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Practical stability of time delay systems: LMI's approach

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Abstract— **In this paper we discuss the practical stability of a class of time delay systems with an unstable equilibrium. We obtain sufficient conditions based on Lyapunov-Krasovskii functionals stated in terms of the feasibility of LMI's. A "Practical" exponential estimate of the solution is also obtained. The result is extended to the practical stabilization and to the case of systems with multiple delays.**

I. INTRODUCTION

In practice, systems usually exhibit nonlinear characteristics and are subject to various forms of disturbances. As a consequence, the uncertainty in the model needs to be taken into account when designing a model-based feedback controller for the process. Furthermore, there is often an interval of time between the application of a stimulus and the system's response, hence problems due to the presence of time delays must also be addressed [4].

The stability and stabilization of systems, with or without delays, is one of the main research topics for the control community. The theoretical definitions of asymptotic stability and of stability in the sense of Lyapunov used in most contributions is too restrictive when considering problems in the real world: a system might be stable or asymptotically stable in theory, however it is actually unstable in practice because the stable domain or the domain of the desired attractor is not large enough. Sometimes the state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state so that its performance is acceptable. It is clear that another notion of stability that is more suitable than Lyapunov stability is needed in such situations. The definition of Practical Stability introduced in La Salle et al. [9] and Lakshmikantham et al. [10] provides indeed a significant performance specification from the engineering point of view. Often, the practical stability is refereed to as ultimate roundedness with a fixed bound.

In the above mentioned publications the definitions and the analysis is restricted to delay free systems. Practical stability conditions for time delay systems were derived in a number of contributions: in Ghunyu Yang et al. [4] Lyapunov-Krasovskii functionals are employed to study a class of time delay systems of neutral type, in Qingling Zhang Chunyu et al. [1] the comparison principle is used for a class of descriptor systems. An analysis based on the fundamental matrix of the system is employed for linear systems with a single delay [2] and for a class of linear non-autonomous systems [3]. There, sufficient practical stability conditions are obtained.

In this paper, following the ideas presented in the work of Poznyak [7] on the inequality the Lyapunov function associated to a system should satisfy for achieving practical stability, we propose a new approach for the analysis of the case of time delay systems. First, the definition of practical stability for a general class of time delay systems is given. Second, sufficient practical stability conditions in the Lyapunov-Krasovskii stability framework are obtained [5]: the characterization is given in terms of feasible LMI conditions derived from the Lyapunov-Krasovskii functional proposed in Mondié and Kharitonov [8]. Third, we extend straightforwardly this result to the practical stabilization and to the case of systems with multiple delays. It should be mentioned that because of the type of functional we use, the feasibility of the LMI's implies a "practical" exponential convergence of the solutions of the time delay system. Finally, a mechanical system with self exited oscillation validates our approach.

II. PREVIOUS DEFINITIONS AND RESULTS

We consider time delay systems of the form

$$\begin{aligned} \dot{x}(t) &= f(t, x_t) \\ x(\theta) &= \varphi(\theta), \quad \theta \in [-h, 0] \end{aligned} \tag{1}$$

where $x(t,\varphi)$ is the solution of the system with initial function φ, $x_t(\varphi)$: segment $\{x(t+\theta,\varphi) \mid \theta \in [-h,0]\}$ and φ is a continuous functions in the Banach space $\mathfrak{C} := C([-h,0],\mathbb{R}^n)$ with norm $\|\varphi\|_h := \max_{\theta\in[-h,0]} \|\varphi(\theta)\|$.

Definition 1: (Practical stability of time delay systems) The system (1) is said to be μ-practically stable if for some $\mu > 0$ there exists $T = T(\mu,\varphi)$ such that $\|x(t)\| \leq \mu$ for $t \geq T$.

