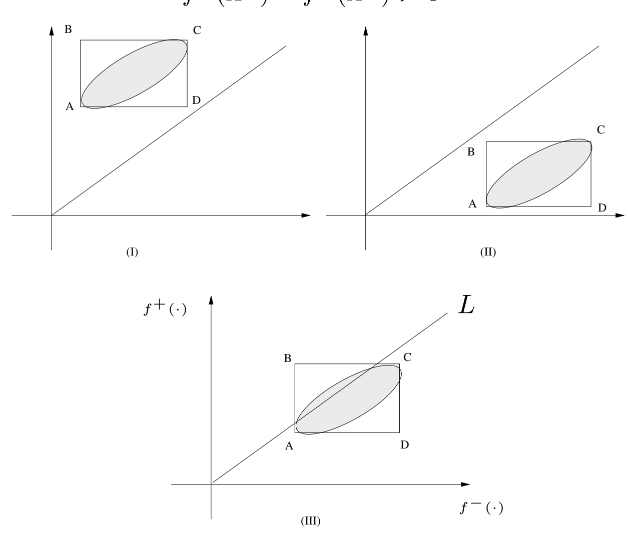

Fig. 3. Three possible relationships between the line L and ABCD

$$f^+(\mathbf{x}^-) - f^-(\mathbf{x}^+) < 0 \tag{12}$$

and it is not possible to conclude robust positivity or negativity. In this case, the box $\mathcal{B}$ can be decomposed into smaller boxes $\mathcal{B}_k, k = 1, 2, \cdots, m$ so that

$$\mathcal{B} = \cup_{k=1}^m \mathcal{B}_k \tag{13}$$

and the above test applied to each $\mathcal{B}_k$. This can be repeated recursively to generate subsets $\mathcal{B}^+$ and $\mathcal{B}^-$ of $\mathcal{B}$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{B}^+$ and $f(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathcal{B}^-$. In general, $\mathcal{B}^+$ or $\mathcal{B}^-$ are unions of boxes but are not necessarily box like or even connected. If a number of functions $f_i(\mathbf{x})$ are to be robustly positive, one can determine the corresponding $\mathcal{B}_i^+$ and find $\cap_i \mathcal{B}_i^+$.

III. MAIN RESULTS

Consider the polynomial family

$$\mathcal{P} := \{P(s, \mathbf{x}) : \mathbf{x} \in \mathcal{B}\} \tag{14}$$

where $\mathcal{B}$ is a box in the first orthant. A typical element of the family is

$$P(s) = a_0(\mathbf{x}) + a_1(\mathbf{x})s + a_2(\mathbf{x})s^2 + a_3(\mathbf{x})s^3 + a_4(\mathbf{x})s^4 + \cdots$$

where $a_i(\mathbf{x})$ are polynomial functions of $\mathbf{x}$ for $i = 0, 1, \cdots, n$ and admit the decomposition

$$a_i(\mathbf{x}) = a_i^+(\mathbf{x}) - a_i^-(\mathbf{x}). \tag{15}$$

We assume throughout that

$$a_n(\mathbf{x}) \neq 0, \quad \text{for all } \ \mathbf{x} \in \mathcal{B}. \tag{16}$$

Since $\mathbf{x} \in \mathcal{B}$ and $\mathbf{B}$ is in the first orthant, the above decomposition is sign definite:

$$a_i^+(\mathbf{x}),\ a_i^-(\mathbf{x}) > 0, \quad \text{for all } \mathbf{x} \in \mathcal{B}.$$

Now define

$$\begin{aligned}
P_{\text{even}}^+(s^2, \mathbf{x}) &:= a_0^+(\mathbf{x}) - a_2^-(\mathbf{x})s^2 + a_4^+(\mathbf{x})s^4 - \cdots \\
P_{\text{even}}^-(s^2, \mathbf{x}) &:= a_0^-(\mathbf{x}) - a_2^+(\mathbf{x})s^2 + a_4^-(\mathbf{x})s^4 - \cdots \\
sP_{\text{odd}}^+(s^2, \mathbf{x}) &:= s\left[a_1^+(\mathbf{x}) - a_3^-(\mathbf{x})s^2 + a_5^+(\mathbf{x})s^4 - \cdots\right] \\
sP_{\text{odd}}^-(s^2, \mathbf{x}) &:= s\left[a_1^-(\mathbf{x}) - a_3^+(\mathbf{x})s^2 + a_5^-(\mathbf{x})s^4 - \cdots\right]
\end{aligned}$$

and

$$\begin{aligned}
P_{\text{even}}(s^2, \mathbf{x}) &:= P_{\text{even}}^+(s^2, \mathbf{x}) - P_{\text{even}}^-(s^2, \mathbf{x}) \\
sP_{\text{odd}}(s^2, \mathbf{x}) &:= sP_{\text{odd}}^+(s^2, \mathbf{x}) - sP_{\text{odd}}^-(s^2, \mathbf{x}).
\end{aligned}$$

