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Observation of a class of quasilinear systems by quasi-continuous high-order sliding modes

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Abstract— **High-order sliding-mode observers are designed for some classes of non linear systems with unknown inputs. Conditions for the feasibility of the proposed approach are given for both the SISO and MIMO cases under the conditions of strong observability or strong detectability. Simulations confirm the theoretical results.**

I. Introduction

Sliding-mode-based robust state observation was successfully developed in recent years [1], [2], [3], [4], [5], and the corresponding implementation issues were extensively studied in [6] and [7]. Under appropriate conditions on the system matrices the sliding-mode-based observation possesses such an attractive feature of **insensitivity** (which is more than robustness) with respect to unknown inputs.

Step-by-step vector-state reconstruction by means of sliding modes is studied by [1], [8], [9], [10]. These observers are based on a system transformation into a triangular form and successive estimation of the state vector using the equivalent output injection. Unfortunately, the realization of step-by-step observers via conventional sliding modes requires *filtration* at each step which produces an intrinsic observation error which cannot be removed.

To overcome this drawback, hierarchical observers based on super-twisting algorithm were developed [11] which avoid any filtration process. A modified version of the super-twisting controller is also used in the step-by-step observer by [10]. In [5] it is presented a different high-order sliding-mode observer based on the reconstruction of the estimation error by the robust exact sliding-mode differentiator [12]. This observer provides **global** finite time convergence of the estimation error for systems with well defined relative degree [13].

Here we study the observation problem for a class of quasilinear systems with bounded but unknown nonlinear functions using the quasicontinuous controller [14], [15]. The notions of strong observability and strong detectability with respect to the unknown inputs are involved. In this paper we provide:

- global finite-time exact state observation for single output strongly-observable quasilinear systems;
- asymptotic state observation for single output quasilinear strongly detectable systems;
- extension of these methods for multiple outputs strongly observable and strongly detectable quasilinear systems.

The overall observer scheme combines a linear Luenberger observer for the known part of the system dynamics, a finite time converging high-order sliding mode differentiator, and another observer fed by a nonlinear compensation term designed according to the quasi-continuous high order sliding mode approach [14], [15].

The paper is structured as follows: Section II introduces the main notions that will be used along the paper. Section III studies the single output case while Section IV presents the generalization of the presented methods to the multiple outputs case. Section V presents some simulation results and Section VI gives some concluding remarks.

II. Fundamentals

Consider system

$$\begin{aligned} \dot{x} &= Ax + Bu + D\gamma(t,x), \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^q$ is the known input vector, $\gamma(t,x) \in \mathbb{R}^m$ is the unknown nonlinear functions vector, $y \in \mathbb{R}^p$ is the output, and the known matrices A, B, C, D have appropriate dimension. The distribution matrix D corresponding to the unknown nonlinear functions vector $\gamma(t,x)$ is used to define the main concepts used along the paper.

Some definitions and important implications concerning the properties of strong observability and strong detectability are recalled in this section. Since such properties are not affected by the nonlinear functions vector $\gamma(t,x)$, only the triple $\{A, D, C\}$ is involved in the next definitions, and the known inputs can be customarily set to zero with no loss of generality.

Definition 1: [16] $s_0 \in \mathbb{C}$ is called an invariant zero of the triple $\{A, D, C\}$ if $\operatorname{rank} R(s_0) < n + \operatorname{rank}(D)$, where R is the Rosenbrock matrix of system (1)

$$R = \begin{bmatrix} sI - A, & -D \\ C, & 0 \end{bmatrix}. \tag{2}$$

Definition 2: [17]. System (1) is said to be strongly observable if for any initial state $x(0)$ and for any unknown

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input (i.e., the nonlinear function $\gamma(t,x)$), the condition $y(t) \equiv 0 \;\; \forall t \geq 0$ implies that $x(t) \equiv 0 \;\; \forall t \geq 0$.

The following statements are equivalent ([17]).

(i) The system (1) is strongly observable.

(ii) The triple $\{A, C, D\}$ has no invariant zeros.

Definition 3: [17]. System (1) is said to be strongly detectable, for any initial state $x(0)$ and for any unknown input (i.e., the nonlinear function $\gamma(t,x)$), the condition $y(t) \equiv 0$ $\forall t \geq 0$ implies that $x(t) \to 0$ with $t \to \infty$.

The following statements are equivalent ([17]).

(i) The system (1) is strongly detectable.

(ii) The system (1) is minimum phase (i.e. the invariant zeroes of the triple $\{A, C, D\}$ satisfy Re $s < 0$).

Obviously, when $D = 0$ (i.e. with no unknown inputs acting on the system) the notions of strong observability and strong detectability coincide, respectively, with the standard observability and detectability properties.