Lemma 2: Let a time delay system of the form (1) be given. If there exists a functional $v(x_t)$ such that

$$\alpha_1 \|x(t)\|^2 \leq v(x_t) \leq \alpha_2 \|x_t\|_h^2 \tag{2}$$

and that

$$\frac{d}{dt} v(x_t) \leq -2\sigma v(x_t) + \kappa \sqrt{v(x_t)} \tag{3}$$

for positive constants α_1, α_2, σ and κ, then for a given initial condition φ, the solution of the system (1) satisfies:

$$\|x(t)\| \leq \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}} e^{-\sigma t} \|\varphi\|_h + \frac{\kappa}{2\sigma\sqrt{\alpha_1}} (1 - e^{-\sigma t}). \tag{4}$$

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Furthermore, the system (1) is μ-practically stable with $\mu > \frac{\kappa}{2\sigma\sqrt{\alpha_1}}$, and

$$T = \begin{cases} 0, & \text{if } \|\varphi\|_h \leq \frac{k}{2\sigma\sqrt{\alpha_2}}; \\ \frac{1}{\sigma}\ln\left(\frac{2\sigma\sqrt{\alpha_2}\|\varphi\|_h-\kappa}{2\sigma\sqrt{\alpha_1}\mu-\kappa}\right), & \text{elsewhere.} \end{cases} \quad (5)$$

Proof: Premultiplication of (3) by $e^{2\sigma\theta}$ yields

$$\frac{\frac{d}{d\theta}\left(e^{2\sigma\theta}v(x_\theta)\right)}{\sqrt{e^{2\sigma\theta}v(x_\theta)}} \leq \kappa e^{\sigma\theta}.$$

Integration from 0 to t gives

$$\int_0^t \frac{\frac{d}{d\theta}\left(e^{2\sigma\theta}v(x_\theta)\right)}{\sqrt{e^{2\sigma\theta}v(x_\theta)}}d\theta \leq \int_0^t \kappa e^{\sigma\theta}d\theta,$$

equivalently,

$$\sqrt{e^{2\sigma t}v(x_t)} - \sqrt{v(\varphi)} \leq \frac{\kappa}{2\sigma}(e^{\sigma t}-1)$$

hence

$$e^{\sigma t}\sqrt{v(x_t)} \leq \frac{\kappa}{2\sigma}(e^{\sigma t}-1) + \sqrt{v(\varphi)}$$

or

$$\sqrt{v(x_t)} \leq \frac{\kappa}{2\sigma}(1-e^{-\sigma t}) + e^{-\sigma t}\sqrt{v(\varphi)}.$$

Now, it follows from (2) that

$$\|x(t)\| \leq \frac{\kappa}{2\sigma\sqrt{\alpha_1}} + e^{-\sigma t}\left(\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\|\varphi\|_h - \frac{\kappa}{2\sigma\sqrt{\alpha_1}}\right).$$

Observe that for an initial conditions φ such that $\|\varphi\|_h \leq \frac{\kappa}{2\sigma\sqrt{\alpha_2}}$ we have that

$$\|x(t)\| \leq \frac{\kappa}{2\sigma\sqrt{\alpha_1}}, \ \forall\ t \geq 0.$$

For an initial conditions φ such that $\|\varphi\|_h > \frac{\kappa}{2\sigma\sqrt{\alpha_2}}$ it follows that

$$\|x(t)\| \leq \mu \ \forall\ t \geq T(\mu,\varphi)$$

where $\mu > \frac{\kappa}{2\sigma\sqrt{\alpha_1}}$ and the time $T(\mu,\varphi)$ is obtained from the condition

$$0 < e^{-\sigma t}\left(\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\|\varphi\|_h - \frac{\kappa}{2\sigma\sqrt{\alpha_1}}\right) \leq \mu - \frac{\kappa}{2\sigma\sqrt{\alpha_1}}$$

hence,

$$T \geq \frac{1}{\sigma}\ln\left(\frac{2\sigma\sqrt{\alpha_2}\|\varphi\|_h-\kappa}{2\sigma\sqrt{\alpha_1}\mu-\kappa}\right).$$

We observe that if $\|\varphi\|_h > \frac{\kappa}{2\sigma\sqrt{\alpha_2}}$ and $\mu > \frac{\kappa}{2\sigma\sqrt{\alpha_1}}$, then $2\sigma\sqrt{\alpha_2}\|\varphi\|_h - \kappa > 0$, $2\sigma\sqrt{\alpha_1}\mu - \kappa > 0$, and $2\sigma\sqrt{\alpha_2}\|\varphi\|_h - \kappa > 2\sigma\sqrt{\alpha_1}\mu - \kappa$ hence T exists. ■

III. Systems with a single delay

In this section, we consider the system

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + n(t) \quad (6)$$

where $A_0,\ A_1 \in R^{n\times n}$, h$\geq$ 0 is the time delay, $n(t)$ is an external signal such that $\|n(t)\| \leq \gamma,\ t \geq 0$ and the initial condition is $\varphi(\theta),\ \theta \in [-h,0]$.

