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Global Hierarchical Observer for Linear Systems with Unknown Inputs

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***Abstract*— A sliding-mode based observer is suggested in order to achieve global, finite-time reconstruction of the state vector for a class of linear systems with unknown inputs. In the paper we show that, by using a recently proposed global second-order sliding mode differentiation algorithm, the necessary and sufficient structural conditions for the observer design are preserved, with respect to previous works; meanwhile, the class of allowed unknown inputs is generalized significantly. A numerical example illustrates the effectiveness of the suggested technique.**

I. INTRODUCTION

A. Antecedents and motivations

The problem of state observation for systems with unknown inputs was widely addressed in the control literature during the last two decades. The specific features of the majority of the existing approaches are:

1. The number of unknown inputs must be less than the number of outputs, and, moreover, additional structural requirements on the system to be observed are met (see, e.g., [1] and [2]). Those conditions turn out to be rather restrictive. For instance they cannot cover the simplest class of mechanical systems with unknown inputs wherein only the position is measurable. In [3] it was suggested a more complicated adaptive observer ensuring an exponential convergence of the estimation error to a small neighborhood of zero.

2. Only asymptotic convergence to zero of the observation and error is guaranteed ([4]) in the smooth observation scheme. However, for instance, for hybrid systems the finite time exact observation is quite important since it is necessary to ensure that the time of observation convergence is less than the dwell time; for example, in the case of walking robots ([5], [6]).

The problem of observation has been actively developed within *Variable Structure Systems Theory* using the *Sliding-Mode Control* approach. Sliding mode observers (see, e.g., the corresponding chapters in the textbooks [7], [8], and the recent tutorials [9], [10] and [11]) are widely used due to their attractive features, namely: a) insensitivity (which is stronger than mere robustness) with respect to some classes of unknown inputs; b) possibility to use the equivalent output injection concept for the unknown inputs identification.

In [12], [13], [14] and [15] a step-by-step sliding mode observers design was proposed. Such an approach is based on the possibility to transform the actual system into a block-observable form and after which the sequential estimation of each transformed state is made by means of the concept of equivalent output injection. Unfortunately, the realization of such observation schemes demands obligatory filtration, which causes an intrinsic error in the observed states that cannot be eliminated. Furthermore, the system structure must be such that the transformation to the triangular form can be performed. A new generation of observers based on higher-order sliding-mode differentiators ([16],,[17]) has been recently studied in the literature [18], [19], [20], [21], [22], [23], [24], [25], and [26]. This sort of observers preserve the advantages of the first order sliding mode observers, but avoid the filtration process, allowing the finite-time convergence to zero for the estimation error. Generally in those papers the unknown inputs were supposed uniformly bounded ([27] and [28]), which allows to stabilize the observation error with some linear observer to a neighborhood of the zero point and, after that, to use a robust exact differentiator that ensures the finite time exact reconstruction of the original state.

B. Main contribution

By using the recently developed global exact differentiator [29], in this paper we propose a scheme for designing a robust observer providing a finite-time exact observation for the class of strongly observable linear systems with unknown inputs and/or nonlinear uncertainties bounded by known functions that could be non-uniformly bounded. Thus, it is achieved a global exact finite time convergence in the presence of possible unbounded unknown inputs

II. PLANT MODEL AND STANDING ASSUMPTIONS

Consider the following linear system with unknown inputs

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dw(t), \quad x(0) = x_0 \\ y(t) &= Cx(t), \qquad\qquad t \geq 0 \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ $(1 \leq p < n)$ are the state vector, the control, and the output of the system, respectively. A, B, C, and D are known matrices of suitable dimension with $\mathrm{rank}(C) = p$, and $\mathrm{rank}(D) = q$.

The following definition can be found in several works, see, e.g., [30], [1], [31], and [32].

Definition 1: For $u \equiv 0$, the triple (A, D, C),is called **strongly observable** if, for any initial condition x_0, the

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condition $y(t) = 0$ for all $t \geq 0$ implies that $x(t) = 0$ for all $t \geq 0$ irrespectively of the actual unknown input $w(t)$.

