# Neural Network Output Feedback Control of a Quadrotor UAV<sup>1</sup>

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*Abstract*—A neural network (NN) based output feedback controller for a quadrotor unmanned aerial vehicle (UAV) is proposed. The NNs are utilized in the observer and for generating virtual and actual control inputs, respectively, where the NNs learn the nonlinear dynamics of the UAV online including uncertain nonlinear terms like aerodynamic friction and blade flapping. It is shown using Lyapunov theory that the position, orientation, and velocity tracking errors, the virtual control and observer estimation errors, and the NN weight estimation errors for each NN are all semi-globally uniformly ultimately bounded (*SGUUB*) in the presence of bounded disturbances and NN functional reconstruction errors while simultaneously relaxing the separation principle.

*Index Terms* — Neural network, Quadrotor UAV, Lyapunov method, Output feedback, Observer

## I. INTRODUCTION

Quadrotor helicopters have quickly emerged as a popular unmanned aerial vehicle (UAV) platform in the recent years. Besides surveillance and search and rescue applications, the popularity of this platform has stemmed from its simple construction as compared with conventional helicopters [1].

The dynamics of the quadrotor UAV are nonlinear and under actuated; characteristics which can make the platform difficult to control. The UAV has six degrees of freedom (DOF) and only four control inputs consisting of thrust and the three rotational torque inputs. To solve the quadrotor UAV tracking control problem, many techniques have been proposed [2-9] where the control objective is to track three desired Cartesian positions and a desired yaw angle.

In [2], [3], and [4], state feedback controllers were proposed based on state-dependent Riccati equations and the small angle approximations, backstepping, and saturation functions, respectively. A drawback of these controllers is the need for full state measurement and knowledge of the dynamics beside the aerodynamic friction are either simplified or ignored altogether. However, the above simplifications are valid at very low speeds such as hovering, and that the impact of the aerodynamic effects can become significant even at moderate velocities [1].

On the other hand, in [5] and [6], sliding mode observers are introduced to estimate the translational and angular velocities of the UAV. The authors in [6] propose a sliding mode estimator for external disturbances such as wind and model uncertainties. In [7], an output feedback controller is developed by strategically introducing an additional constant term into the filtered tracking error. The constant term is then utilized in the design of an auxiliary control input for the translational velocities. In addition, the nonlinearities were assumed be known and linearly parameterized.

In [8], an adaptive-fuzzy control method is applied using supervised training. In [9], the approximation property of NN [10] is applied to learn the dynamics of the quadrotor UAV using offline training with experimentally collected data. However, offline NN training is a major drawback.

In contrast, the work in this paper seeks to remove the assumptions of full state measurement and knowledge of the UAV dynamics. First, a NN observer is utilized to estimate the velocities of the UAV so that an output feedback control law can be realized. Then, a novel virtual NN control input is developed for the roll and pitch which ensures the UAV tracks a desired translational velocity while maintaining a stable flight configuration. The virtual control input is well defined and provides a means of controlling all six DOF using only four control inputs. The physical meaning of the virtual control inputs can be linked to the types of trajectories that can be successfully tracked as well. Finally, the inputs of the dynamical system are calculated by utilizing the approximation properties of NN to learn the complete dynamics of the UAV online, including unmodeled dynamics like aerodynamic damping and blade flapping [1] and by relaxing the linearly parameterized representation.

All NNs are tuned online in order to accommodate the change in the UAV dynamics and the operating environment. It is shown using Lyapunov theory that the position, orientation, and velocity tracking errors, the virtual control observer estimation errors, and the NN weight estimation errors of each NN are all *semi-globally uniformly ultimately bounded (SGUUB)* while simultaneously relaxing the separation principle. Although not shown, numerical results confirm the theoretical conjectures.

#### II. BACKGROUND

# A. Quadrotor UAV Dynamics

Consider a quadrotor UAV with six DOF defined in the inertial coordinate frame,  $E^a$ , as  $[x, y, z, \phi, \theta, \psi]^T \in E^a$  where  $\rho = [x, y, z]^T \in E^a$  are the position coordinates of the UAV and  $\Theta = [\phi, \theta, \psi]^T \in E^a$  describe its orientation referred to as roll, pitch, and yaw, respectively. The translational and angular velocities are expressed in the body fixed frame attached to the center of mass of the UAV,  $E^b$ , and the dynamics of the UAV in the body fixed frame can be written as [7]

$$M\begin{bmatrix} \dot{v}\\ \dot{\omega} \end{bmatrix} = \overline{S}(\omega)\begin{bmatrix} v\\ \omega \end{bmatrix} + \begin{bmatrix} N_1(v)\\ N_2(\omega) \end{bmatrix} + \begin{bmatrix} G(R)\\ 0_{3x1} \end{bmatrix} + U + \tau_d, \qquad (1)$$

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where

$$M = \begin{bmatrix} mI_3 & 0_{3x3} \\ 0_{3x3} & J \end{bmatrix} \in \mathfrak{R}^{6x6}, \quad \overline{S}(\omega) = \begin{bmatrix} -mS(\omega) & 0_{3x3} \\ 0_{3x3} & S(J\omega) \end{bmatrix} \in \mathfrak{R}^{6x6},$$
$$U = \begin{bmatrix} 0 & 0 & u_1 & u_2^T \end{bmatrix}^T \in \mathfrak{R}^6$$

and *m* is a positive scalar that represents the total mass of the UAV,  $J \in \Re^{3x^3}$  represents the positive definite inertia  $v(t) = [v_{xb}, v_{yb}, v_{zb}]^T \in \Re^3$  represents matrix, the velocity,  $\omega(t) = \left[\omega_{xb}, \omega_{yb}, \omega_{zb}\right]^T \in \Re^3$ translational represents the angular velocity,  $N_i(\bullet) \in \Re^{3x1}$ , i = 1, 2, are the nonlinear aerodynamic effects,  $u_1 \in \Re^1$  provides the thrust along the z-direction,  $u_2 \in \Re^3$  provides the rotational torques,  $\tau_d = [\tau_{d1}^T, \tau_{d2}^T]^T \in \Re^6$  and  $\tau_{di} \in \Re^3, i = 1, 2$ represents unknown, bounded disturbances such that  $\|\tau_d\| < \tau_M$  for all time t, with  $\tau_M$  being a known positive constant,  $I_{nxn} \in \Re^{nxn}$  is an nxn identity matrix, and  $0_{mxl} \in \Re^{mxl}$  represents an mxl matrix of all zeros. Furthermore,  $G(R) \in \Re^3$  represents the gravity vector defined as  $G(R) = mgR^{T}(\Theta)E_{z}$  where  $E_{z} = [0,0,1]^{T}$  is a unit vector in the inertial coordinate frame,  $g = 9.81 m/s^2$ , and  $S(\bullet) \in \Re^{3x3}$  is the general form of a skew symmetric matrix defined as [7]. It is important to highlight  $w^T S(\gamma) w = 0$  for any vector  $w \in \Re^3$ , and this property is commonly referred to as the skew symmetric property [10].