Theorem 3: If there exist positive definite matrices $P,Q \in R^{n\times n}$ and a positive constant σ such that the inequality

$$\mathscr{M}(P,Q) + 2\sigma\mathscr{N}(P) < 0 \quad (7)$$

holds, where

$$\mathscr{M}(P,Q) = \begin{bmatrix} PA_0 + A_0^TP + Q & PA_1 \\ A_1^TP & -e^{-2\sigma h}Q \end{bmatrix}, \quad (8)$$

$$\mathscr{N}(P) = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \quad (9)$$

then the system (6) is μ-practically stable with

$$\mu > \frac{\gamma\lambda_{\max}(P)}{\sigma\lambda_{\min}(P)}, \quad (10)$$

and

$$T = \begin{cases} 0, & \text{if } \ \|\varphi\|_h \leq \frac{\gamma\lambda_{max}(P)}{\sigma\sqrt{\lambda_{min}(P)[\lambda_{max}(P)+h\lambda_{max}(Q)]}} \\ \frac{1}{\sigma}\ln\left(\frac{\sigma\sqrt{\lambda_{min}(P)[\lambda_{max}(P)+h\lambda_{max}(Q)]}\|\varphi\|_h-\gamma\lambda_{max}(P)}{\sigma\mu\lambda_{min}(P)-\gamma\lambda_{max}(P)}\right), & \text{elsewhere.} \end{cases} \quad (11)$$

Furthermore,

$$\|x(t)\| \leq \frac{\sqrt{\lambda_{\max}(P)+h\lambda_{\max}(Q)}}{\sqrt{\lambda_{\min}(P)}}e^{-\sigma t}\|\varphi\|_h + \frac{\gamma\lambda_{max}(P)}{\sigma\lambda_{min}(P)}(1-e^{-\sigma t}). \quad (12)$$

Proof: Consider the Lyapunov-Krasovskii functional

$$v(x_t) = x^T(t)Px(t) + \int_{-h}^0 x^T(t+\theta)e^{2\sigma\theta}Qx(t+\theta)d\theta \quad (13)$$

where P and Q are the positive definite matrices of Theorem 3.
First, it is straightforward to verify that the functional (13) satisfies

$$\alpha_1\|x(t)\|^2 \leq v(x_t) \leq \alpha_2\|x_t\|_h^2, \quad (14)$$

where

$$\alpha_1 = \lambda_{\min}(P),$$
$$\alpha_2 = \lambda_{\max}(P) + h\lambda_{\max}(Q).$$

Next, the time derivative of $v(x_t)$ along the trajectories of system (6) is

$$\begin{aligned}\frac{d}{dt}v(x_t) = {} & 2x^T(t)P[A_0x(t) + A_1x(t-h) + n(t)] + \\ & +x^T(t)Qx(t) - x^T(t-h)e^{-2\sigma h}Qx(t-h) \\ & -2\sigma\int_{-h}^0 x^T(t+\theta)e^{2\sigma\theta}Qx(t+\theta)d\theta.\end{aligned}$$

The majorization of the term $2x^T(t)Pn(t)$ gives

$$2x^T(t)Pn(t) \leq 2\|x(t)\|\,\|P\|\,\|n(t)\|,$$

now, it follows from first inequality in (14) that $\|x(t)\| \leq \frac{1}{\sqrt{\alpha_1}}\sqrt{v(x_t)}$. Therefore $\|n(t)\| \leq \gamma$ implies

$$2x^T(t)Pn(t) \leq 2\frac{\gamma\|P\|}{\sqrt{\alpha_1}}\sqrt{v(x_t)}.$$

Thus, we have that

$$\begin{aligned}\frac{d}{dt}v(x_t) \leq & \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \mathscr{M}(P,Q) \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \\ & -2\sigma\int_{-h}^{0} x^T(t+\theta)e^{2\sigma\theta}Qx(t+\theta)d\theta \\ & +2\frac{\gamma\|P\|}{\sqrt{\alpha_1}}\sqrt{v(x_t)}.\end{aligned}$$