Finally, let

$$\begin{aligned}
\bar{P}_{\text{even}}(s^2) &:= P_{\text{even}}^+(s^2, \mathbf{x}^+) - P_{\text{even}}^-(s^2, \mathbf{x}^-) \\
\underline{P}_{\text{even}}(s^2) &:= P_{\text{even}}^+(s^2, \mathbf{x}^-) - P_{\text{even}}^-(s^2, \mathbf{x}^+) \\
s\bar{P}_{\text{odd}}(s^2) &:= sP_{\text{odd}}^+(s^2, \mathbf{x}^+) - sP_{\text{odd}}^-(s^2, \mathbf{x}^-) \\
s\underline{P}_{\text{odd}}(s^2) &:= sP_{\text{odd}}^+(s^2, \mathbf{x}^-) - sP_{\text{odd}}^-(s^2, \mathbf{x}^+).
\end{aligned} \tag{17}$$

Theorem 1 *The family $\mathcal{P}$ is robustly Hurwitz stable if the following four fixed polynomials are Hurwitz stable.*

$$\begin{aligned}
P_1(s) &= \underline{P}_{\text{even}}(s^2) + s\underline{P}_{\text{odd}}(s^2) \\
P_2(s) &= \underline{P}_{\text{even}}(s^2) + s\bar{P}_{\text{odd}}(s^2) \\
P_3(s) &= \bar{P}_{\text{even}}(s^2) + s\bar{P}_{\text{odd}}(s^2) \\
P_4(s) &= \bar{P}_{\text{even}}(s^2) + s\underline{P}_{\text{odd}}(s^2)
\end{aligned} \tag{18}$$

To prove the theorem, we require the following. Let $co\{v_1, v_2, \cdots, v_k\}$ denote the convex hull of the complex plane points $v_1, v_2, \cdot, v_k$.

Lemma 4

$$\{P(j\omega, \mathbf{x}), \quad \mathbf{x} \in \mathcal{B}\} \subset co\{P_1(j\omega), \cdots, P_4(j\omega)\}.$$

Proof: We have

$$P(j\omega, \mathbf{x}) = P_{\text{even}}(-\omega^2, \mathbf{x}) + j\omega P_{\text{odd}}(-\omega^2, \mathbf{x}), \quad \mathbf{x} \in \mathcal{B}$$

where

$$\begin{aligned}
P_{\text{even}}(-\omega^2, \mathbf{x}) &= P_{\text{even}}^+(-\omega^2, \mathbf{x}) - j\omega P_{\text{even}}^-(-\omega^2) \\
P_{\text{odd}}(-\omega^2, \mathbf{x}) &= P_{\text{odd}}^+(-\omega^2, \mathbf{x}) - j\omega P_{\text{odd}}^-(-\omega^2)
\end{aligned}$$

The real part is bounded by

$$\begin{aligned}
&\underbrace{P_{\text{even}}^+(-\omega^2, \mathbf{x}^-) - P_{\text{even}}^-(-\omega^2, \mathbf{x}^+)}_{\underline{P}_{\text{even}}(-\omega^2)} \leq \\
&P_{\text{even}}(-\omega^2, \mathbf{x}) \leq \underbrace{P_{\text{even}}^+(-\omega^2, \mathbf{x}^+) - P_{\text{even}}^-(-\omega^2, \mathbf{x}^-)}_{\bar{P}_{\text{even}}(-\omega^2)}
\end{aligned} \tag{19}$$