The matrix  $R(\Theta) \in \Re^{3x3}$  is the translational rotation matrix which is used to relate a vector in the body fixed frame to the inertial coordinate frame defined as [2]

$$R(\Theta) = R = \begin{vmatrix} c_{\theta}c_{\psi} & s_{\phi}s_{\theta}c_{\psi} - c_{\phi}s_{\psi} & c_{\phi}s_{\theta}c_{\psi} + s_{\phi}s_{\psi} \\ c_{\theta}s_{\psi} & s_{\phi}s_{\theta}s_{\psi} + c_{\phi}c_{\psi} & c_{\phi}s_{\theta}s_{\psi} - s_{\phi}c_{\psi} \\ -s_{\theta} & s_{\phi}c_{\theta} & c_{\phi}c_{\theta} \end{vmatrix}$$
(2)

where the abbreviations  $s_{(\bullet)}$  and  $c_{(\bullet)}$  have been used for  $\sin(\bullet)$  and  $\cos(\bullet)$ , respectively. It is important to note that  $R^{-1} = R^T$ ,  $\dot{R} = RS(\omega)$  and  $\dot{R}^T = -S(\omega)R^T$ . It is also necessary to define a rotational transformation matrix from the fixed body to the inertial coordinate frame as in [7]

$$T(\Theta) = T = \begin{bmatrix} 1 & s_{\phi}t_{\theta} & c_{\phi}t_{\theta} \\ 0 & c_{\phi} & -s_{\phi} \\ 0 & \frac{s_{\phi}}{c_{\theta}} & \frac{c_{\phi}}{c_{\theta}} \end{bmatrix}$$
(3)

where the abbreviation  $t_{(\bullet)}$  has been used for  $\tan(\bullet)$ . The transformation matrices R and T are nonsingular as long as  $-(\pi/2) < \phi < (\pi/2), \quad -(\pi/2) < \theta < (\pi/2)$  and  $-\pi \le \psi \le \pi$ . These regions will be assumed throughout the development of this work, and will be referred to as the *stable operation regions* of the UAV. Under these flight conditions, it is observed that  $||R||_F = R_{\max}$  and  $||T||_F < T_{\max}$  for known constants  $R_{\max}$  and  $T_{\max}$  [12].

$$\dot{\rho} = Rv \tag{4}$$
$$\dot{\Theta} = T\omega$$

The control inputs to the UAV,  $u_1$  and  $u_2$ , represent the generated thrust and torques, respectively, generated by the angular speeds of rotors,  $\varpi_i$ , i = 1, 2, 3, 4, and are related to the thrust and drag factors by the following relationship [4]

$$u_{1} = -c_{t} \sum_{i=1}^{4} \varpi_{i}^{2}$$

$$u_{2} = \left[ dc_{t} (\varpi_{2}^{2} - \varpi_{4}^{2}), dc_{t} (\varpi_{1}^{2} - \varpi_{3}^{2}), c_{d} \sum_{i=1}^{4} (-1)^{i+1} \varpi_{i}^{2} \right]^{T}$$
(5)

where *d* is a positive scalar representing the distance from the epicenter of the quadrotor to the rotor axes,  $c_i$  is a positive scalar representing the thrust factor, and  $c_d$  is a positive scalar representing the drag factor.

Once the control inputs to the UAV have been determined, the relationship in (5) can be used to determine the required rotor speeds in order to achieve the desired thrust and rotational torques. In, [3] and [6], the tracking control of the rotor speeds was considered; however, in this work, we are concerned with deriving the required thrust and rotational torques as in [2], [4], [7], and [8], respectively.

The nonlinear aerodynamic effects which are considered in this effort are in the form of aerodynamic damping [7] and blade flapping [1] where the aerodynamic damping terms are modeled as in [7]. In [1], aerodynamic effects, like blade flapping were studied and revealed to have significant impact on the tracking ability of a quadrotor UAV. The flapping of the rotor blades tilts the rotor plane away from the direction of motion, thus affecting the thrust and rotational torques of the UAV. For complete details on blade flapping and its full effects, please see [1].

**Remark 1:**  $\|\cdot\|$  and  $\|\cdot\|_{F}$  will be used interchangeably as the Frobenius vector and matrix norms [10]. Next semi-global uniformly ultimately boundedness is defined.

Definition 1: The equilibrium point  $x_e$  is said to be semiglobal uniformly ultimately bounded (SGUUB) if there exists a compact set  $S \subset \mathbb{R}^n$  so that for all  $x_0 \in S$  there exists a bound B > 0 and a time  $T(B, x_0)$  such that  $||x(t) - x_e|| \le B$  for all  $t \ge t_0 + T[10]$ .

#### III. OUTPUT FEEDBACK TRACKING CONTROL

The overall control objective for the UAV is to track a desired trajectory,  $\rho_d = [x_d, y_d, z_d]^T$ , and a desired yaw  $\psi_d$  while maintaining a stable flight configuration. The velocity  $v_{zb}$  is directly controllable with the thrust input. However, in order to control the translational velocities  $v_{xb}$  and  $v_{yb}$ , the pitch and roll must be controlled, respectively, thus redirecting the thrust. With these objectives in mind, the control problem statement can be defined as follows.

Given the desired position,  $\rho_d$ , find the control velocity  $v_d \in E^b$  such that  $\rho \to \rho_d$ . Then, find the desired pitch,

 $\theta_d$ , and roll,  $\phi_d$ , as well as the thrust,  $u_1$  such that  $v \to v_d$ . Next, given  $\Theta_d = [\phi_d \ \theta_d \ \psi_d]^T$ , find the desired angular velocity control  $\omega_d \in E^b$  such that  $\Theta \to \Theta_d$ . Finally, find the rotational torques,  $u_2$ , such that  $\omega \to \omega_d$ . To complete the control objective, complete knowledge of the UAV dynamics and velocity information is required whereas this information is considered not available. The constant total mass and moments of inertia of the UAV are known similar to the other works [2]-[7]. Therefore, the *universal approximation property* of NN is utilized in the design of the observer, virtual control inputs, and the controller.

# A. NN Observer Design

To relax the need for velocity measurements and knowledge of the dynamics of the quadrotor UAV, a NN observer will be utilized. To begin the observer development, define new variables  $\Theta^b = T^{-1}\Theta \in E^b$ ,  $X = [\rho^T (\Theta^b)^T]^T \in \Re^6$  and  $V = [v^T \ \omega^T]^T \in \Re^6$  whose dynamics are given by (4) and (1), respectively, and rewritten as

$$\dot{X} = A(t)V + \xi_{1}$$

$$\dot{V} = M^{-1} \left( \overline{S}(\omega)V + \begin{bmatrix} N_{1}(v) \\ N_{2}(\omega) \end{bmatrix} + \begin{bmatrix} G(R) \\ 0_{3x1} \end{bmatrix} \right) + M^{-1}U + \overline{\tau}_{d}$$
(6)

where  $\xi_1 \in \Re^6$  represents bounded sensor measurement noise such that  $\|\xi_1\| \leq \xi_{1M}$  for a known constant  $\xi_{1M}$ ,  $\overline{\tau}_d = [\overline{\tau}_{d1}^T \ \overline{\tau}_{d2}^T]^T = M^{-1}(\tau_d) \in \Re^6$  represents disturbances,  $A(t) = A = \begin{bmatrix} R & 0_{3x3} \\ 0_{3x3} & I_{3x3} \end{bmatrix}$ .

From the previously defined properties of R,  $A^{-1} = A^T$  and  $||A||_F \le A_M$  for a positive computable constant  $A_M$ . Additionally, it is straight forward to verify  $w^T \dot{A}^T A w = 0$  for any vector  $w \in \Re^6$ .