Observe that the functional (13) can be rewritten as

$$\begin{aligned}v(x_t) &= \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \\ & + \int_{-h}^{0} x^T(t+\theta)e^{2\sigma\theta}Qx(t+\theta)d\theta.\end{aligned}$$

Clearly, we have that

$$\frac{d}{dt}v(x_t) + 2\sigma v(x_t) - 2\frac{\gamma\|P\|}{\sqrt{\alpha_1}}\sqrt{v(x_t)} \leq$$
$$\begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \{\mathscr{M}(P,Q) + 2\sigma\mathscr{N}(P)\} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

with $\mathscr{M}(P,Q)$ and $\mathscr{N}(P)$ respectively given by (8) and (9) and we can conclude that if the condition (7) holds, then

$$\frac{d}{dt}v(x_t) \leq -2\sigma v(x_t) + 2\frac{\gamma\|P\|}{\sqrt{\alpha_1}}\sqrt{v(x_t)}.$$

Clearly, $v(x_t)$ satisfies the conditions of Lemma 2 and it follows that the system is μ-practically stable with

$$\mu > \frac{2\frac{\gamma\|P\|}{\sqrt{\alpha_1}}}{2\sigma\sqrt{\alpha_1}} = \frac{\gamma\|P\|}{\sigma\alpha_1} = \frac{\gamma\lambda_{\max}(P)}{\sigma\lambda_{\min}(P)},$$

and

$$T = \frac{1}{\sigma}\ln\left(\frac{\sigma\sqrt{\lambda_{min}(P)[\lambda_{max}(P) + h\lambda_{max}(Q)]}\|\varphi\|_h - \gamma\lambda_{max}(P)}{\sigma\mu\lambda_{min}(P) - \gamma\lambda_{max}(P)}\right).$$

Now, it follows from (4) that the solution of system (6) satisfies:

$$\begin{aligned}\|x(t)\| \leq & \frac{\sqrt{\lambda_{\max}(P) + h\lambda_{\max}(Q)}}{\sqrt{\lambda_{\min}(P)}}e^{-\sigma t}\|\varphi\|_h \\ & + \frac{\gamma\lambda_{max}(P)}{\sigma\lambda_{min}(P)}(1 - e^{-\sigma t}),\ \forall\ t \geq 0.\end{aligned}$$

■

It is straightforward to extend this result to the practical stabilization of a class of non autonomous systems with delay.

Corollary 4: Consider a system of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + Bu(t) + n(t) \tag{15}$$

where A_0, A_1, $B \in R^{n\times n}$, $h \geq 0$ is the time delay, $n(t)$ is an external signal such that $\|n(t)\| \leq \gamma$, $t \geq 0$. If there exist positive definite matrices Q_0, $Q_1 \in R^{n\times n}$, a positive constant σ and a matrix $Y \in R^{n\times n}$ such that

$$\mathscr{M}(Q_0,Q_1) + 2\sigma\mathscr{N}(Q_0) < 0 \tag{16}$$

holds, where

$$\mathscr{M}(Q_0,Q_1) = \begin{bmatrix} A_0Q_0 + BY + Q_0A_0^T + Y^TB + Q_1 & A_1Q_0 \\ Q_0A_1^T & -e^{-2\sigma h}Q_1 \end{bmatrix},$$
$$\mathscr{N}(Q_0) = \begin{bmatrix} Q_0 & 0 \\ 0 & 0 \end{bmatrix},$$

then the feedback control law

$$u(t) = YQ_0^{-1}x(t) \tag{17}$$

μ-practically stabilizes the system (15) with

$$\mu > \frac{\gamma\lambda_{\max}(Q_0^{-1})}{\sigma\lambda_{\min}(Q_0^{-1})}, \tag{18}$$

and

$$T = \begin{cases} 0, & \text{if } \|\varphi\|_h \leq c_1 \\ \frac{1}{\sigma}\ln(c_2), & \text{elsewhere,} \end{cases} \tag{19}$$

where

$$c_1 = \frac{\gamma\lambda_{max}(Q_0^{-1})}{\sigma\sqrt{\lambda_{min}(Q_0^{-1})[\lambda_{max}(Q_0^{-1}) + h\lambda_{max}(Q_0^{-1}Q_1Q_0^{-1})]}}$$

and

$$c_2 = \frac{\sigma\sqrt{\lambda_{min}(Q_0^{-1})[\lambda_{max}(Q_0^{-1}) + h\lambda_{max}(Q_0^{-1}Q_1Q_0^{-1})]}\|\varphi\|_h - \gamma\lambda_{max}(Q_0^{-1})}{\sigma\mu\lambda_{min}(Q_0^{-1}) - \gamma\lambda_{max}(Q_0^{-1})}.$$