It is worth to note if the system is not strongly observable, then there exists an initial condition $x_0 = \xi$ and an unknown input $w(t)$ such that $y(t) = 0$ for all $t \geq 0$ and $x(t)$ being not equal to zero for all $t \geq 0$. Therefore, in such a case it would be clearly impossible to reconstruct the system state $x(t)$ from the output measurement. It means that strong observability property is a necessary structural condition for the reconstruction of the state vector.

Hence, throughout the paper we shall make the assumptions:

A1. For $u \equiv 0$, the triple (A, D, C) is strongly observable (SO)

A2. There exist a known constant k and a known class-K_∞ function $F : [0, \infty) \to [0, \infty)$ such that

$$\|w(t)\| \leq k + F(\|y\|) \tag{2}$$

In the next Section we present a fundamental result, based on the condition A1, that the system state can be expressed as a function (simple linear combination) of the system outputs and a certain number of their derivatives.

III. Preliminaries

A. Auxiliary system

Implement the following linear observer

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu + L(y(t) - \tilde{y}(t)) \tag{3}$$

$$\tilde{y}(t) = C\tilde{x}(t) \tag{4}$$

with L chosen so that $\tilde{A} = (A - LC)$ is Hurwitz. Notice that A1 guarantees that the pair (A, C) is observable. Let us define $e(t) := x(t) - \tilde{x}(t)$. Clearly the estimation error will not vanish. Indeed the error dynamics is

$$\dot{e}(t) = \tilde{A}e(t) + Dw(t) \tag{5}$$

Knowing e and $\tilde{x}$ it is immediate to recover the original system state by $x = \tilde{x} + e$. Below we apply a procedure to reconstruct $e(t)$. Unlike the original system (1), the error system (5) has always a stable eigenvalues. This makes possible to compute an explicit upperbound for the norm of the error vector e. It is the subject of the Lemma 2.

Lemma 2 Consider system (5) having an Hurwitz characteristic matrix $\tilde{A}$ and with the unknown input w satisfying (2). Let $P = P^T > 0$ be the symmetric positive definite (SPD) solution of the Lyapunov equation $\tilde{A}^T P + P\tilde{A} = -Q$ for an arbitrary SPD matrix Q.

Define

$$\Gamma(\|y\|, t) := b + \mu \left(\int_0^t \exp^{-\delta(t-\tau)} F^2(\|y\|) d\tau \right)^{1/2} \tag{6}$$

where

$$b > 0,\ \mu = \left(\frac{\|P\| \|D\|}{\epsilon \lambda_{\min}(P)} \right)^{1/2},\ \delta = \frac{(\lambda_{\min}(Q) - \epsilon \|P\| \|D\|)}{\lambda_{\max}(P)} \tag{7}$$

and ϵ is small enough so that $\delta > 0$. Then, there is a time $T_1 > 0$ such that the following conditions hold for all $t \geq T_1$

$$\|e(t)\| \leq \Gamma(\|y\|, t), \quad \|\dot{e}(t)\| \leq \varphi(t) \tag{8}$$

$$\text{where } \varphi(t) := \Big(\|\tilde{A}\| \Gamma(\|y\|, t) + \|D\| F(\|y\|) \Big)$$

with F defined by (2).

Proof of Lemma 2 See the Appendix.

B. Reconstruction of the state error $e(t)$

For a matrix $Y \in \mathbb{R}^{r \times q}$ having $\operatorname{rank} Y = h$, we select $Y^\perp \in \mathbb{R}^{r-h \times r}$ as one of the matrices so that $Y^\perp Y = 0$ and $\operatorname{rank} Y^\perp = r - h$. Note that $Y^\perp$ always exists and that it is not unique for a given matrix[1] Y.

The sequence of transformations we are going to introduce is aimed at expressing the error e as an algebraic function of the outputs and a finite number of their derivatives.