Next, define a change of variable as Z = AV, whose derivative with respect to time is given by  $\dot{Z} = \dot{A}V + A\dot{V}$ , and after simplification, written as

$$\dot{Z} = Af_o(x_o) + \overline{G} + AM^{-1}U + A\,\overline{\tau}_d$$

where  $x_o = V$  and

 $f_o(x_o) = f_o = \left[ (1/m)(N_1(v))^T \left( J^{-1}N_2(\omega) + J^{-1}S(J\omega)\omega \right)^T \right]^T \in \Re^6$ (7) defines the unknown nonlinear dynamics, and with  $\overline{G} = [0 \ 0 \ g \ 0 \ 0 \ 0]^T \in \Re^6$  a constant vector containing the gravity term. Then, define the NN observer estimates  $\hat{X}$  and  $\hat{Z}$  as well as the observer estimation error  $\widetilde{X} = X - \hat{X}$ , and for convenience, the proposed observer takes the form of

$$\hat{X} = \hat{Z} + K_{o1}\tilde{X}$$

$$\hat{Z} = A\hat{f}_{o} + \overline{G} + K_{o2}\tilde{X} + AM^{-1}U$$
(8)

where  $K_{o1}$ ,  $K_{o2}$  positive design constants. The observer velocity estimate  $\hat{V}$  is then written as

$$\hat{V} = \begin{bmatrix} \hat{v} \\ \hat{\omega} \end{bmatrix} = A^T (\hat{Z} + K_{o3} \widetilde{X})$$
<sup>(9)</sup>

where  $K_{o3}$  is a positive design constant.

In (8), the universal approximation property of NN [10] has been utilized to estimate the unknown function  $f_o(x_o)$  by constant ideal bounded weights  $W_o^T, V_o^T$ such that  $\|W_o\|_F \leq W_{Mo}$  for a known constant  $W_{Mo}$ , and written as  $f_o(x_o) = W_o^T \sigma(V_o^T x_o) + \varepsilon_o$  where  $\varepsilon_o$  is the bounded NN approximation error such that  $\|\varepsilon_o\| \leq \varepsilon_{Mo}$  for a known constant  $\varepsilon_{Mo}$ . The NN estimate of  $f_o$  is written as  $\hat{f}_o = \hat{W}_o^T \sigma(V_o^T \hat{x}_o) = \hat{W}_o^T \hat{\sigma}_o$  where  $\hat{W}_o^T$  is the estimate of  $W_o^T$ , and  $\hat{x}_o$  is the NN input written in terms of the observer velocity estimates as  $\hat{x}_o = [1 \ \hat{X}^T \ \hat{V}^T]^T$ .

Noting  $\hat{Z} = A\hat{V} - K_{o3}\tilde{X}$  from (9) and the definition of  $\dot{Z}$  above, the observer error dynamics of (8) can be formulated as

$$\begin{split} \tilde{\widetilde{X}} &= A\widetilde{V} - (K_{o1} - K_{o3})\widetilde{X} + \xi_1 \\ \tilde{\widetilde{Z}} &= A \left( f_o - \hat{f}_o \right) - K_{o2}\widetilde{X} + A \overline{\tau}_d \end{split}$$
(10)

Similarly, noting

$$\widetilde{V} = V - \widehat{V} = A^{T} \left( \widetilde{Z} - K_{o3} \widetilde{X} \right), \qquad (11)$$

adding and subtracting  $W_o^T \sigma(V_o^T \hat{x}_o)$ , and using (10), the observer estimation error dynamics of (9) take the form of  $\hat{V} = \tilde{f}_o - A^T (K_{o2} - K_{o3}(K_{o1} - K_{o3})) \tilde{X} - (K_{o3}I_{6x6} - \dot{A}^T A) \tilde{V} + \xi_2 (12)$  where  $\tilde{f}_o = \tilde{W}_o \hat{\sigma}_o$ ,  $\tilde{W}_o = W_o - \hat{W}_o$ ,  $\tilde{\sigma}_o = \sigma_o - \hat{\sigma}_o$ , and  $\xi_2 = \varepsilon_o + \overline{\tau}_d - K_{o3}A^T \xi_1 + W_o \tilde{\sigma}_o \in \Re^6$ , Further,  $\|\xi_2\| \le \xi_{2M}$  where  $\xi_{2M} = \varepsilon_{Mo} + M_M \tau_M + K_{o3}A_M \xi_{1M} + 2W_{Mo} \sqrt{N_o}$  is a positive computable constant with  $N_o$  the number of hidden layer neurons and where  $M_M = \|M^{-1}\|_r$ .

Examining the error dynamics of (10) and (12) reveals that  $\widetilde{X}, \widetilde{V}$ , and  $\widetilde{f}_o$  are equilibrium points when  $\|\xi_1\| = 0$ ,  $\|\xi_2\| = 0$  and  $\|\overline{\tau}_d\| = 0$ . Next the following theorem is stated.

*Theorem 1: (NN Observer Boundedness)* Let the NN observer be defined by (8) and (9), respectively, with the NN update law for the observer given by

$$\dot{\hat{W}}_o = F_o \hat{\sigma}_o \tilde{X}^T - \kappa_{o1} F_o \hat{W}_o \tag{13}$$

where  $F_o = F_o^T > 0$  and  $\kappa_{o1} > 0$  are design parameters. Then there exists constant positive design parameters  $K_{o1}, K_{o2}$ , and  $K_{o3}$  where  $K_{o1} > K_{o3}$ ,  $K_{o2} > K_{o3}(K_{o1} - K_{o3})$ , and  $K_{o3} > (2N_o)/\kappa_{o1}$ , such that the observer estimation errors  $\widetilde{X}$ ,  $\widetilde{V}$  and the NN observer weight estimation errors,  $\widetilde{W}_o$ , are *SGUUB*.

*Proof:* Due to page length constraints, proof of *Theorem 1* is consider during proof of *Theorem 2* using

$$V_{o} = \frac{1}{2}\widetilde{X}^{T} \left( K_{o2} - K_{o3} (K_{o1} - K_{o3}) \right) \widetilde{X} + \frac{1}{2} \widetilde{V}^{T} \widetilde{V} + \frac{1}{2} tr \{ \widetilde{W}_{o}^{T} F_{o}^{-1} \widetilde{W}_{o} \}$$
(14)

#### B. NN Virtual Control Input Development

In this section, a series of virtual control inputs for the backstepping control law will be defined to ensure that the quadrotor UAV follows a specified trajectory. To begin the development of the UAV tracking controller, we first define the tracking errors for the position and translational velocity.

For the position, define

$$e_{\rho} = \rho_d - \rho \in E^a \,. \tag{15}$$

Differentiating (15) and substitution of (4) yields the position error dynamics

$$\dot{e}_{\rho} = \dot{\rho}_d - Rv \,. \tag{16}$$

Next, select the desired velocity to stabilize the position error dynamics as

$$v_{d} = [v_{dx} \ v_{dy} \ v_{dz}]^{T} = R^{T} (\dot{\rho}_{d} + K_{\rho} e_{\rho}) \in E^{b}$$
(17)

where  $K_{\rho} = diag\{k_{\rho x}, k_{\rho y}, k_{\rho z}\} \in \Re^{3x3}$  is a diagonal positive

definite design matrix all with positive design constants. Next, the translational velocity tracking error system is defined as

$$e_{v} = \begin{bmatrix} e_{vx} \\ e_{vy} \\ e_{vz} \end{bmatrix} = \begin{bmatrix} v_{dx} \\ v_{dy} \\ v_{dz} \end{bmatrix} - \begin{bmatrix} v_{xb} \\ v_{yb} \\ v_{zb} \end{bmatrix} = v_{d} - v.$$
(18)