Similarly, the imaginary part is bounded by

$$\begin{aligned}
&\underbrace{P_{\text{odd}}^+(-\omega^2, \mathbf{x}^-) - P_{\text{odd}}^-(-\omega^2, \mathbf{x}^+)}_{\underline{P}_{\text{odd}}(-\omega^2, \mathbf{x})} \leq \\
&P_{\text{odd}}(-\omega^2) \leq \underbrace{P_{\text{odd}}^+(-\omega^2, \mathbf{x}^+) - P_{\text{odd}}^-(-\omega^2, \mathbf{x}^-)}_{\bar{P}_{\text{odd}}(-\omega^2)}
\end{aligned} \tag{20}$$

This is depicted in Fig. 4. ■

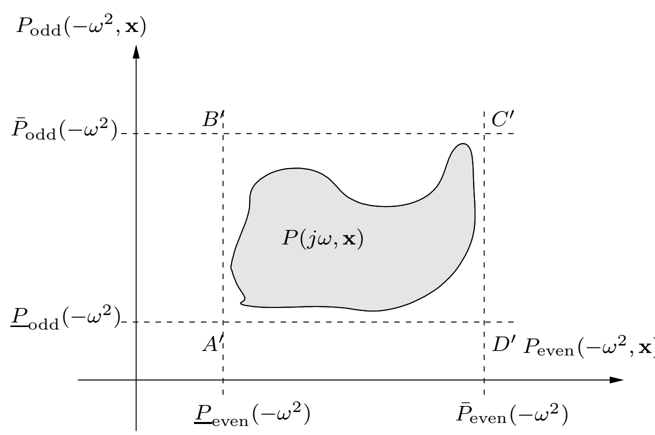

Fig. 4. A bounded image set

We require another technical lemma before giving the proof of Theorem 1.

Lemma 5 *The convex combinations*

$$\begin{aligned}&\lambda_1 P_1(s)+(1-\lambda_1)P_2(s),\\&\lambda_2 P_2(s)+(1-\lambda_2)P_3(s),\\&\lambda_3 P_3(s)+(1-\lambda_3)P_4(s),\\&\lambda_3 P_4(s)+(1-\lambda_4)P_1(s),\end{aligned}$$

$\lambda_i \in [0,1]$ *are Hurwitz stable if and only if* $P_1(s)$*,* $P_2(s)$*,* $P_3(s)$*,* $P_4(s)$ *are Hurwitz stable.*

Proof: Following the Vertex Lemma [16], the above segments are Hurwitz stable since in each case the even and odd part of the endpoint are the same. ■

We now give the proof of Theorem 1.

Proof: (**Theorem 1**) Consider an arbitrary polynomial $P(s,\mathbf{x}^*)$ in the family $\mathcal{P}$ with $\mathbf{x}^* \in \mathcal{B}$. It is clear from Lemma 4 that the image $P(j\omega,\mathbf{x}^*)$ is contained in the rectangle $(ABCD)$ for every ω. Since the vertices are Hurwitz stable, the rectangle will pass through n quadrants as ω runs 0 to ∞ and so does the image $P(j\omega,\mathbf{x}^*)$. Therefore, $P(s,\mathbf{x}^*)$is Hurwitz and the theorem is proved. ■

Example 2 (Kharitonov's Theorem [17]) Consider the interval family of polynomials

$$P(s)=x_0+x_1s+x_2s^2+x_3s^3+x_4s^4+x_5s^5+x_6s^6+x_7s^7+\cdots$$

where

$$0<x_i^-<x_i<x_i^+.$$

Note that, using the previous notations,

$$a_i=x_i=a_i^+ \quad \text{and} \quad a_i^-=0.$$

$$\begin{aligned}P^+_{\text{even}}(s^2)&=x_0+x_4s^4+x_8s^8+\cdots\\P^-_{\text{even}}(s^2)&=-x_2s^2-x_6s^6-x_{10}s^{10}+\cdots\\sP^+_{\text{odd}}(s^2)&=x_1s+x_5s^5+x_9s^9+\cdots\\sP^-_{\text{odd}}(s^2)&=-x_3s^3-x_7s^7-x_{11}s^{11}+\cdots\end{aligned}$$