Consider the following algorithm:

Step 1

Define $M_1 = C$. Thus, $y_e := y - \tilde{y} = M_1 e$. Write down the derivative of y, $\dot{y}_e = M_1 \dot{e} = M_1 e + M_1 Dw$. We want to find a transformed output whose first derivative in not affected by w. Consider the transformed output $\overline{y}^1 = (M_1 D)^\perp y_e(t)$ and its time derivative

$$\dot{\overline{y}}^1 = (M_1 D)^\perp \dot{y}_e(t) = (M_1 D)^\perp M_1 \tilde{A} e \tag{9}$$

Now construct the extended vector

$$\xi^1 := \begin{bmatrix} \dot{\overline{y}}^1 \\ y_e \end{bmatrix} = \begin{bmatrix} (M_1 D)^\perp M_1 \tilde{A} \\ M_1 \end{bmatrix} e \equiv M_2 e \tag{10}$$

with implicitly defined matrix M_2. Note that, from (10) and (9), the vector ξ^1 can be expressed as function of the output vector and its first derivative.

Step 2

Consider the transformed output $\overline{y}^2 = (M_2 D)^\perp \xi^1$ and its time derivative

$$\dot{\overline{y}}^2 = (M_2 D)^\perp \dot{\xi}^1 = (M_2 D)^\perp M_2 \tilde{A} e \tag{11}$$

Now construct the extended vector

$$\xi^2 := \begin{bmatrix} \dot{\overline{y}}^2 \\ y_e \end{bmatrix} = \begin{bmatrix} (M_2 D)^\perp M_2 \tilde{A} \\ M_1 \end{bmatrix} e \equiv M_3 e \tag{12}$$

with implicitly defined matrix M_3.

Note that, from (12), (11), (10) and (9), the vector ξ^2 can be expressed as function of the output vector and its first and second derivative.

Step k. $k = 3, 4, \ldots, n-1$

Consider the transformed output $\overline{y}^{(k)} = (M_k D)^\perp \xi^{(k-1)}$ and its time derivative

$$\dot{\overline{y}}^{(k)} = (M_k D)^\perp \dot{\xi}^{(k-1)} = (M_k D)^\perp M_k \tilde{A} e \tag{13}$$

Now construct the extended vector

$$\xi^{(k)} = \begin{bmatrix} \dot{\overline{y}}^{(k)} \\ y_e \end{bmatrix} = \begin{bmatrix} (M_k D)^\perp M_k \tilde{A} \\ M_1 \end{bmatrix} e \equiv M_{k+1} e \tag{14}$$

[1] A Matlab code for computing $B = F^\perp$ is >> $B = (null((F)'))'$;

with implicitly defined matrix M_{k+1}. It is easy to see that the vector $\xi^{(k)}$ can be expressed as function of the output vector and its derivatives up to the (k-1)-th order.

At the end of the k-th step, the matrix M_{k+1} and the term $\xi^{(k)}$ are available such that

$$\xi^{(k)} = M_{k+1} e(t) \tag{15}$$

Suppose that for some l there is a matrix M_l such that $\operatorname{rank} M_l = n$. Then the algebraic equation (15) would have a unique solution

$$e(t) = M_l^+ \xi^{(l-1)} \tag{16}$$

where $M_l^+ = \left(M_l^T M_l\right)^{-1} M_l^T$. It means that for $\operatorname{rank} M_l = n$, the state $e(t)$ could be estimated using a linear combination of the output and its derivatives up to the $(l-1)$-th order. Since (A, D, C) is SO if and only $\left(\tilde{A}, D, C\right)$ is SO, it holds that strong observability property is a necessary and sufficient condition to have $\operatorname{rank} M_n = n$. Indeed (see e.g., [30], [31], [32])

$$\text{The triple } \left(\tilde{A}, D, C\right) \text{ is SO iff } \operatorname{rank} M_n = n. \tag{17}$$

Thus, from A1 and (17), we conclude that for the considered class of systems we have that $\operatorname{rank} M_n = n$. This means that the number of iterations of the above procedure will be at most $n-1$.