Introduce the observability matrix of the pair $(C,\, A)$

$$P = \left[\begin{array}{cccc} C^T & (CA)^T & \cdots & (CA^{n-1})^T \end{array} \right]^T. \tag{3}$$

Recall that system (1) is observable (in the absence of the unknown input) if and only if the observability matrix P has the full rank. In that case the matrix $A - LC$ can be assigned any arbitrary spectrum by choosing an appropriate column matrix parameter L. The integer $n_O = rank(P)$ is called the *observability index* of the system.

III. SINGLE OUTPUT CASE

III-A. *Problem statement*

Consider the quasilinear system (1) with scalar output, i.e. $p = 1$. The nominal matrices A, B, C and D have suitable dimensions and are assumed known. In order to provide for possibility to observe the systems with discontinuous unknown inputs it is considered that the solution of the equations are understood in the Filippov sense [18] and exists for all $t > 0$.

Definition 4: [13] The relative degree of the system (1) with respect to the nonlinearity is the number r such that

$$CA^j D = 0,\; j = 1, ..., r-2, \quad CA^{r-1} D \neq 0. \tag{4}$$

Recall that the nonlinearity of the system is considered as unknown. The next theorems were proven in [5].

Theorem 1: [5]. The system (1) is strongly observable if and only if the output of the system (1) has relative degree n with respect to the unknown input (i.e., the nonlinearity $\gamma(t,x)$).

Theorem 2: [5]. The system (1) is strongly detectable if and only if the relative degree with respect to the unknown input (i.e., the nonlinearity $\gamma(t,x)$) exists, and the system is minimum-phase. In that case also $r \leq n_O$ is ensured.

The task is to build an observer providing for the asymptotic (or, preferably, finite-time converging) estimation of the system state.

III-B. *Strongly observable case*

System (1) is supposed to satisfy the following assumptions.

Assumption 1: The system (1) is strongly observable.

Assumption 2: The scalar nonlinear function $\gamma(t,x)$ is a bounded Lebesgue-measurable function for all t and $x \in \mathcal{X}$, $|\gamma(t,x)| \leq \gamma^+$.

The observer is built in the form

$$\begin{aligned} \dot{z} &= Az + Bu + L(y - Cz), && (5)\\ \dot{\hat{\xi}} &= (A - LC)\hat{\xi} + Gv(t), && (6)\\ \hat{x} &= \hat{\xi} + z, && (7) \end{aligned}$$

where $z \in \mathbb{R}^n$ is the state of the auxiliary Luenberger-like observer (5), $\hat{x} \in \mathbb{R}^n$ is the estimate of x, the column vector $L = [l_1, l_2, ..., l_n]^T \in \mathbb{R}^n$ is chosen so that the eigenvalues of the matrix $A - LC$ have negative real parts, which exists due to Assumption 1. The matrix G, and the nonlinear discontinuous function $v(t)$ (observer compensation input) will be designed along the paper.

The proposed observer is actually composed of two parts. Equation (5) is a Luenberger observer providing a "wrong.[es]timate z of x affected by a bounded error $\xi = x - z$ due to the presence of the unknown bounded nonlinear function $\gamma(\cdot)$. System (6) will be designed to ensure the finite time convergence of $\hat{\xi}$ to ξ. The algebraic equation (7) will thus ensure that the estimation error $x - \hat{x}$ converges to zero in finite time.

Suppose that Assumptions 1 and 2 hold, respectively. Since the pair (C, A) is observable, stable eigenvalues can be arbitrarily assigned to the matrix $(A - LC)$ by choosing an appropriate column gain vector L ([19]). Obviously the pair $(C, A - LC)$ is also observable, and its observability matrix

$$\tilde{P} = \begin{bmatrix} C \\ C(A-LC) \\ \vdots \\ C(A-LC)^{n-2} \\ C(A-LC)^{n-1} \end{bmatrix} \tag{8}$$

is not singular. Set the gain matrix G as the unique solution of the next equation

$$\tilde{P}\, G = [0, 0, \ldots, 0, 1]^T \tag{9}$$

The "quasi-continuous.[a]rbitrary-order sliding mode controller was suggested in [14], [15].

Define $\xi = x - z$, $\xi_y = C\xi$, $\tilde{\xi}_y = \xi_y - \hat{\xi}_y$.