The desired velocity  $v_d$  is a virtual control input to (16), and applying (17) to (16) while observing  $v = v_d - e_v$ , reveals the closed loop position error dynamics to be rewritten as

$$\dot{e}_{\rho} = -K_{\rho}e_{\rho} + Re_{v}. \tag{19}$$

Next, the translational velocity tracking error dynamics are developed. Since the velocity vector is not measurable in this work, it is desirable to rewrite (18) in terms of the observer velocity estimates as

$$\hat{e}_{v} = [\hat{e}_{vx} \ \hat{e}_{vy} \ \hat{e}_{vz}]^{T} = v_{d} - \hat{v} = v_{d} - (v - \widetilde{v})$$
(20)

where  $\tilde{v}$  is the observer estimation error for the translational velocity. Now, differentiating (20), observing  $\dot{v}_d = -S(\omega)v_d + R^T(\ddot{\rho}_d + K_\rho(\dot{\rho}_d - Rv))$  and substituting the translational velocity dynamics in (1) as well as adding and subtracting  $S(\omega)\tilde{v}$  reveals

$$\dot{\hat{e}}_{\nu} = -S(\omega)\hat{e}_{\nu} - \frac{1}{m}N_{1}(\nu) - \frac{1}{m}G(R) - \frac{1}{m}u_{1}E_{z}$$

$$+ R^{T}(\ddot{\rho}_{d} + K_{\rho}\dot{\rho}_{d} - K_{\rho}R\nu) - \bar{\tau}_{d1} - \dot{\tilde{\nu}} + S(\omega)\tilde{\nu}$$
(21)

and the observer error dynamics become apart of the velocity tracking error dynamics. Next, we rewrite (2) in terms of the desired orientation angles,  $\Theta_d$ , define  $R_d = R(\Theta_d)$ , and add and subtract  $G(R_d)/m$  and  $R_d^T(\ddot{\rho}_d + K_\rho \dot{\rho}_d - K_\rho R \hat{v})$  to  $\dot{\hat{e}}_v$  to yield

$$\dot{\hat{e}}_{v} = -S(\omega)\hat{e}_{v} - \frac{1}{m}G(R_{d}) - \frac{1}{m}u_{1}E_{z} - \bar{\tau}_{d1}$$

$$+ R_{d}^{T}(\ddot{\rho}_{d} + K_{\rho}\dot{\rho}_{d} - K_{\rho}R\hat{v} + f_{c1}(x_{c1}))$$
(22)

where

$$f_{c1}(x_{c1}) = R_d \left( -(G(R) - G(R_d)) / m + (R - R_d)^T (\dot{\rho}_d + K_\rho \dot{\rho}_d) \right) + R_d \left( S(\omega) \widetilde{\nu} - R^T K_\rho R \nu + R_d^T K_\rho R \hat{\nu} - \dot{\widetilde{\nu}} - N_1(\nu) / m \right)$$

is an unknown function which can be rewritten as  $f_{c1}(x_{c1}) = [f_{c11} \ f_{c12} \ f_{c13}]^T \in \Re^3$ . In the forthcoming development, the approximation properties of NN will be utilized to estimate the unknown function  $f_{c1}(x_{c1})$  by bounded ideal weights  $W_{c1}^T, V_{c1}^T$  such that  $||W_{c1}||_F \leq W_{Mc1}$  for a known constant  $W_{Mc1}$ , and written as  $f_{c1}(x_{c1}) = W_{c1}^T \sigma(V_{c1}^T x_{c1}) + \varepsilon_{c1}$  where  $\varepsilon_{c1}$  is the bounded NN approximation error. The NN

estimate of  $f_{c1}$  is written as  $\hat{f}_{c1} = \hat{W}_{c1}^T \sigma (V_{c1}^T \hat{x}_{c1}) = \hat{W}_{c1}^T \hat{\sigma}_{c1}$ =  $[\hat{W}_{c11}^T \hat{\sigma}_{c1} \ \hat{W}_{c12}^T \hat{\sigma}_{c1} \ \hat{W}_{c13}^T \hat{\sigma}_{c1}]^T$  where  $\hat{W}_{c1}^T$  is the NN estimate of  $W_{c1}^T$ ,  $\hat{W}_{c1i}^T$ , i = 1, 2, 3 is the  $i^{th}$  row of  $\hat{W}_{c1}^T$ , and  $\hat{x}_{c1}$  is the NN input written in terms of the observer estimates, UAV orientation, and derivatives of the desired trajectory.

The key step in the development of the virtual control input for the dynamic system is identifying the *desired* closed loop velocity tracking error dynamics. For convenience, the *desired* translational velocity closed loop system is selected as

$$\dot{\hat{e}}_{v} = -(K_{v} + S(\omega))\hat{e}_{v} - \bar{\tau}_{d1}$$
<sup>(23)</sup>

where  $K_v = diag\{k_{v1}\cos(\theta_d), k_{v2}\cos(\phi_d), k_{v3}\}$  is a diagonal positive definite design matrix with each  $k_{vi} > 0$ , i = 1, 2, 3. In the following development, it will be shown that  $\theta_d \in (-\pi/2, \pi/2)$  and  $\phi_d \in (-\pi/2, \pi/2)$ ; therefore, it is clear that  $K_v > 0$ . Then, equating (22) and (23) while considering the only the first two velocity error states reveals  $[-s_{at}] [k_v \hat{e}_w c_{at}]$ 

$$-g \begin{bmatrix} c_{\theta d} s_{\phi d} \\ c_{\theta d} s_{\phi d} \end{bmatrix} + \begin{bmatrix} v_{1} + v_{2} - a_{0} \\ k_{v2} \hat{e}_{vy} c_{\phi d} \end{bmatrix} +$$

$$- \begin{bmatrix} c_{\theta d} c_{\psi d} & c_{\theta d} s_{\psi d} \\ -s_{\phi d} s_{\theta d} c_{\psi d} - c_{\phi d} s_{\psi d} \\ -s_{\phi d} s_{\theta d} c_{\theta d} \end{bmatrix} \begin{bmatrix} \ddot{x}_{d} + k_{\rho x} \dot{x}_{d} - \hat{v}_{R1} + f_{c11} \\ \ddot{y}_{d} + k_{\rho y} \dot{y}_{d} - \hat{v}_{R2} + f_{c12} \\ \ddot{z}_{d} + k_{\rho z} \dot{z}_{d} - \hat{v}_{R3} + f_{c13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(24)$$

where  $\hat{v}_R = [\hat{v}_{R1} \ \hat{v}_{R2} \ \hat{v}_{R3}]^T = K_\rho R \hat{v}$ . Then, applying basic math operations, the first line of (24) can be solved for the desired pitch  $\theta_d$  while the second line reveals the desired roll  $\phi_d$ . Using the NN estimates,  $\hat{f}_{c1}$ , The desired pitch  $\theta_d$  can be written as

$$\theta_d = a \tan\left(\frac{N_{\theta d}}{D_{\theta d}}\right) \tag{25}$$

where

$$\begin{split} N_{\theta d} &= c_{\psi d} \left( \ddot{x}_{d} + k_{\rho x} \dot{x}_{d} - \hat{v}_{R1} + \hat{f}_{c11} \right) + s_{\psi d} \left( \ddot{y}_{d} + k_{\rho y} \dot{y}_{d} - \hat{v}_{R2} + \hat{f}_{c12} \right) + k_{\nu 1} \hat{e}_{\nu y} \\ D_{\theta d} &= \ddot{z}_{d} + k_{\rho x} \dot{z}_{d} - \hat{v}_{R3} - g + \hat{f}_{c13} \end{split}$$