and

$$\begin{aligned}\bar{P}_{\text{even}}(s^2)&=P^+_{\text{even}}(s^2,\mathbf{x}^+)-P^-_{\text{even}}(s^2,\mathbf{x}^-)\\&=x_0^++x_2^-s^2+x_4^+s^4+x_6^-s^6+x_8^+s^8+\cdots\\\underline{P}_{\text{even}}(s^2)&=P^+_{\text{even}}(s^2,\mathbf{x}^-)-P^-_{\text{even}}(s^2,\mathbf{x}^+)\\&=x_0^-+x_2^+s^2+x_4^-s^4+x_6^+s^6+x_8^-s^8+\cdots\\s\bar{P}_{\text{odd}}(s^2)&=sP^+_{\text{odd}}(s^2,\mathbf{x}^+)-sP^-_{\text{odd}}(s^2,\mathbf{x}^-)\\&=x_1^+s+x_3^-s^3+x_5^+s^5+x_7^-s^7+x_9^+s^9+\cdots\\s\underline{P}_{\text{odd}}(s^2)&=sP^+_{\text{odd}}(s^2,\mathbf{x}^-)-sP^-_{\text{odd}}(s^2,\mathbf{x}^+)\\&=x_1^-s+x_3^+s^3+x_5^-s^5+x_7^+s^7+x_9^-s^9+\cdots\end{aligned}$$

Therefore,

$$\begin{aligned}P^1(s)&=\underline{P}_{\text{even}}(s^2)+s\underline{P}_{\text{odd}}(s^2)\\&=x_0^-+x_1^-s+x_2^+s^2+x_3^+s^3+x_4^-s^4+\cdots\\P^2(s)&=\underline{P}_{\text{even}}(s^2)+s\bar{P}_{\text{odd}}(s^2)\\&=x_0^-+x_1^+s+x_2^+s^2+x_3^-s^3+x_4^-s^4+\cdots\\P^3(s)&=\bar{P}_{\text{even}}(s^2)+s\bar{P}_{\text{odd}}(s^2)\\&=x_0^++x_1^+s+x_2^-s^2+x_3^-s^3+x_4^+s^4+\cdots\\P^4(s)&=\bar{P}_{\text{even}}(s^2)+s\underline{P}_{\text{odd}}(s^2)\\&=x_0^++x_1^-s+x_2^-x^2+x_3^+s^3+x_4^+s^4+\cdots\end{aligned}$$

Therefore, Kharitonov's theorem has been recovered from Theorem 1.

IV. CONTROLLER SYNTHESIS

In this conference paper, it is best to illustrate the use of Theorem 1 in controller synthesis by examples.

Example 3 Consider the feedback system with the plant and controller transfer function matrix

$$G(s)=\begin{bmatrix}\dfrac{s-5}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)}\\[2ex]\dfrac{s+6}{(s+3)(s+4)} & \dfrac{s-7}{(s+3)(s+4)}\end{bmatrix}$$

and

$$C(s)=\begin{bmatrix}K_1 & 0\\0 & K_2\end{bmatrix}.$$

We now have

$$G(s)=D_g(s)^{-1}N_g(s) \quad \text{and} \quad C(s)=N_c(s)D_c(s)^{-1}$$

where

$$\begin{aligned}D_g(s)&=\begin{bmatrix}(s+1)(s+2) & 0\\0 & (s+3)(s+4)\end{bmatrix},\\N_g(s)&=\begin{bmatrix}s-5 & s\\s+6 & s-7\end{bmatrix}\end{aligned}$$

and

$$D_c(s)=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\quad N_c(s)=\begin{bmatrix}K_1 & 0\\0 & K_2\end{bmatrix}.$$

The characteristic polynomial of the closed-loop system is

$$\begin{aligned}P(s,K_1,K_2)=\,&s^4+(10+K_2+K_1)s^3\\&+(-4K_2+2K_1+35)s^2\\&+(50-18K_2K_1-23K_1-19K_2)s\\&+(24+35K_2K_1-60K_1-14K_2)\end{aligned}$$

We begin searching for the set of stabilizing parameters (K_1,K_2) inside the box $(K_1,K_2)\in[0,1]\times[0,1]$. Let $K_1\in[K_1^-,K_1^+]$ and $K_2\in[K_2^-,K_2^+]$. Also denote

$$\mathbf{x}:=[K_1 \quad K_2].$$

We now have

$$P^+_{\text{even}}(s^2,\mathbf{x}) = s^4 - 4K_2 s^2 + (24 + 35K_2K_1)$$
$$P^-_{\text{even}}(s^2,\mathbf{x}) = -(2K_1 + 35)s^2 + (60K_1 + 14K_2)$$
$$sP^+_{\text{odd}}(s^2,\mathbf{x}) = 50s$$
$$sP^-_{\text{odd}}(s^2,\mathbf{x}) = -(10 + K_2 + K_1)s^3 + (18K_2K_1 + 23K_1 + 19K_2)s.$$