Let $i = 1, \ldots, n-1$ and denote

$$\begin{aligned} \varphi_{0,n} &= \tilde{\xi}_y, \;\; N_{0,n} = |\tilde{\xi}_y| && (10)\\ \Psi_{0,n} &= \varphi_{0,n}/N_{0,n} = \text{ sign } \tilde{\xi}_y, && (11)\\ \varphi_{i,n} &= \tilde{\xi}_y^{(i)} + \beta_i N_{i-1,n}^{(n-i)/(n-i+1)} \Psi_{i-1,n}, && (12)\\ N_{i,n} &= |\tilde{\xi}_y^{(i)}| + \beta_i N_{i-1,n}^{(n-i)(n-i+1)}, && (13)\\ \Psi_{i,n} &= \varphi_{i,n}/N_{i,n} && (14) \end{aligned}$$

where β_1, ..., β_{n-1} are positive numbers.

The corrective term v of the observer (6) is set according to the quasi-continuous n-sliding controller [14], [15]

$$v(t) = -\alpha\Psi_{n-1,n}(\tilde{\xi}_y, \dot{\tilde{\xi}}_y, ..., \tilde{\xi}_y^{(n-1)}). \tag{15}$$

It can be stated the following Theorem:

Theorem 3: Let Assumptions 1 and 2 be satisfied. Then with sufficiently large α and β_i, the state x of the system is estimated in finite time by the observer (5)-(7), (9)-(14) .

Proof: Consider the linear Luenberger part of the observer (5). Denote $\xi = x - z$, $\xi_y = C\xi(t)$, then obtain

$$\begin{aligned} \dot{\xi} &= (A - LC)\xi(t) + D\gamma(t, x), && (16)\\ \xi_y &= C\xi(t). && (17) \end{aligned}$$

Recall that the matrix $A - LC$ is Hurwitz. Considerthe Lyapunov function $V = \frac{1}{2}\xi^T H\xi$, where H is the symmetric positive-definite matrix solution of the Lyapunov equation $H(A - LC) + (A - LC)^T H = -I$. Its derivative

$$\begin{aligned} \dot{V} = \xi^T(H(A - LC) + (A - LC)^T H)\xi +\\ (\xi^T HD + D^T H\xi)\gamma(t, x) \end{aligned}$$

is negative definite with $\gamma(t, x) = 0$. Due to the boundedness of $\gamma(t, x)$, the same properties hold with sufficiently large $\|\xi\|$. Thus obtain that the estimation error ξ converges to a bounded vicinity of the origin.

Define now the output error vector ε, containing the output error $\tilde{\xi}_y = \hat{\xi}_y - \xi_y$ and its first $n - 1$ derivatives:

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} \tilde{\xi}_y \\ \dot{\tilde{\xi}}_y \\ \vdots \\ \tilde{\xi}_y^{(n-1)} \end{bmatrix}. \tag{18}$$

Let us derive the explicit form for the successive derivatives of the output estimation error $\tilde{\xi}_y$ up to the order n. Considering Theorem 1, equation (9), and defining $\tilde{\xi} = \hat{\xi} - \xi$, it yields

$$\begin{aligned} \tilde{\xi}_y^{(j)} &= C(A - LC)^j\tilde{\xi}, \quad 1 \leq j \leq n - 1 && (19)\\ \tilde{\xi}_y^{(n)} &= C(A - LC)^n\tilde{\xi} - CA^{n-1}D\gamma(t, x) + v(t), && (20) \end{aligned}$$

Notice that $C(A - LC)^{n-1}D = CA^{n-1}D$.

The equations (18)-(19) lead to the following mapping

$$\varepsilon = \tilde{P}\tilde{\xi} \tag{21}$$

From the Assumption 1, we get that matrix $\tilde{P}$ is nonsingular thus bijective implication $\tilde{\xi} = 0 \Leftrightarrow \varepsilon = 0$ holds.

It has been shown that, under the Assumption 1, the system (6), (9) can reconstruct the coordinates of the system (16) exactly and in finite time, provided that the observer input $v(t)$ is selected in such a way that the vector ε is steered to zero in finite time.

The ε dynamics takes the following Brunowsky chain-of-integrators canonical form

$$\begin{cases} \dot{\varepsilon}_1 = \dot{\varepsilon}_2 \\ \dot{\varepsilon}_2 = \dot{\varepsilon}_3 \\ \dots \\ \dot{\varepsilon}_n = C(A - LC)^n\tilde{\xi} - CA^{n-1}D\gamma(t, x) + v(t) \end{cases} \tag{22}$$

where $\tilde{\xi}_y = \varepsilon_1$. The previously proven boundedness of vector ξ together with assumption 2 allow us to guarantee the existence of a known constant Γ such that

$$|C(A - LC)^n\tilde{\xi} + CA^{n-1}D\gamma(x)| < \Gamma \tag{23}$$

It was shown in [14], [15] that provided that the tuning parameters β_1, ..., β_{n-1}, α are chosen sufficiently large *in the given order* then the control law defined by (10)-(15) stabilizes the error system $\tilde{\xi} = \xi - \hat{\xi}$ in finite time.