Similarly, the desired roll angle,  $\phi_d$ , is found to be

$$\phi_d = a \tan\left(\frac{N_{\phi d}}{D_{\phi d}}\right) \tag{26}$$

where

$$\begin{split} N_{\phi d} &= s_{\psi d} \left( \ddot{x}_{d} + k_{\rho x} \dot{x}_{d} - \hat{v}_{R1} + \hat{f}_{c11} \right) - c_{\psi d} \left( \ddot{y}_{d} + k_{\rho y} \dot{y}_{d} - \hat{v}_{R2} + \hat{f}_{c12} \right) - k_{v2} \hat{e}_{vy} \\ D_{\phi d} &= c_{\partial d} \left( \ddot{z}_{d} + k_{\rho x} \dot{z}_{d} - \hat{v}_{R3} - g + \hat{f}_{c13} \right) + s_{\partial d} c_{\psi d} \left( \ddot{x}_{d} + k_{\rho x} \dot{x}_{d} - \hat{v}_{R1} + \hat{f}_{c11} \right) \\ &+ s_{\partial d} s_{\psi d} \left( \ddot{y}_{d} + k_{\rho y} \dot{y}_{d} - \hat{v}_{R2} + \hat{f}_{c12} \right) \end{split}$$

**Remark 3:** The expressions for the desired pitch and roll in (25) and (26) lend themselves very well to the control of quadrotor UAV. The expressions will always produce desired values in the *stable operation regions* of the UAV. Finally, the virtual control inputs reveal the types of desired trajectories, which can be tracked in the steady state. That is, that there exist desired trajectories which will result in operating regions near the unstable operating points of the UAV since  $a \tan(\bullet)$  approaches  $\pm \pi/2$  as its argument increases. Additionally, large values of  $k_{\rho x}$  and  $k_{\rho y}$  can push an UAV toward instability.

Now that the desired orientation has been found, next define the attitude tracking error as

$$e_{\Theta} = \Theta_d - \Theta \in E^a \tag{27}$$

where dynamics are found using (4) to be  $\dot{e}_{\Theta} = \dot{\Theta}_d - T\omega$ . In order to drive the orientation errors (27) to zero, the desired angular velocity,  $\omega_d$ , is selected as

$$\omega_d = T^{-1}(\dot{\Theta}_d + K_{\Theta}e_{\Theta}) \in E^a$$
(28)

where  $K_{\Theta} = diag\{k_{\Theta 1}, k_{\Theta 2}, k_{\Theta 3}\} \in \Re^{3x3}$  is a diagonal positive definite design matrix all with positive design constants. Define the angular velocity tracking error as

$$e_{\omega} = \omega_d - \omega, \qquad (29)$$

and observing  $\omega = \omega_d - e_{\omega}$ , the closed loop orientation tracking error system can be written as

$$\dot{e}_{\Theta} = -K_{\Theta}e_{\Theta} + Te_{\omega}. \tag{30}$$

Examining (28), calculation of the desired angular velocity requires knowledge of  $\dot{\Theta}_d$ ; however,  $\dot{\Theta}_d$  is not known in view of the fact that  $\dot{\hat{v}}$  and  $\dot{\hat{f}}_{c1}$  are not available. Further, development of  $u_2$  in the following section, will reveal  $\dot{\omega}_d$  is required which in turn implies  $\ddot{\hat{v}}$  and  $\ddot{\hat{f}}_{c1}$  must be known. Since these requirements are not practical, the *universal approximation property* of NN is invoked to estimate  $\omega_d$  and  $\dot{\omega}_d$ .

To aid in the NN virtual control development, the desired orientation,  $\Theta_d \in E^a$ , is reconsidered in the fixed body frame,  $E^b$ , using the relation  $\Theta_d^b = T^{-1}\Theta_d$ . Rearranging (28), the dynamics of the proposed virtual controller when the all dynamics are known are revealed to be

$$\hat{\Theta}_{d}^{b} = \omega_{d} - T^{-1} K_{\Theta} e_{\Theta} \qquad (31)$$
$$\hat{\omega}_{d} = \dot{T}^{-1} (\dot{\Theta}_{d} + K_{\Theta} e_{\Theta}) + T^{-1} (\ddot{\Theta}_{d} + K_{\Theta} \dot{e}_{\Theta})$$

For convenience, we define a change of variable as  $\Omega_d = \omega_d - T^{-1}K_{\Theta}e_{\Theta}$ , and the dynamics (31) become

$$\dot{\Theta}_{d}^{b} = \Omega_{d}, \qquad \dot{\Omega}_{d} = \dot{T}^{-1}\dot{\Theta}_{d} + T^{-1}\ddot{\Theta}_{d} = f_{\Omega}(x_{\Omega}) = f_{\Omega} \quad (32)$$

Defining the estimates of  $\Theta_d^b$  and  $\Omega_d$  to be  $\hat{\Theta}_d^b$  and  $\hat{\Omega}_d$ , respectively, and the estimation error  $\widetilde{\Theta}_d^b = \Theta_d^b - \hat{\Theta}_d^b$ , the dynamics of the proposed NN virtual control inputs become

$$\dot{\widehat{\Theta}}_{d}^{b} = \widehat{\Omega}_{d} + K_{\Omega 1} \widetilde{\Theta}_{d}^{b}, \qquad \dot{\widehat{\Omega}}_{d} = \widehat{f}_{\Omega} + K_{\Omega 2} \widetilde{\Theta}_{d}^{b}$$
(33)

where  $K_{\Omega 1}$  and  $K_{\Omega 2}$  are positive constants. The estimate  $\hat{\omega}_d$  is then written as

$$\hat{\omega}_d = \hat{\Omega}_d + K_{\Omega 3} \widetilde{\Theta}_d^b + T^{-1} K_{\Theta} e_{\Theta}$$
(34)

where  $K_{\Omega_3}$  is a positive constant.

In (33), universal approximation property of NN has been utilized to estimate the unknown function  $f_{\Omega}(x_{\Omega}) = \dot{T}^{-1} \dot{\Theta}_{d} + T^{-1} \ddot{\Theta}_{d}$ by bounded ideal weights  $W_{\Omega}^{T}, V_{\Omega}^{T}$  such that  $\|W_{\Omega}\|_{T} \leq W_{M\Omega}$  for a known constant  $W_{M\Omega}$ , and written as  $f_{\Omega}(x_{\Omega}) = W_{\Omega}^{T} \sigma (V_{\Omega}^{T} x_{\Omega}) + \varepsilon_{\Omega}$  where  $\varepsilon_{\Omega}$  is NN approximation error the bounded such that  $\|\varepsilon_{\Omega}\| \leq \varepsilon_{\Omega M}$  for a known constant  $\varepsilon_{\Omega M}$ . The NN estimate of  $f_{\Omega}$  is written as  $\hat{f}_{\Omega} = \hat{W}_{\Omega}^T \sigma (V_{\Omega}^T \hat{x}_{\Omega}) = \hat{W}_{\Omega}^T \hat{\sigma}_{\Omega}$  where  $\hat{W}_{\Omega}^{T}$  is the NN estimate of  $W_{\Omega}^{T}$  and  $\hat{x}_{\Omega}$  is the NN input written in terms of the virtual control input estimates and the NN observer velocity estimates. The NN input is chosen to take the form of  $\hat{x}_{\Omega} = [1 \rho_d \dot{\rho}_d^T \ddot{\rho}_d^T \ddot{\rho}_d^T (\Theta_d^b)^T \hat{\Omega}_d^T \hat{V}^T]^T$ .