Thus,

$$P^1(s) = s^4 - (10 + K_2^- + K_1^-)s^3 + (-4K_2^+ + 2K_1^- + 35)s^2 + (50 - 18K_2^-K_1^- - 23K_1^- - 19K_2^-)s + (24 + 35K_2^+K_1^+ - 60K_1^- - 14K_2^-)$$
$$P^2(s) = s^4 - (10 + K_2^+ + K_1^+)s^3 + (-4K_2^+ + 2K_1^- + 35)s^2 + (50 - 18K_2^+K_1^+ - 23K_1^+ - 19K_2^+)s + (24 + 35K_2^+K_1^+ - 60K_1^- - 14K_2^-)$$
$$P^3(s) = s^4 - (10 + K_2^- + K_1^-)s^3 + (-4K_2^- + 2K_1^+ + 35)s^2 + (50 - 18K_2^-K_1^- - 23K_1^- - 19K_2^-)s + (24 + 35K_2^-K_1^- - 60K_1^+ - 14K_2^+)$$
$$P^4(s) = s^4 - (10 + K_2^+ + K_1^+)s^3 + (-4K_2^- + 2K_1^+ + 35)s^2 + (50 - 18K_2^+K_1^+ - 23K_1^+ - 19K_2^+)s + (24 + 35K_2^-K_1^- - 60K_1^+ - 14K_2^+).$$

As stated above, stability of the above four polynomials is a sufficient condition for robust stability of the feedback system. The search for the stabilizing region will be done by bisecting the box whenever the sufficient condition fails. We finally obtained the stabilizing region within the given box as:

$$(K_1, K_2) \in [0, 0.399] \times [0, 1].$$

Fig. 5 shows that the box with its four corners at $P^i(j\omega)$, $i = 1, 2, 3, 4$ clearly contains the image set at a fixed frequency. Fig. 6 illustrates the evolution of the image set over the range of frequency.

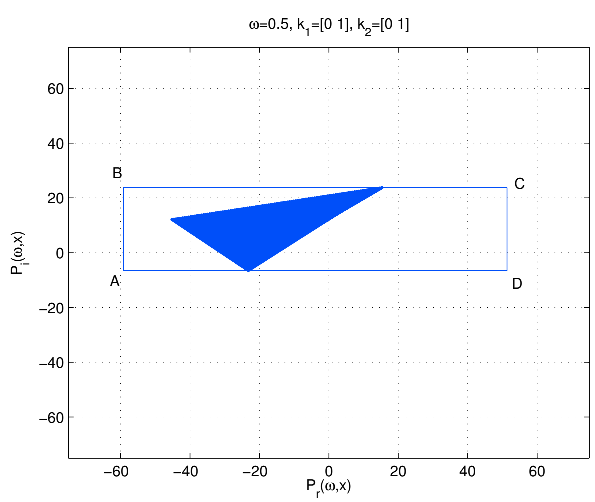

Fig. 5. Showing the sufficiency

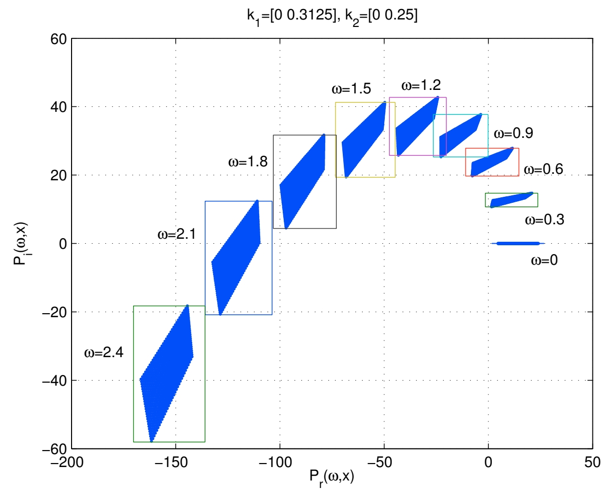

Fig. 6. Showing the sufficiency for the range of frequency

Example 4 Consider the plant and controller transfer function matrices

$$G(s) = \begin{bmatrix} \dfrac{s-a}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \\ \dfrac{s+6}{(s+3)(s+4)} & \dfrac{s-b}{(s+3)(s+4)} \end{bmatrix}$$

and

$$C(s) = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$

Then, we have the characteristic polynomial of the closed-loop system

$$P(s, K_1, K_2, a, b) = s^4 + (10 + K_2 + K_1)s^3 + (-K_2b + 3K_2 - K_1a + 7K_1 + 35)s^2 + (-K_1K_2b - 6K_1K_2 - K_1K_2a - 3K_2b + 2K_2 - 7K_1a + 12K_1 + 50)s^2 + (K_1K_2ab - 2K_2b - 12K_1a + 24)$$