Now let us extract from the above algorithm the equations for computing the sequence of the matrices M_1, M_2, ..., M_n.

$$M_1 = C,\ M_{k+1} = \begin{bmatrix} (M_k D)^\perp M_k \tilde{A} \\ C \end{bmatrix},\ k = 1, \ldots, n-1 \tag{18}$$

IV. Hierarchical Second Order Sliding Observation Concept

To obtain the required output derivatives we will use the Global Sub-optimal Differentiator presented in [29] (see the Appendix A). The design of the Global Sub-optimal Differentiator requires the explicit knowledge of an instantaneous upperbound to the second derivative of the signal to derive.

A. State observation by means of a global differentiator

The algorithm outlined in III-B is now developed. The recovery of $M_2 e(t)$ will be based on the design of a *sliding surface* $s^{(1)}$ and its corresponding *output injection* $v^{(1)}$ using the *"global sub-optimal" algorithm (see [29]) described in the Appendix A.*

Let us define a sliding vector variable $s^{(1)}$ as follows:

$$s^{(1)}(t) = \pi^{(1)}(t) - \int_0^t v^{(1)}(\tau)\, d\tau \tag{19}$$

$$\pi^{(1)}(t) = \begin{bmatrix} (M_1 D)^\perp y_e(t) \\ \int_0^t y_e(t) d\tau \end{bmatrix} \tag{20}$$

taking the time derivative of $s^{(1)}$, and noting that $\frac{d}{dt}\pi^{(1)}(t) = M_2 e$, we get the following expression for the time derivative of $s^{(1)}$:

$$\dot{s}^{(1)} = M_2 e - v^{(1)} \tag{21}$$

In order to steer to zero vector $s^{(1)}$ and its unmeasurable derivative $\dot{s}^{(1)}$, the components of vector $v^{(1)}$ can be defined as follows according to the compact notation introduced in the Appendix A:

$$\dot{v}^{(1)} = \mathbf{GSO}\left(s^{(1)}, \|M_2\| \varphi(t)\right) \tag{22}$$

where the scalar function φ is an instantaneous upper-bound to $\|\dot{e}(t)\|$, given in (8). GSO denotes the Global Sub-Optimal Algorithm. As shown in the appendix (see also [29]), when the algorithm is applied by using the sliding quantity $s^{(1)}$ constructed as above it implements a real time differentiator, in the sense that $v^{(1)}$ converges in finite time to the derivative of $\pi^{(1)}$. As shown in [29], there is a reaching time t_1 such that $s^{(1)}(t) = \dot{s}^{(1)}(t) = 0$, for all $t \geq t_1$, which implies that

$$v^{(1)}(t) = \frac{d}{dt}\pi^{(1)}(t) = M_2 e(t), \quad \text{for all } t \geq t_1. \tag{23}$$

Now, for recovering $M_3 e(t)$ we design $s^{(2)}$ and its corresponding *output injection* $v^{(2)}$ by a similar procedure, detailed as follows. The variable $s^{(2)}$ is given by the formula

$$s^{(2)}(t) = \pi^{(2)}(t) - \int_0^t v^{(2)}(\tau)\, d\tau \tag{24}$$

$$\pi^{(2)}(t) := \begin{bmatrix} (M_2 D)^\perp v^{(1)}(t) \\ \int_0^t y_e(t) d\tau \end{bmatrix} = \begin{bmatrix} (M_2 D)^\perp M_2 e \\ \int_0^t y_e(t) d\tau \end{bmatrix} \tag{25}$$

In the last equation it was considered (23). Since $\frac{d}{dt}\pi^{(2)}(t) = M_3 e(t)$, we get the following expression for $\dot{s}^{(2)}$:

$$\dot{s}^{(2)}(t) = M_3 e - v^{(2)}(t) \tag{26}$$

To steer to zero $s^{(2)}$ and its unmeasurable derivative $\dot{s}^{(2)}$, the components of vector $v^{(2)}$ can be defined as $\dot{v}^{(2)} = \mathbf{GSO}\left(s^{(2)}, \|M_3\|\, \varphi(t)\right)$. As before, there exists a finite time t_2 such that $s^{(2)}(t) = \dot{s}^{(2)}(t) = 0$, for all $t \geq t_2 \geq t_1$. Therefore considering (26) we have