Observing  $\widetilde{\omega}_d = \omega_d - \hat{\omega}_d = \widetilde{\Omega}_d - K_{\Omega \Im} \widetilde{\Theta}_d^b$ , subtracting (33) from (32) and adding and subtracting  $W_{\Omega}^T \hat{\sigma}_{\Omega}$ , the virtual controller estimation error dynamics are found to be

$$\begin{split} & \dot{\Theta}_{d}^{b} = \widetilde{\omega}_{d} - (K_{\Omega 1} - K_{\Omega 3})\widetilde{\Theta}_{d}^{b}, \qquad \dot{\Omega}_{d} = \widetilde{f}_{\Omega} - K_{\Omega 2}\widetilde{\Theta}_{d}^{b} + \xi_{\Omega} \quad (35) \\ & \text{where } \widetilde{\Omega}_{d} = \Omega_{d} - \hat{\Omega}_{d}, \widetilde{f}_{\Omega} = \widetilde{W}_{\Omega}^{T}\widehat{\sigma}_{\Omega}, \quad \widetilde{W}_{\Omega}^{T} = W_{\Omega}^{T} - \hat{W}_{\Omega}^{T}, \quad \xi_{\Omega} = \varepsilon_{\Omega} + W_{\Omega}^{T}\widetilde{\sigma}_{\Omega}, \\ & \text{and } \widetilde{\sigma}_{\Omega} = \sigma_{\Omega} - \hat{\sigma}_{\Omega}. \quad \text{Furthermore,} \|\xi_{\Omega}\| \leq \xi_{\Omega M} \text{ with } \xi_{\Omega M} = \varepsilon_{\Omega M} + 2W_{M\Omega}\sqrt{N_{\Omega}} \quad \text{a} \\ & \text{positive computable constant and } N_{\Omega} \text{ the number of hidden} \\ & \text{layer neurons in the virtual control NN. Similarly, the} \\ & \text{estimation error dynamics of (34) are found to be} \end{split}$$

 $\dot{\omega}_{d} = -K_{\Omega 3}\widetilde{\omega}_{d} + \widetilde{f}_{\Omega} - (K_{\Omega 2} - K_{\Omega 3}(K_{\Omega 1} - K_{\Omega 3}))\widetilde{\Theta}_{d}^{b} + \xi_{\Omega} \quad (36)$ Examination of (35) and (36) reveals  $\widetilde{\Theta}_{d}^{b}$ ,  $\widetilde{\omega}_{d}$ , and  $\widetilde{f}_{\Omega}$  to be equilibrium points of the estimation error dynamics when  $\|\xi_{\Omega}\| = 0$ .

# C. NN Output Feedback Control Law

In the previous section, the desired translational velocity was formulated to ensure the quadrotor UAV tracked a desired trajectory, and the roll and pitch angles were determined to guarantee the desired translational velocities  $v_{dx}$ ,  $v_{dy}$  were tracked. Then, using the NN virtual controller, the desired angular velocity was found so that the desired orientation of the UAV is tracked. In this section, the actual inputs  $u_1$  and  $u_2$  to the dynamic system (1) are calculated so that the desired lift velocity  $v_{dz}$  and desired angular velocity

# $\omega_d$ are tracked and the overall control objective is met.

First, the thrust control input,  $u_1$ , will be addressed. Consider again the translational velocity tracking error dynamics written in terms of the observer velocity estimates (22). Considering the dynamics of the third error state  $\dot{\hat{e}}_{vz}$  in (22), the thrust control input is found to be

$$u_{1} = mc_{\theta d} c_{\theta d} \left( \ddot{z}_{d} + k_{\rho x} \dot{z}_{d} - \hat{v}_{R3} - g + \hat{f}_{c13} \right) + m \left( c_{\phi d} s_{\theta d} s_{\psi d} - s_{\phi d} c_{\psi d} \right) \left( \ddot{y}_{d} + k_{\rho y} \dot{y}_{d} - \hat{v}_{R2} + \hat{f}_{c12} \right) + m k_{\nu 3} \hat{e}_{\nu 3} + m \left( c_{\phi d} s_{\theta d} c_{\psi d} + s_{\phi d} s_{\psi d} \right) \left( \ddot{x}_{d} + k_{\rho x} \dot{x}_{d} - \hat{v}_{R1} + \hat{f}_{c11} \right)$$
(37)

where  $\hat{f}_{c1}$  is the NN estimate previously defined in *Section III.B.* Next, substituting the virtual control inputs (25) and (26) as well as the thrust (37) into (22) reveals the closed loop translational velocity tracking dynamics to be

$$\dot{\hat{e}}_{v} = -(K_{v} + S(\omega))\hat{e}_{v} + R_{d}^{T}\tilde{W}_{c1}^{T}\hat{\sigma}_{c1}^{T} + \xi_{c1}, \quad (38)$$
after adding and subtracting  $R_{d}^{T}W_{c1}^{T}\hat{\sigma}_{c1}^{T}, \quad \text{where}$ 

$$\xi_{c1} = R_{d}^{T}W_{c1}^{T}\tilde{\sigma}_{c1}^{T} + \varepsilon_{c1} - \bar{\tau}_{d1}, \quad \tilde{W}_{c1} = W_{c1} - \hat{W}_{c1}, \quad \text{and}$$

$$\tilde{\sigma}_{c1} = \sigma_{c1} - \hat{\sigma}_{c1}. \quad \text{Further,} \quad \left\|R_{d}\right\|_{F} = R_{d\max} \text{ for a known}$$
constant  $R_{d\max}$ , and  $\left\|\xi_{c1}\right\| \le \xi_{Mc1}$  for a computable constant  $\xi_{Mc1} = \varepsilon_{Mc1} + 2R_{d\max}W_{Mc1}\sqrt{N_{c}} + M_{M}\tau_{M}$  where  $M_{M}$  was defined in *Section III.A*, and  $N_{c1}$  is the number of hidden layer neurons.

Next, the rotational torques,  $u_2$ , will be addressed. Consider again the angular velocity tracking error (29). Similar to (22), the angular velocity tracking error is rewritten in terms of the NN virtual control estimate of  $\hat{\omega}_d$  in (34) and NN observer estimate of  $\hat{\omega}$  in (9) as

$$\hat{e}_{\omega} = \hat{\omega}_d - \hat{\omega}. \tag{39}$$

Multiplying both sides of (39) by *J*, the angular velocity tracking error dynamics become

$$\dot{\dot{P}}_{\omega} = f_{c2}(x_{c2}) - u_2 - \tau_{d2}$$
(40)

where  $f_{c2}(x_{c2}) = f_{c2} = J\dot{\omega}_d - S(J\omega)\omega - N_2(\omega) - J\dot{\widetilde{\omega}}_d + J\dot{\widetilde{\omega}} \in \Re^3$ , and unknown. Therefore, the universal approximation property of NN is utilized to estimate the function  $f_{c2}(x_{c2})$ by bounded ideal weights  $W_{c2}^T, V_{c2}^T$  such  $\|W_{c2}\|_{c} \leq W_{Mc2}$  for a known constant  $W_{Mc2}$  and written as  $f_{c2}(x_{c2}) = W_{c2}^T \sigma(V_{c2}^T x_{c2}) + \varepsilon_{c2}$  where  $\varepsilon_{c2}$  is the bounded NN functional reconstruction error such that  $\|\mathcal{E}_{c2}\| \leq \mathcal{E}_{Mc2}$  for a known constant  $\varepsilon_{Mc2}$ . The NN estimate of  $f_{c2}$  is given by  $\hat{f}_{c2} = \hat{W}_{c2}^T \sigma(V_{c2}^T \hat{x}_{c2}) = \hat{W}_{c2}^T \hat{\sigma}_{c2}$  where  $\hat{W}_{c2}^T$  is the NN estimate of  $W_{c2}^T$  and  $\hat{x}_{2c} = \begin{bmatrix} 1 & \hat{\omega}^T & \hat{\Omega}_d^T & \Theta_d^T \end{bmatrix}^T$  is the input to the NN written in terms of the observer and virtual controller estimates. By the construction of the virtual controller,  $\hat{\omega}_d$  is not directly available; therefore, observing (44), the terms  $\dot{\Omega}_d^T$ ,  $\tilde{\Theta}_d^{bT}$ , and  $e_{\Theta}^T$  have been included instead.