The objective of the design is to find the region of stabilizing controller parameters (K_1, K_2) inside the box $(K_1, K_2) \in [0,\ \ 1] \times [0,\ \ 1]$ so that the closed-loop system is robustly stable under all plant parameter variations

$$a \in [a^-,\ \ a^+] \quad \text{and} \quad b \in [b^-,\ \ b^+].$$

We now have

$$P^+_{\text{even}}(s^2,\mathbf{x}) = s^4 - (K_2b + K_1a)s^2 + (24 + K_1K_2ab)$$
$$P^-_{\text{even}}(s^2,\mathbf{x}) = -(35 + 3K_2 + 7K_1)s^2 + (2K_2b + 12K_1a)$$
$$sP^+_{\text{odd}}(s^2,\mathbf{x}) = (2K_2 + 12K_1 + 50)s$$
$$sP^-_{\text{odd}}(s^2,\mathbf{x}) = -(K_1 + K_2 + 10)s^3 + (K_1K_2b + 3K_2b + 7K_1a + K_1K_2a + 6K_1K_2)s.$$

Thus,

$$\begin{aligned} P^1(s) &= s^4 + (K_1^+ + K_2^+ + 10)s^3 - (K_2^- b^- + K_1^- a^- \\ &+35 + 3K_2^+ + 7K_1^+)s^2 + (2K_2^- + 12K_1^- + 50 \\ &-K_1^+K_2^+b^+ - 3K_2^+b^+ - 7K_1^+a^+ - K_1^+K_2^+a^+ \\ &-6K_1^+K_2^+)s + (24 + K_1^-K_2^-a^-b^- - 2K_2^+b^+ \\ &-12K_1^+a^+) \\ P^2(s) &= s^4 + (K_1^- + K_2^- + 10)s^3 - (K_2^- b^- + K_1^- a^- \\ &+35 + 3K_2^+ + 7K_1^+)s^2 + (2K_2^+ + 12K_1^+ + 50 \\ &-K_1^-K_2^-b^- - 3K_2^-b^- - 7K_1^-a^- - K_1^-K_2^-a^- \\ &-6K_1^-K_2^-)s + (24 + K_1^-K_2^-a^-b^- - 2K_2^+b^+ \\ &-12K_1^+a^+) \\ P^3(s) &= s^4 + (K_1^- + K_2^- + 10)s^3 - (K_2^+ b^+ + K_1^+ a^+ \\ &+35 + 3K_2^- + 7K_1^-)s^2 + (2K_2^+ + 12K_1^+ + 50 \\ &-K_1^-K_2^-b^- - 3K_2^-b^- - 7K_1^-a^- - K_1^-K_2^-a^- \\ &-6K_1^-K_2^-)s + (24 + K_1^+K_2^+a^+b^+ - 2K_2^-b^- \\ &-12K_1^-a^-) \\ P^4(s) &= s^4 + (K_1^+ + K_2^+ + 10)s^3 - (K_2^+ b^+ + K_1^+ a^+ \\ &+35 + 3K_2^- + 7K_1^-)s^2 + (2K_2^- + 12K_1^- + 50 \\ &-K_1^+K_2^+b^+ - 3K_2^+b^+ - 7K_1^+a^+ - K_1^+K_2^+a^+ \\ &-6K_1^+K_2^+)s + (24 + K_1^+K_2^+a^+b^+ - 2K_2^-b^- \\ &-12K_1^-a^-). \end{aligned}$$

Keeping the box of plant parameters that represent robustness requirement, the algorithm bisects the controller parameter box whenever the sufficient condition fails. After several iterations, we obtain the box

$$(K_1, K_2) \in [0, 0.333] \times [0, 1]$$

that robustly stabilizes the closed-loop system under the plant parameter perturbations in $(a, b) \in [4, 6] \times [6, 8]$.