$$v^{(2)} = M_3 e(t) \quad t \geq t_2 \geq t_1$$

Let us define $l \leq n$ as the least integer so that $\operatorname{rank} M_l = n$. Thus, we can resume the general design of sliding surfaces with their corresponding output injection terms. Design the *output injection* $v^{(k)}$ *at the k-th level* as

$$\dot{v}^{(k)} = \begin{cases} \mathbf{GSO}\left(s^{(k)}, \|M_{k+1}\| \varphi(t)\right) & 1 \leq k \leq l-2 \\ \mathbf{GSO}\left(s^{(k)}, \varphi(t)\right) & k = l-1 \end{cases} \tag{27}$$

with

$$s^{(k)} = \pi^{(k)} - \int_0^t v^{(k)}(\tau)\, d\tau \tag{28}$$

$$\pi^{(k)} = \begin{cases} \begin{bmatrix} (M_{\tilde{A},1} D)^\perp y_e(t) \\ \int_0^t y_e(t) d\tau \end{bmatrix} & k = 1 \\ \begin{bmatrix} (M_{\tilde{A},k} D)^\perp v^{(k-1)}(t) \\ \int_0^t y_e(t) d\tau \end{bmatrix} & 1 < k < l-2 \\ M_{\tilde{A},l}^+ \begin{bmatrix} (M_{\tilde{A},l-1} D)^\perp v^{(l-2)}(t) \\ \int_0^t y_e(t) d\tau \end{bmatrix} & k = l-1 \end{cases} \tag{29}$$

We included $M_l^+ = \left[M_l^T M_l\right]^{-1} M_l^T$ in the last variable $\pi^{(l-1)}$ of (29) in order to recover directly the state vector $e(t)$ by means of the output injection signal $v_i^{(k)}$.

Theorem 1: Consider system (1) satisfying the assumptions A1-A2. Implement the auxiliary observer (5) and let $e = x - \tilde{x}$ and $y_e = y - \tilde{y}$. Implement the output injection terms $v^{(k)}$ $(k = 1, 2, \ldots, l-1)$ according to (27)-(29). Then there exist $t_1 < t_2 < ... < t_{l-1}$ such that

$$\begin{array}{lll} v^{(k)} = M_{k+1} e(t), & t \geq t_k & k = 1, .., l-2 \\ v^{(l-1)} = x(t) - \tilde{x}(t) & & t \geq t_{l-1} \end{array} \quad (30)$$

which implies that the state vector x can be reconstructed exactly at $t \geq t_{l-1}$ by means of the following relationship

$$x = v^{(l-1)} + \tilde{x} \quad (31)$$

Proof: Since it has already been proven for $k = 1$ and $k = 2$, the remaining of the proof can be done by induction. ■

V. Example

A fifth-order system with unknown inputs is studied:

$$\begin{array}{ll} \dot{x}(t) = Ax(t) + Dw(t), & x(0) = (1, 3, -1, 4, 5) \\ y(t) = Cx(t), & t \geq 0 \end{array} \quad (32)$$

The matrices in (32) take the form

$$A = \begin{bmatrix} -1.49 & -0.17 & -0.42 & 0.27 & 0.31 \\ -1.08 & -0.83 & -0.66 & 0.02 & 0.12 \\ -2.26 & -0.12 & -1.00 & 0.09 & -0.08 \\ -2.78 & -0.37 & -0.89 & -0.58 & 0.27 \\ -1.55 & -0.22 & -0.90 & 0.18 & -0.49 \end{bmatrix}$$

$$D^T = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 \\ -1 & -1 & -1 & 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The unknown input terms are $w^T = \begin{bmatrix} 1 + x_2 \sin(t) & 0.9 \cos(0.85t) \end{bmatrix}$. Let us implement the method suggested in this manuscript. The auxiliary observer (4) is implemented with the matrix L chosen in order that the eigenvalues of $\tilde{A} = A - LC$ are $\{-0.27, -1.539 + 1.007i, -1.539 - 1.007i, -1.635, -1.217\}$.