Using the NN estimate  $\hat{f}_{c2}$ , the rotational torque control input is written as

$$u_2 = \hat{f}_{c2} + K_\omega \hat{e}_\omega, \tag{41}$$

and substituting the control input (41) into the angular velocity dynamics (40) as well as adding and subtracting  $W_{c2}^T \hat{\sigma}_c$ , the closed loop dynamics become

$$J\dot{\hat{e}}_{\omega} = -K_{\omega}\hat{e}_{\omega} + \widetilde{W}_{c2}^{T}\hat{\sigma}_{c2} + \xi_{c2}$$
(42)

where  $\widetilde{W}_{c2}^{T} = W_{c2}^{T} - \hat{W}_{c2}^{T}$ ,  $\xi_{c2} = \varepsilon_{c2} + W_{c2}^{T} \widetilde{\sigma}_{c} - \tau_{d2}$ , and  $\widetilde{\sigma}_{c2} = \sigma_{c2} - \hat{\sigma}_{c2}$ . Further,  $\|\xi_{c2}\| \le \xi_{Mc2}$  for a computable constant  $\xi_{Mc2} = \varepsilon_{Mc2} + 2W_{Mc2}\sqrt{N_{c2}} + \tau_{dM}$  where  $N_{c2}$  is the number of hidden layer neurons.

As a final step, we define an augmented translational and angular velocity error system as  $\hat{e}_s = \begin{bmatrix} \hat{e}_v^T & \hat{e}_\omega^T \end{bmatrix}^T$  whose closed loop dynamics are described by (38) and (42), respectively, and written as

$$\overline{J}\dot{\hat{e}}_{s} = A_{d}^{T}\widetilde{f}_{c} - (K_{s} + S_{s}(\omega))\hat{e}_{s} + \xi_{c}$$
(43)  
where  $\overline{J} = \begin{bmatrix} I_{3\sqrt{3}} & 0_{3\sqrt{3}} \end{bmatrix} \in \Re^{6x6}$  is a constant,

$$\begin{split} K_{S} &= [K_{v} \ 0_{3x3}; 0_{3x3} \ K_{\omega}] > 0 \in \Re^{6x6}, A_{d} = [R_{d} \ 0_{3x3}; 0_{3x3} \ I_{3x3}] \in \Re^{6x6} \\ S_{S}(\omega) &= [S(\omega) \ 0_{3x3}; 0_{3x3} \ 0_{3x3}] \in \Re^{6x6}, \qquad \hat{e}_{S}^{T} S_{S}(\omega) \hat{e}_{S} = 0, \\ \xi_{c} &= [\xi_{c1}^{T} \ \xi_{c2}^{T}]^{T} \in \Re^{6}, \text{ and } \|\xi_{c}\| \leq \xi_{Mc} \text{ for a positive computable} \\ \text{constant } \xi_{Mc} &= \sqrt{\xi_{Mc1}^{2} + \xi_{Mc2}^{2}}. \text{ Additionally, } \tilde{f}_{c} = \tilde{W}_{c}^{T} \hat{\sigma}_{c} \in \Re^{6} \\ \text{where} \end{split}$$

$$\widetilde{W}_{c} = \begin{bmatrix} \widetilde{W}_{c1} & 0 \\ 0 & \widetilde{W}_{c2} \end{bmatrix} \text{ and } \hat{\sigma}_{c} = [\hat{\sigma}_{c1}^{T} \ \hat{\sigma}_{c2}^{T}]^{T}.$$

Examining (43) reveals  $\hat{e}_s$  and  $\tilde{f}_c$  to be equilibrium points of the augmented error dynamics when  $\|\xi_c\| = 0$ . Further, a single NN is utilized to estimate  $\hat{f}_c = [\hat{f}_{c1}^T \ \hat{f}_{c2}^T]^T \in \Re^6$ .

In the final theorem, the stability of the entire system is considered. In other words, the position, orientation, and velocity tracking errors are considered along with the estimation errors of the observer and virtual controller and the NN weight estimation errors of each NN. Considering the entire system in a single Lyapunov candidate allows the separation principle to be relaxed.

Theorem 2: (Quadrotor UAV System Stability) Given the dynamic system of a quadrotor UAV in (1), let the NN observer be defined by (8) and (9), respectively, with the NN update law for the observer given by (13). Given a smooth desired trajectory, let the desired translational velocity for the UAV to track be defined by (17) with the desired pitch and roll defined by (25) and (26), respectively. Let the NN virtual controller be defined by (33) and (34), respectively, with the NN update law given by

$$\dot{\hat{W}}_{\Omega} = F_{\Omega} \hat{\sigma}_{\Omega} \left( \widetilde{\Theta}_{d}^{b} \right)^{T} - \kappa_{\Omega I} F_{\Omega} \hat{W}_{\Omega}$$
(44)

where  $F_{\Omega} = F_{\Omega}^{T} > 0$  and  $\kappa_{\Omega 1} > 0$  are design parameters. Let the dynamic NN controller be defined by (37) and (41), respectively, with the NN update given by

$$\dot{\hat{W}}_c = F_c \hat{\sigma}_c \left( A_d \hat{e}_S \right)^T - \kappa_{c1} F_c \hat{W}_c$$
(45)

where  $F_c = F_c^T > 0$  and  $\kappa_{c1} > 0$  are constant design parameters. Then there exists positive design constants  $K_{o1}, K_{o2}, K_{o3}, K_{\Omega 1}, K_{\Omega 2}, K_{\Omega 3}$ , and positive definite constant design matrices  $K_{\rho}, K_{\Theta}, K_{\nu}, K_{\omega}$ , such that observer estimation errors  $\widetilde{X}, \widetilde{V}$  and the NN observer weight estimation errors,  $\widetilde{W}_o$ , the virtual controller estimation errors  $\widetilde{\Theta}_{d}^{b}$ ,  $\widetilde{\omega}_{d}$  and the virtual control NN weight estimation errors,  $\widetilde{W}_{\Omega}$ , the position, orientation, and translational and angular velocity tracking errors,  $e_{\rho}$ ,  $e_{\Theta}$ ,  $\hat{e}_{s}$ , respectively, and the dynamic controller NN weight estimation errors,  $\widetilde{W}_{c}$ , are all *SGUUB*.