Obtain now that $\hat{\xi} = x - z$, then, by simple algebraic manipulation, the state estimate $\hat{x} = z + \hat{\xi}$ will satisfy the equality $\hat{x} = x$ starting from the moment when $\varepsilon = 0$. ■

III-B.0.a. Output error derivatives estimation: The nth order quasi-continuous controller requires the availability of the successive derivatives of the output estimation error up to the order $n-1$. In order to reconstruct such derivatives exactly and in finite time, the well known Arbitrary-Order sliding-mode differentiator by A. Levant [12] can be used.

The separation results relevant to the combined use of the above differentiator and any n-sliding homogenous controller were discussed in [12].

III-C. *Strongly detectable case*

Introduce a new Assumption 3 which generalizes the Assumption 1.

Assumption 3: System (1) is minimum phase and it has relative degree r with respect to the unknown input (i.e., nonlinearity $\gamma(t, x)$), with $r < n$.

Remark 1: Assumption 3 is equivalent to the stability of the invariant zeros of the system, and also means that the system is strongly detectable in the sense of Definition 3. Note also that Assumption 1 is obtained with $r = n$.

Let $r \leq n$ be the relative degree of system (1) with respect to the nonlinearity $\gamma(t, x)$, which means that

$$CA^iD = 0,\ i = 0, ..., r - 2 \qquad CA^{r-1}D \neq 0. \tag{24}$$

As previously made, define the Luenberger-like observer

$$\dot{z} = Az + Bu + L(y - Cz). \tag{25}$$

and consider the dynamics of the error vector $\tilde{e} = x - z$:

$$\begin{aligned} \dot{\tilde{e}} = (A - LC)\tilde{e} + D\gamma(t, x)\\ \xi_y = C\tilde{e} \end{aligned} \tag{26}$$

Under Assumption 3 it can be found a transformation matrix T and a transformed state vector

$$(\xi_1^T, \xi_2^T)^T = T\tilde{e} = [T_1\ T_2]^T\tilde{e}$$

such that

$$T(A-LC)T^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad TD = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \qquad CT^{-1} = \begin{bmatrix} C_1 & 0 \end{bmatrix} \tag{27}$$

where $A_{21} \in \mathbb{R}^{(n-r)\times r}$, $A_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$, and with the matrices $A_{11} \in \mathbb{R}^{r\times r}$, $A_{12} \in \mathbb{R}^{r\times(n-r)}$, C_1^T and $D_1 \in \mathbb{R}^r$ of the form

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_r \end{bmatrix} \tag{28}$$

$$A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{r+1} & a_{r+2} & \cdots & a_n \end{bmatrix} \tag{29}$$

$$D_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^{r-1}D \end{bmatrix} \quad CA^{r-1}D \neq 0$$

$$, C_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}.$$

By the minimum phase assumption it follows that matrix A_{22} is Hurwitz. The transformed system dynamics are

$$\dot{\xi}_1 = A_{11}\xi_1 + A_{12}\xi_2 + D_1\gamma(t,x) \tag{30}$$

$$\dot{\xi}_2 = A_{21}\xi_1 + A_{22}\xi_2 \tag{31}$$

$$\hat{\xi}_y = C_1\hat{\xi}_1 \tag{32}$$

and the overall observer is constructed as

$$\dot{\hat{\xi}}_1 = A_{11}\hat{\xi}_1 + Gv(t) \tag{33}$$

$$\dot{\hat{\xi}}_2 = A_{21}\hat{\xi}_1 + A_{22}\hat{\xi}_2 \tag{34}$$

$$\hat{x} = z + T^{-1}\begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} \tag{35}$$

$$G = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \tag{36}$$

Define

$$\tilde{\xi}_y = \hat{\xi}_y - \xi_y \tag{37}$$

The compensation term $v(t)$ in (33) is designed by means of the same "quasi-continuous.ªrbitrary-order sliding mode controller used in the previous subsection, but with order r instead of n. Thus the form of v will be

$$v(t) = -\alpha\Psi_{r-2,r-1}(\tilde{\xi}_y, \dot{\tilde{\xi}}_y, ..., \tilde{\xi}_y^{(r-1)}). \tag{38}$$

The observer control (38) provides for the finite time attainment of condition $\tilde{\xi}_y = \dot{\tilde{\xi}}_y = \ldots, \tilde{\xi}_y^{(r-1)} = 0$. Thus, unfortunately, in the actual case only the first r transformed coordinates ξ_1 are reconstructed exactly in finite time, which coincide with the first r components of vector Tx. The next Theorem shows that the "missing"vector state component ξ_2 can be reconstructed asymptotically thanks to the minimum phase property of the system.