As for the assumption A2, the following **bound to the unknown inputs** are considered in the observer design $\|w\| \leq F(\|y\|) = 2 + |y_2|$. The matrix $M_{\tilde{A},2}$ has full rank equal 5. Thus, we must design only one sliding surface for the construction of the observer. In order to reconstruct $M_{\tilde{A},2} e$, the signal $v^{(1)}$ is implemented according to

$$\dot{v}^{(1)} = \mathbf{GSO}\left(s^{(1)}, \|M_2\| \, \varphi(t)\right) \quad (33)$$

$$s^{(1)}(t) = M_2^{-1} \begin{bmatrix} (M_1 D)^{\perp} y_e(t) \\ \int_0^t y_e(t) d\tau \end{bmatrix} - \int_0^t v^{(1)}(t) d\tau \quad (34)$$

With the matrices M_1 and M_2 designed according to (18).

The term $\varphi(t)$ is constructed as specified in the Lemma 2 with the resulting parameters

$$b = 0.1, \;\; \mu = 17.75, \;\; \epsilon = 0.08, \;\; \delta = 2e - 5 \quad (35)$$

After a finite time we get that $v^{(1)}(t) = e(t)$. So that the state is reconstructed as

$$x = \tilde{x} + v^{(1)} \quad (36)$$

The algorithm is run with zero initial condition for all internal variables. Simulations are performed by discretizing the system and observer by Euler method with sampling step $T_s = 10^{-5} s$. Since x_1 and x_2 are already known, the state reconstruction is shown for the states x_3, x_4, x_5. The figures 1, 2, 3 show the trajectories of the actual and estimated states x_3, x_4, x_5, with corresponding zoom on the steady state.

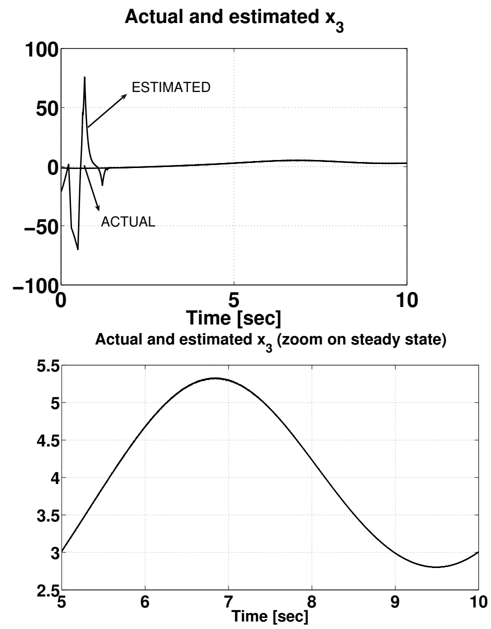

Fig. 1. The actual and estimated state x_3.

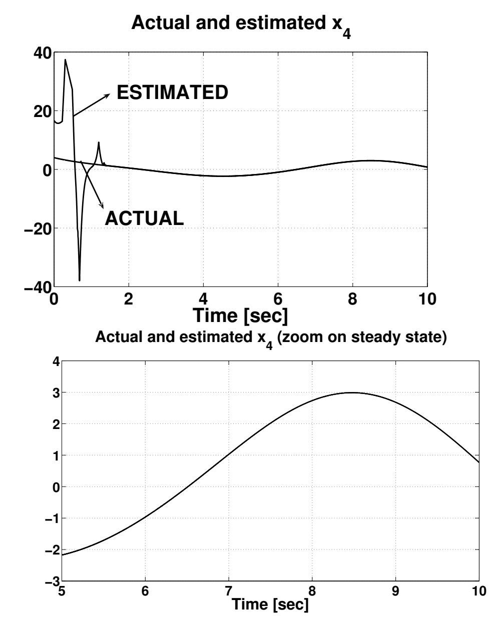

Fig. 2. The actual and estimated state x_4.