*Proof:* Consider the following positive definite Lyapunov candidate

$$V_{UAV} = V_o + V_{\Omega} + V_{\Omega}$$

where  $V_{a}$  was defined in (14) and

$$V_{\Omega} = \frac{1}{2} \widetilde{\Theta}_{d}^{b^{T}} \left( K_{\Omega 2} - K_{\Omega 3} (K_{\Omega 1} - K_{\Omega 3}) \right) \widetilde{\Theta}_{d}^{b} + \frac{1}{2} \widetilde{\omega}_{d}^{T} \widetilde{\omega}_{d} + \frac{1}{2} tr \{ \widetilde{W}_{\Omega}^{T} F_{\Omega}^{-1} \widetilde{W}_{\Omega} \}$$
(46)

$$V_{c} = \frac{1}{2} e_{\rho}^{T} e_{\rho} + \frac{1}{2} e_{\Theta}^{T} e_{\Theta} + \frac{1}{2} \hat{e}_{S}^{T} \overline{J} \hat{e}_{S} + \frac{1}{2} tr \left\{ \widetilde{W}_{c}^{T} F_{c}^{-1} \widetilde{W}_{c} \right\}.$$
(47)

The first derivative of  $V_{UAV}$  with respect to time is given by  $\dot{V}_{UAV} = \dot{V}_o + \dot{V}_{\Omega} + \dot{V}_c$ , and substitution of the closed loop observer and virtual control estimation error dynamics (10), (12), (35) and (36), respectively, the closed loop position, orientation, and velocity tracking errors (19), (30), and (43), respectively, as well as the NN update laws (13), (44), and (45) reveals the upper bound of  $\dot{V}_{UAV}$  can be written as

$$\begin{split} \dot{V}_{UAV} &\leq -\left(\frac{\overline{K}_o}{2} - \frac{N_0}{\kappa_{o1}}\right) \|\widetilde{X}\|^2 - \left(\frac{K_{o3}}{2} - \frac{N_0}{\kappa_{o1}} - \frac{\Pi_{\max}^2}{2K_{K\min}}\right) \|\widetilde{V}\|^2 \\ &- \left(\frac{\kappa_{o1}}{4} - \frac{N_o}{K_{S\min}}\right) \|\widetilde{W}_o\|_F^2 - \left(\overline{K}_\Omega - \frac{N_\Omega}{\kappa_{\Omega1}}\right) \|\widetilde{\Theta}_d^b\|^2 \\ &- \left(\frac{K_{\Omega3}}{2} - \frac{N_\Omega}{\kappa_{\Omega1}} - \frac{\Pi_{\max}^2}{2K_{K\min}}\right) \|\widetilde{\omega}_d\|^2 - \frac{\kappa_{\Omega1}}{4} \|\widetilde{W}_\Omega\|_F^2 - \frac{K_{K\min}}{4} \|e_K\|^2 \\ &- \left(\frac{K_{S\min}}{2} - \frac{\Pi_{\max}^2}{2K_{K\min}}\right) \|\hat{e}_S\|^2 - \frac{\kappa_{c1}}{2} \|\widetilde{W}_c\|_F^2 + \eta_{UAV} \end{split}$$

$$(48)$$

where  $e_{\rm K} = [e_{\rho}^T \ e_{\Theta}^T]^T$ ,  $K_{\rm K} = [K_{\rho} \ 0_{3x3}; 0_{3x3} \ K_{\Theta}]$ ,  $\Pi = [R \ 0_{3x3}; 0_{3x3} \ T]$ with  $\|\Pi\|_F < \Pi_{\rm max}$ ,  $K_{\rm K min}$  and  $K_{\rm S min}$  are the minimum singular values of  $K_K$  and  $K_S$ , respectively, and  $\eta_{UAV}$  is a computable constant dependent on  $\xi_{1M}$ ,  $\xi_{2M}$ ,  $\xi_{\Omega M}$ , and  $\xi_{Mc}$ . Finally,  $\dot{V}_{UAV}$  is less than zero provided the controller gains are selected so that the following inequalities are satisfied

$$\overline{K}_{o} > \frac{2N_{0}}{\kappa_{ol}}, \quad K_{o3} > \frac{2N_{0}}{\kappa_{ol}} + \frac{\Pi_{\max}^{2}}{2K_{K\min}}, \quad \kappa_{o1} > \frac{4N_{o}}{K_{S\min}}, \quad \overline{K}_{\Omega} > \frac{N_{\Omega}}{\kappa_{\Omega l}}, \quad (49)$$

$$K_{\Omega 3} > \frac{2N_{\Omega}}{\kappa_{\Omega l}} + \frac{\Pi_{\max}^{2}}{2K_{K\min}} \quad \text{and} \quad K_{S\min} > \frac{\Pi_{\max}^{2}}{K_{K\min}}$$

Therefore, it can be concluded that  $\dot{V}_{UAV}$  is less than zero provided the following inequalities hold:

$$\begin{split} \|\widetilde{X}\| &> \sqrt{\frac{\eta_{UAV}}{(\overline{K}_o/2 - N_o/\kappa_{o1})}} \quad or \quad \|\widetilde{\Theta}_d^b\| > \sqrt{\frac{\eta_{UAV}}{\overline{K}_\Omega - N_\Omega/\kappa_{\Omega1}}} \quad or \\ \|\widetilde{W}_o\|_F &> \sqrt{\frac{\eta_{UAV}}{\kappa_{o1}/4 - N_o/K_{S\,\min}}} \quad or \quad \|e_K\| > \sqrt{\frac{4\eta_{UAV}}{K_{K\,\min}}} \quad or \quad \|\widetilde{W}_c\|_F > \sqrt{\frac{2\eta_{UAV}}{\kappa_{c1}}} \quad or \\ \|\widetilde{\omega}_d\| &> \sqrt{\frac{\eta_{UAV}}{K_{\Omega3}/2 - N_\Omega/\kappa_{\Omega1} - \Pi_{\max}^2/2K_{K\,\min}}} \quad or \quad \|\hat{e}_S\| > \sqrt{\frac{2\eta_{UAV}}{K_{S\,\min} - \Pi_{\max}^2/K_{K\,\min}}} \quad or \\ \|\widetilde{W}_o\| &> \sqrt{\frac{4\eta_{UAV}}{K_{\Omega3}/2 - N_\Omega/\kappa_{\Omega1} - \Pi_{\max}^2/2K_{K\,\min}}} \quad or \quad \|\widetilde{e}_S\| > \sqrt{\frac{2\eta_{UAV}}{K_{S\,\min} - \Pi_{\max}^2/K_{K\,\min}}} \quad or \end{split}$$

Thus, using standard Lyapunov extensions [10]  $\dot{V}_{UAV}$  is

less than zero outside of a compact set revealing that all the signals are *SGUUB*.

**Remark 4:** The above error bounds can be decreased through the appropriate selection of the design parameters. Further, using definitions of  $\tilde{v}, \tilde{\omega}$ , and  $\tilde{\omega}_d$ , the velocity tracking errors (18) and (29) can be rewritten as

 $e_v = v_d - v = v_d - \hat{v} - \tilde{v} = \hat{e}_v - \tilde{v}$ 

$$e_{\omega} = \omega_d - \omega = \hat{\omega}_d + \tilde{\omega}_d - \hat{\omega} - \tilde{\omega} = \hat{e}_{\omega} + \tilde{\omega}_d - \tilde{\omega}$$

From *Theorem 2*,  $\tilde{v}, \tilde{\omega}, \hat{e}_v, \hat{e}_\omega, \tilde{\omega}_d$  are all *SGUUB*; therefore it can be concluded  $e_v, e_\omega$  are also *SGUUB*. Thus,  $v \to v_d$ ,  $\omega \to \omega_d$ .

#### IV. CONCLUSIONS

A new NN output feedback control law was developed for an underactuated quadrotor UAV which utilizes the natural constraints of the underactuated system to generate virtual control inputs to guarantee the UAV tracks a desired trajectory without the knowledge of dynamics. All six DOF are successfully tracked using only four control inputs while in the presence of unmodeled dynamics and bounded disturbances. Lyapunov analysis guarantees *SGUUB* of all the signals while relaxing the separation principle. Although not shown, numerical results confirm the theoretical conjectures [11].

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