Theorem 4: Consider system (1) and let Assumptions 3 and 2 be satisfied. Then the state x of the system is asymptotically estimated by the observer (25)-(35). In particular, the first r coordinates of vector Tx are estimated exactly and in finite time, while the remaining $n-r$ state observation errors tend exponentially to zero.

Proof: The boundedness of the estimation error $\tilde{e}$ governed by the dynamic equations (26) follows from the stability of the matrix $A-LC$ exactly as in the proof of Theorem 3. Furthermore, the triple $(A-LC, D, C)$ is minimum-phase. Let us now show the convergence of $\hat{\xi}_1$ and $\hat{\xi}_2$ to the actual values of ξ_1 and ξ_2 respectively. By (30)-(34) the dynamics of the error variable $\tilde{\xi}_1 = \hat{\xi}_1 - \xi_1$ is

$$\dot{\tilde{\xi}}_1 = A_{11}\tilde{\xi}_1 - A_{12}\xi_2 - D_1\gamma(t,x) + G(v(t)), \quad \tilde{\xi}_y = C_1\tilde{\xi}_1 \tag{39}$$

The dynamic of the rth coordinate of the vector $\tilde{\xi}_1$ takes the form

$$\dot{\tilde{\xi}}_{1_r} = A_{11}^{(r)}\tilde{\xi}_1 - A_{12}^{(r)}\xi_2 - CA^{r-1}D\gamma(t,x) + v(t),$$

where the superindex (r) is used to denote the rth row of the corresponding matrix. Similarly to the proof of Theorem 3, the boundedness of $\tilde{\xi}$ and $\gamma(t,x)$ guarantees that there exist a a known constant Γ such that

$$|A_{11}^{(r)}\tilde{\xi}_1 - A_{12}^{(r)}\xi_2 - CA^{r-1}D\gamma(t,x)| < \Gamma \tag{40}$$

Thus, under adequate selection of its parameter gains, the high-order sliding-mode compensation term (38) ensures the finite time attainment of condition $\tilde{\xi}_y = \dot{\tilde{\xi}}_y = \ldots, \tilde{\xi}_y^{(r-1)} = 0$ which implies that $\hat{\xi}_1 = \xi_1$ after a finite transient.

Denote $\tilde{\xi}_2 = \hat{\xi}_2 - \xi_2$. The dynamics of $\tilde{\xi}_2$ is easily derived by considering (30)-(34) as

$$\dot{\tilde{\xi}}_2 = A_{21}\tilde{\xi}_1 + A_22\tilde{\xi}_2$$

Since the convergence of $\tilde{\xi}_1$ to zero is achieved after a finite time, and A_{22} is Hurwitz, thus also $\tilde{\xi}_2 \to 0$, which proves the *asymptotic* convergence of $\hat{\xi}$ to ξ. Such equality yields that $\hat{\xi} = T(x-z)$, and, pre-multiplying both terms by the inverse of T, is possible to write down the expression $x = z + T^{-1}\hat{\xi}$ which concludes the proof. ■

IV. MIMO CASE

Let us study in this Section the general case represented by the system (1), by assuming that $\gamma(t,x)$ is a nonlinear functions vector with arbitrary dimension $m > 1$. Rewrite the observability matrix P (given in 3) of system (1) in the form

$$P = \begin{bmatrix} P_1^T & P_2^T & \cdots P_m^T \end{bmatrix}$$

where P_i is the observability matrix of the pair (c_i, A) and c_i $(i = 1, ..., m)$ is the i-th row of the matrix C.

Here we restrict ourselves to the case when the system has a **well defined vector relative degree** according to the following definition.

Definition 5: [13] System (1) is said to have a well defined vector relative degree $(r_1, ..., r_p)$ with respect to the nonlinear vector $\gamma(t, x)$ if

$$\begin{array}{l} c_i A^h D_j = 0 \quad i,j = 1,2,...,p,\ h = 0,1,...,r_i - 2, \\ c_i A^{r_i - 1} D_j \neq 0 \end{array} \tag{41}$$

and

$$rank\, Q = m,\ Q = \begin{bmatrix} c_1 A^{r_1-1} D_1 & ... & c_1 A^{r_1-1} D_m \\ & ... & \\ c_p A^{r_m-1} D_1 & ... & c_p A^{r_m-1} D_m \end{bmatrix}. \tag{42}$$

Lemma 5: Let the output y of (1) have the vector relative degree $r = (r_1, ..., r_m)$ with respect to the nonlinear functions vector $\gamma(t, x)$. Then the vectors c_1, ...,$c_1 A^{r_1-1}$, ..., c_m, ..., $c_m A^{r_m-1}$ are linearly independent.