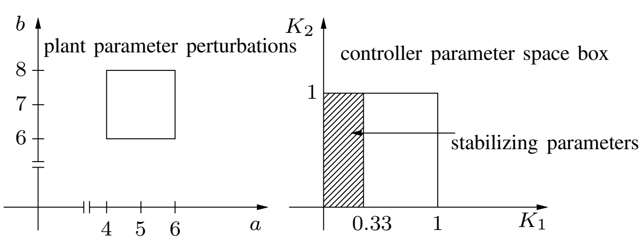

Fig. 7. Parameter spaces

V. CONCLUDING REMARKS

The algorithm given here can be enhanced by adding some necessary conditions that give outer approximations of the stabilizing sets. These conditions could for example be robust positivity of the coefficients. An issue of importance for further research is the search for performance attaining subsets. It would also be important and nice to obtain data based, model free versions of the current results.

REFERENCES

[1] Special Issue on PID2006, *IEEE Control Systems Magazine*, February, 2006.

[2] S. Hara, D. Shiokata, and T. Iwasaki, “Fixed order controller design via generalized KYP lemma” *Proceedings of the IEEE Conference on Control Applications*, Taipei, Taiwan, September 2-4, 2004

[3] D. Shiokata, S. Hara, and T. Iwasaki, “From Nyquist/Bode to GKYP design: design algorithms and CACSD tools,” *Proceedings of the American Control Conference*, Boston, MA, June 30-July 2, 2004.

[4] S. Hara, T. Iwasaki, and D. Shiokata, “Robust PID control using generalized KYP synthesis: Direct open-loop shaping in multiple frequency ranges,” *IEEE Control Systems Magazine*, vol. 26, pp. 80 – 91, February, 2006.

[5] D. Henrion, A. Hansson, and R. Wallin, “Reduced LMIs for fixed-order polynomial controller design,” *Proceedings of the Symposium on Mathematical Theory of Networks and Systems (MTNS)*, Leuven, Belgium, July 5-9, 2004.

[6] T. Iwasaki and R.E. Skelton, “All fixed order H_∞ controllers: observer based structure and covariance bounds,” *IEEE Transactions on Automatic Control*, Vol. 40, pp. 512 - 516, March 1995.

[7] P. Dorato, “Quantified multivariable polynomial inequalities: The mathematics of (almost) all practical design problems,” *Proceedings of the Sixth IEEE Mediterranean Conference on Control and Systems*, Alghero, Italy, June 9 - 11, 1998.

[8] E.N. Gryazina and B.T. Polyak, “Stability regions in the parameter space: D-decomposition revisited,” *Automatica*, vol. 42, no. 1, pp. 13 – 26, January, 2006.

[9] Yu. I. Neimark, *Stability of Linearized Systems* (in Russian), LKVVIA, Leningrad, 1949.

[10] D.D. Šiljak, *Nonlinear Systems: The Parameter Analysis and Design*, Wiley, New York, 1969.

[11] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SAIM, Philadelphia, PA, 1994.

[12] R.R.E. DeGaston and M.G. Safonov, “Exact calculation of the multiloop stability margin,” *IEEE Transactions on Automatic Control*, vol. AC-33, no. 2, pp. 156 - 171, February, 1988.

[13] C. Elizondo-Gonzalez, “Necessary and sufficient conditions for robust positivity of polynomic functions via sign decomposition,” Proceedings of the 3rd IFAC Symposium on Robust Control Design (ROCOND 2000), Prague, Czech Republic, 2000.

[14] D.D. Šiljak and D.M. Stipanovic, “SPR criteria for uncertain rational matrices via polynomial positivity and Bernstein’s expansions,” *IEEE Transactions of Circuits and Systems*, vol.48, no. 11, pp. 1366 - 1369, 2001

[15] D.D. Šiljak and D.M. Stipanovic, “Robust D-stability via positivity,” *Automatica*, vol. 35, no. 8, pp. 1477 - 1484, 1999.

[16] S.P. Bhattacharyya, H. Chapellat, and L.H. Keel, *Robust Control: The Parametric Approach*, Prentice Hall PTR, Upper Saddle River, NJ, 1995.

[17] V.L. Kharitonov, “Asymptotic stability of an equilibrium position of a family of systems of linear differential equations,” *Differential Uravnen*, vol. 14, pp. 2086 - 2088, 1978. English translation in *Differential Equations*, vol. 14, pp. 1483 - 1485, 1979.