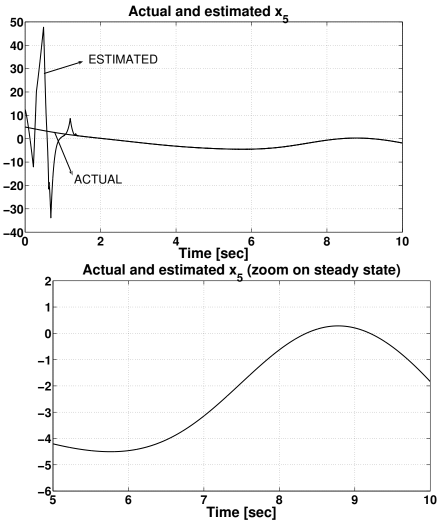

Fig. 3. The actual and estimated state x_5.

VI. Conclusions

Without requiring the system to be expressed in (or being transformed to) any normal form, it is shown that strong observability is a necessary and sufficient condition for state estimation in the presence of unknown inputs which are bounded by unknown functions which may be non-uniformly bounded. By using a global version of a second-order sliding-mode control algorithm, the global boundedness assumptions on the unknown inputs, often made in the related literature, are dispensed with. The suggested observer design ensures the insensitivity of the observer with respect to the unknown inputs, and, furthermore, under an additional smoothness condition on the unknown inputs, it also offers the possibility of reconstructing them exactly, and in finite time.

Appendix I
Global suboptimal real-time differentiator [29]

Say $x(t)$ some scalar signal to be differentiated. Let a known bounded function be available such that

$$|\ddot{x}| \leq \Phi_1(t) \tag{37}$$

The considered system is described by the simple model

$$\dot{\theta}_1 = \theta_2, \quad \dot{\theta}_2 = u, \quad \hat{\dot{x}} = \theta_2 \tag{38}$$

where $\theta_1, \theta_2, u \in R$ and u is an observer control signal to be specified. If the available sliding quantity $\gamma_1 = \theta_1 - x$ and its unmeasurable derivative $\gamma_2 = \theta_2 - \dot{x}$ are steered to zero in a finite time $\overline{T}$, then it turns out that

$$\theta_1(t) = x(t), \quad \theta_2(t) = \dot{x}(t) \quad t \geq \overline{T} \tag{39}$$

Thus, the differentiator design reduces to the problem of finding a control signal u providing for the finite-time stabilization of the following second-order uncertain system

$$\dot{\gamma}_1 = \gamma_2, \quad \dot{\gamma}_2 = -\ddot{x} + u(t) \tag{40}$$

where γ_2 is not available and $\ddot{x}$ is uncertain. The solution strictly depends on the assumed conditions regarding $\ddot{x}$. Consider now the finite-time stabilization problem for system (40) under the general assumption (37). The GSO should be considered as the solution of this control problem. The differentiator here discussed thus represents only a specific application of the GSO algorithm.

Consider the next Lemma 1.

Lemma 1 ([29]): Consider a measurable quantity $x(t)$, satisfying condition (37), and system (38). Let $\gamma_1 = \theta_1 - x$ and $\gamma_2 = \dot{\gamma}_1$. In order to guarantee that, for some finite $\overline{T} > 0$, the following condition holds

$$\theta_2(t) = \dot{x}(t) \qquad t \geq \overline{T} \tag{41}$$

it is sufficient to apply the control signal

$$u(t) = \begin{cases} -[\Phi_1(t)+\chi]\,\mathrm{sign}\,(\gamma_1(t)-\gamma_1(t_0)), & t_0 \leq t \leq t_{M_1} \\ -[\Phi_1(t)+\chi]\,\mathrm{sign}\,(\gamma_1(t_{M_i})), & t_{M_i} < t \leq t_{c_i} \\ \left[\Phi_1(t)+\frac{1}{3}+\chi\right]\mathrm{sign}\,(\gamma_1(t_{M_i})), & t_{c_i} < t \leq t_{M_{i+1}} \end{cases} \tag{42}$$