Proof: Suppose that the contrary is true, i.e.

$$\lambda_{11} c_1 + \lambda_{12} c_1 A + ... + \lambda_{1r_1} c_1 A^{r_1-1} + ... + \lambda_{m1} c_m + \lambda_{m2} c_m A + ... + \lambda_{mr_m} c_m A^{r_m-1} = 0. \tag{43}$$

Prove that $\lambda_{ij} = 0$. Multiply (43) by D_1, ..., D_m and obtain m equalities. Due to (41) obtain that the rows of the matrix Q are linearly dependent with the dependence coefficients λ_{1r_1}, ..., λ_{mr_m}, which contradicts to (42). Thus $\lambda_{1r_1} = ... = \lambda_{mr_m} = 0$. Now multiply (43) by AD_1, ..., AD_m and obtain new m equalities. Taking into account (41) obtain new linear dependence of the rows of Q with the coefficients λ_{1r_1-1}, ..., λ_{mr_m-1}. In that case the rows $(c_i A^{r_i-1} D_1, ..., c_i A^{r_i-1} D_m)$ corresponding to $r_j = 1$ do not appear. Continuing this process obtain that all $\lambda_{ij} = 0$. ■

We met the following Assumptions

Assumption 4: System (1) is minimum phase and it has a well defined vector relative degree $(r_1, ..., r_m)$ with respect to the nonlinear functions vector.

Assumption 4 implies the strong detectability of the system, and, according to Lemma 5, the total relative degree $r = r_1 + ... + r_m$ with respect to the nonlinear functions vector $\gamma(t, x)$ does not exceed the observability index $n_O = \operatorname{rank} P$.

Assumption 5: The nonlinear functions vector $\gamma_i(t, x)$ $(i = 1, 2, ..., m)$ are bounded functions fulfilling the restriction $|\gamma_i(t, x)| \leq \gamma_i^+$.

Let $r_M = \max r_i$.

Choose a matrix L such that $A - LC$ is an Hurwitz one and implement the following linear, Luenberger-like observer

$$\dot{z} = Az + Bu + L(y - Cz). \tag{44}$$

Like in the SISO case,that the triple $(A - LC, D, C)$ keeps the relative degree, observability index, unobservable subspace $Px = 0$, and the minimum-phase property of the original triple (A, D, C). The corresponding error system is

$$\dot{\tilde{e}} = (A - LC)\tilde{e} + D\gamma(t, x),$$

where $\tilde{e} = x - z$. Then in some new coordinates $\begin{bmatrix} \tilde{e}_C^T & \tilde{e}_N^T \end{bmatrix}^T = [T_C^T\ T_N^T]^T \tilde{e} = T\tilde{e}$ the system takes the normal form

$$\begin{array}{c} \dot{\tilde{e}}_C = A_C \tilde{e}_C + A_{CN} \tilde{e}_N + D_C \gamma(t, x) \\ \dot{\tilde{e}}_N = A_{NC} \tilde{e}_C + A_N \tilde{e}_N \end{array}, \quad \tilde{e}_y = C_C \tilde{e}_C,$$

where the observation errors $\tilde{e}_{Ci} = (\tilde{e}_{Ci1}^T, ..., \tilde{e}_{Cir_i}^T)^T \in \mathbb{R}^{r_i}$ are calculated as $\tilde{e}_{Cij} = c_i A^{j-1}(x - z)$, $\tilde{e}_N \in \mathbb{R}^{n-r}$, and the system matrices have the form

$$\begin{bmatrix} A_C & A_{CN} \\ A_{NC} & A_N \end{bmatrix} = T(A - LC)T^{-1}, \tag{45}$$

$$A_C = \begin{bmatrix} A_{11} & \cdots & A_{1,m} \\ & \ddots & \\ A_{m,1} & \cdots & A_{mm} \end{bmatrix}, \quad A_{CN} = \begin{bmatrix} A_{C1} \\ \vdots \\ A_{Cm} \end{bmatrix}$$

$$D_C = \begin{bmatrix} D_{11} & ... & D_{m1} \\ & \ddots & \\ D_{1m} & ... & D_{mm} \end{bmatrix} \tag{46}$$

$$A_{ii} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ a_{ii,1} & a_{ii,2} & \cdots & a_{ii,r_i} \end{bmatrix}, \tag{47}$$

$$A_{i,j} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{ij,1} & a_{ij,2} & \cdots & a_{ij,r_j} \end{bmatrix},\ i \neq j, \tag{48}$$

$$A_{C,j} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{Cj,1} & a_{Cj,2} & \cdots & a_{Cj,n-r} \end{bmatrix}, \tag{49}$$

$$\begin{array}{l} y_j = C_{Cj} e_{Cj}, \qquad C_{Ci} = \begin{bmatrix} 1 & 0 & ... & 0 \end{bmatrix}, \\ D_{ij} = \begin{bmatrix} 0 & ... & 0 & c_i A^{r_i-1} D_j \end{bmatrix}^T. \end{array} \tag{50}$$

The matrix A_N is Hurwitz by the minimum phase assumption. The nonlinear part of the observer is chosen as

$$\dot{\hat{\xi}}_1 = A_C \hat{\xi}_1 + G_C v(t) \tag{51}$$

$$\dot{\hat{\xi}}_2 = A_{NC} \hat{\xi}_1 + A_N \hat{\xi}_2 \tag{52}$$

$$\hat{x} = z + T^{-1} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} \tag{53}$$

where $v \in \mathbb{R}^m$ and $G_C = diag\{G_1 \quad G_2 \quad ... \quad G_m\}$ and $G_i \in \mathbb{R}^{r_i}$ is given by

$$G_i = \begin{bmatrix} 0 & 0 & ... & 0 & 1 \end{bmatrix}^T,\ i = 1, ..., m$$

The entries v_i of vector v $(i = 1, 2, ..., m)$ are computed as per the quasi-continuous $(r_i - 1)$-th order sliding mode controller as

$$v_i(\tilde{\xi}_{y_i}) = -\alpha \Psi_{r_i-2, r_i-1}(\tilde{\xi}_{y_i}, \dot{\tilde{\xi}}_{y_i}, ..., \tilde{\xi}_{y_i}^{(r-2)}). \tag{54}$$

where $\tilde{\xi}_{y_i}$ denotes the ith row of the vector $\tilde{\xi}_y$.

The following Theorem is proven

Theorem 6: Under Assumptions 4, 5, with properly chosen parameters the observer (44)-(53) provides after a finite-time the convergence to zero of the estimation error $T_C(x - \hat{x})$. The remaining error coordinates $T_N x$ are estimated asymptotically.

The proof can be rather easily developed by following similar procedure as that of Theorem 4, and is omitted for brevity.

V. Example

Consider the Chua's Circuit [20] rewritten in the general form (1) as

$$\dot{x} = \begin{bmatrix} -\alpha c & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} x + \begin{bmatrix} -\alpha \\ 0 \\ 0 \end{bmatrix} \gamma(t,x) \qquad (55)$$
$$y = [0 \quad 0 \quad 1]x$$

where $\gamma(t,x) = x_1^3$. The quasilinear system (55) satisfies Assumption 1. The parameters of (55) are chosen as $\alpha = 10$, $\beta = 16$, $c = -0{,}143$ and the initial conditions are given by $x_0 = [0{,}1 \quad 0{,}1 \quad 0{,}1]^T$. The aforementioned choice of parameters and initial conditions guaranteeing the chaotic behavior of the Chua's circuit, in addition it ensures that both the state variables and its derivatives are bounded for all t (i.e., $\exists \gamma^+ : |x_1^3| < \gamma^+ \; \forall x \in \mathcal{X}$).

The auxiliary Luenberger-like gain is chosen as $L = [-7{,}2203 \quad -0{,}5747 \quad 6{,}43]^T$. The matrix G obtained by the application of (9) takes the form $G = [-0{,}0625 \quad 0 \quad 0]^T$.

The observer control $v(t)$ is selected according to the 3th order quasicontinuous-sliding-mode controller:

$$u = -30 \frac{\ddot{\tilde{\xi}}_y + 2(|\dot{\tilde{\xi}}_y| + |\tilde{\xi}_y|^{2/3})^{-1/2}(\dot{\tilde{\xi}}_y + |\tilde{\xi}_y|^{2/3} \operatorname{sign} \tilde{\xi}_y)}{|\ddot{\tilde{\xi}}_y| + 2(|\dot{\tilde{\xi}}_y| + |\tilde{\xi}_y|^{2/3})^{1/2}} \qquad (56)$$

with the output error derivative estimates $\hat{\dot{\tilde{\xi}}}_y$, $\hat{\ddot{\tilde{\xi}}}_y$ provided by the Levant differentiator with $\kappa_0 = 2(30^{1/3})$, $\kappa_1 = 1{,}5(30^{1/2})$, $\kappa_1 = 33$.

The actual and estimated state variables, and the estimation error, are shown in the Figure 1.