where χ is a positive arbitrary constant customarily set to $\chi = 1$, t_{M_i} $(i = 1, 2, \ldots)$, is the sequence of time instants at which $\gamma_2(t_{M_i}) = 0$, and t_{c_i} $(i = 1, 2, \ldots)$ is the first time instant subsequent t_{M_i} at which one of the following relationships is verified

$$\gamma_1(t_{c_i}) = \frac{1}{2}\gamma_1(t_{M_i}), \quad \overline{\gamma_2}(t_{c_i}) = \sqrt{|\gamma_1(t_{M_i})|}$$

where $\overline{\gamma}_2(t)$ is defined for $t \geq t_{M_1}$ as

$$\begin{aligned} &\overline{\gamma}_2(t_{M_i}) = 0 \quad i = 1, 2, \ldots \\ &\dot{\overline{\gamma}}_2(t) = \begin{cases} 2\Phi_1(x,t) + \chi & t_{M_i} \leq t \leq t_{c_i} \\ 0 & t_{c_i} < t < t_{M_{i+1}} \end{cases} \end{aligned} \tag{43}$$

Proof: See [29]. ■

Remark 1: The control law $u(t)$ in the Lemma 1 is fully determined by the knowledge of γ_1 and $\Phi_1(t)$. Thus it is convenient to express it using the compact notation

$$u(t) = GSO(\gamma_1; \Phi_1(t)) \tag{44}$$

As shown in the Lemma, when the GSO algorithm is applied by using the sliding quantity γ_1 constructed as above it implements a real time differentiator, in the sense that the integral of the discontinuous control signal $u(t)$ converges in finite time to the derivative of $x(t)$.

Appendix II
Proof of Lemma 2

Since $\tilde{A}$ is Hurwitz, for any $Q = Q^T > 0$, there exists a matrix solution $P = P^T > 0$ of the Lyapunov equation $\tilde{A}^T P + P\tilde{A} = -Q$. Select the Lyapunov function as $V = e^T P e$. Its time derivative is $\dot{V} = -e^T Q e + 2e^T P D w$. Thus, it can be written that

$$\dot{V} \leq -e^T Q e + 2\,\|P\|\,\|D\|\,\|w\|\,\|e\| \tag{45}$$

Using the inequality $2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$, which is valid for any real numbers a, b, ϵ with $\epsilon > 0$, it can be written that $2\,\|w\|\,\|e\| \leq \frac{1}{\epsilon}\,\|w\|^2 + \epsilon\,\|e\|^2$. Hence, it derives that

$$\dot{V} \leq -\frac{(\lambda_{\min}(Q) - \epsilon\,\|P\|\,\|D\|)}{\lambda_{\max}(P)}V + \frac{1}{\epsilon}\,\|P\|\,\|D\|\,\|w\|^2 \tag{46}$$

Thus, the magnitude of the Lyapunov function can be overestimated by considering the maximal solution of (46):

$$V(t) \leq \exp^{-\delta t} V(0) + \frac{1}{\epsilon} \|P\| \|D\| \int_0^t \exp^{-\delta(t-\tau)} \|w(\tau)\|^2 d\tau \tag{47}$$

with δ given in (7). Therefore, we get the following bound for the error vector

$$\|e(t)\|^2 \leq \exp^{-\delta t} \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|e(0)\|^2 + \frac{\|P\| \|D\|}{\epsilon \lambda_{\min}(P)} \int_0^t \exp^{-\delta(t-\tau)} \|w(\tau)\|^2 d\tau$$

Finally, by the assumption A2, we have that

$$\|e(t)\| \leq \gamma \exp^{-\frac{\delta}{2} t} \|e(0)\| + \mu \left(\int_0^t \exp^{-\delta(t-\tau)} F^2(\|y\|) d\tau \right)^{\frac{1}{2}}$$

where $\gamma = \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \right)^{1/2}$. Let $T_1 < \infty$ be the time when $\gamma \exp^{-(\delta/2) T_1} \|e(0)\| < b$. Thus it directly follow the condition (8) which proves the Lemma.

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