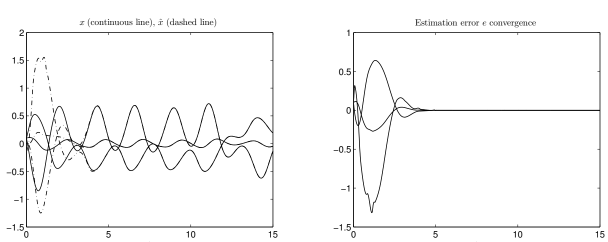

Fig. 1. State reconstruction and estimation errors.

VI. Conclusions

High-order sliding-mode quasi-continuous state observers are proposed for quasilinear systems under conditions of strong observability and strong detectability. The finite-time-convergent exact observation of the state is provided under the conditions of strong observability, while in the case of the strong detectability only a subset of the states are observed in finite time, while other estimations are asymptotically exact. A novel structure combining a Luenberger-like linear term and the quasicontinuous high-order sliding-mode controller [14], [15] has been proposed. The results are first presented for systems with scalar output, and then extended to the multiple output case.

References

[1] V. Utkin, J. Guldner, and J. Shi, *Sliding Modes in Electromechanical Systems*. London: Taylor and Francis, 1999.

[2] Y. Orlov, "Sliding mode observer-based synthesis of state derivative-free model reference adaptive control of distributed parameter systems," *ASME Journal of Dynamic Systems, Measurement, and Control*, vol. 122, no. 4, pp. 725–731, 2000.

[3] C. Edwards and S. Spurgeon, *Sliding Mode Control*. London: Taylor and Francis, 1998.

[4] J. Davila, L. Fridman, and A. Levant, "Second-order sliding-mode observer for mechanical systems," *IEEE Trans. Automat. Contr.*, vol. 50, no. 11, pp. 1785– 1789, November 2005.

[5] L. Fridman, A. Levant, and J. Davila, "Observation of linear systems with unknown inputs via high-order sliding-modes," *Int. J. System Science*, vol. 38, no. 10, pp. 773–791, 2007.

[6] A. Poznyak, "Stochastic output noise effects in sliding mode estimations," *International Journal of Control*, vol. 76, pp. 986–999, 2003.

[7] C. Edwards, S. Spurgeon, and C. P. Tan, "On development and applications of sliding mode observers," in *Variable Structure Systems:Towards 21st Century*, ser. Lecture Notes in Control and Information Science, X. Yu and J. Xu, Eds. Berlin, Germany: Springer Verlag, 2002, pp. 253–282.

[8] J.-X. Xu and H. Hashimoto, "Parameter identification methodologies based on variable structure control," *International Journal of Control*, vol. 57, no. 5, pp. 1202–1220, 1993.

[9] T. Ahmed-Ali and F. Lamnabhi-Lagarrigue, "Sliding observer-controller design for uncertain triangular nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 44, no. 6, pp. 1244–1249, 1999.

[10] T. Floquet and J. Barbot, "A canonical form for the design of unknown input sliding mode observers," in *Advances in variable structure and sliding mode control*, ser. Lecture Notes in Control and Information Science, C. Edwards, E. Fossas, and L. Fridman, Eds. Berlin: Springer Verlag, 2006, pp. 271–292.

[11] F. Bejarano, A. Poznyak, and L. Fridman, "Observer for linear time invariant systems with unknown inputs based on the hierarchical super-twisting concept," in *2006 International Workshop on Variable Structure Systems - VSS'06*, Sardinia, Italy, 2006, pp. 208–213.

[12] A. Levant, "High-order sliding modes: differentiation and output-feedback control," *International Journal of Control*, vol. 76, no. 9-10, pp. 924–941, 2003.

[13] A. Isidori, *Nonlinear Control Systems*. London, England: Springer-Verlag, 1996.

[14] A. Levant, "Quasi-continuous high-order sliding-mode controllers," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1812–1816, 2005.

[15] ——, "Homogeneous quasi-continuous sliding-mode control," *Lecture Notes in Control and Information Sciences*, vol. 334, pp. 143–168, 2006.

[16] H. L. Trentelman, A. A. Stoorvogel, and M. Hautus, *Control theory for linear systems*. London, Great Britain: Springer-Verlag, 2001.

[17] M. Hautus, "Strong detectability and observers," *Linear Algebra and its Applications*, vol. 50, pp. 353–368, 1983.

[18] A. Filippov, *Differential Equations with Discontinuos Right-hand Sides*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1988.

[19] C. Chen, *Linear Systems: theory and design*. New York: Oxford University Press, 1999.

[20] A. Huang, L. Pivka, C. Wu, and M. Franz, "Chua's equation with cubic nonlinearity," *Int. J. of Bifurcation and Chaos*, vol. 6, pp. 2175–2222, 1996.