Furthermore,

$$\begin{aligned}\|x(t)\| \leq & \frac{\sqrt{\lambda_{\max}(Q_0^{-1}) + h\lambda_{\max}(Q_0^{-1}Q_1Q_0^{-1})}}{\sqrt{\lambda_{\min}(Q_0^{-1})}}e^{-\sigma t}\|\varphi\|_h \\ & + \frac{\gamma\lambda_{max}(Q_0^{-1})}{\sigma\lambda_{min}(Q_0^{-1})}(1 - e^{-\sigma t}),\ \forall\ t \geq 0.\end{aligned} \tag{20}$$

Proof: We consider a feedback control law of the form $u(t) = Kx(t)$, $K \in R^{m\times n}$. The closed-loop system is:

$$\dot{x}(t) = (A_0 + BK)x(t) + A_1x(t-h) + n(t).$$

Using the result of Theorem 3 we have that if there exist positive definite matrices P, $Q \in R^{n\times n}$ and a positive constant σ such that

$$\mathscr{M}(P,Q) + 2\sigma\mathscr{N}(P) < 0 \tag{21}$$

holds, where

$$\mathscr{M}(P,Q) = \begin{bmatrix} P(A_0 + BK) + (A_0 + BK)^TP + Q & PA_1 \\ A_1^TP & -e^{-2\sigma h}Q \end{bmatrix},$$
$$\mathscr{N}(P) = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix},$$

then the closed loop system is μ-practically stable. Left and right multiplication of (21) by matrix $\text{diag}\{P^{-1},P^{-1}\}$ and setting $Q_0 = P^{-1}$, $Q_1 = P^{-1}QP^{-1}$ and $Y = KP^{-1}$ we have

$$\begin{pmatrix} A_0Q_0+Q_0A_0^T+BY+Y^TB+Q_1 & A_1Q_0 \\ Q_0A_1^T & -e^{-2\sigma h}Q_1 \end{pmatrix} + 2\sigma \begin{bmatrix} Q_0 & 0 \\ 0 & 0 \end{bmatrix} < 0.$$

Moreover,

$$u(t) = YQ_0^{-1}x(t)$$

and (18), (19) and (20) follow from (10), (11) and (12), respectively. ■

IV. Case of Multiple Delays

The results of the previous section can be extended to the case of systems with multiple delays.

Consider the system

$$\begin{array}{l} \dot{x}(t) = \sum_{i=0}^{m} A_i x(t-h_i) + n(t), \\ x(\theta) = \phi(\theta), \quad \theta \in [-H,0], \end{array} \tag{22}$$

where $A_i \in \mathbb{R}^{n\times n}$, $i = 0,1,\dots,m$, $0 = h_0 < h_1 < \cdots < h_m = H$ are delays, and $n(t)$ is an external signal such that $\|n(t)\| \leq \gamma,\ t \geq 0$.

We would like to have an analogous result to Theorem 3 for the system described by (22).

Theorem 5: Consider the linear time delay system of the form (22). If there exist positive definite matrices $P, Q_i \in R^{n\times n}$, $i = 1,\dots,m$ and a positive constant σ such that the inequality

$$\mathscr{M}(P,Q_1,\dots,Q_m) + 2\sigma\mathscr{N}(P) < 0 \tag{23}$$

holds, where

$$\mathscr{M}(P,Q_1,\dots,Q_m) = \text{diag}\Big\{\sum_{i=0}^{m} Q_i, -e^{-2\sigma,h_1}Q_1, \dots, -e^{-2\sigma h_m}Q_m\Big\} + A^TPE + E^TPA,$$

$$\mathscr{N}(P) = \text{diag}\{P, 0_n, \dots, 0_n\},$$

with $A = [A_0\ A_1\ \cdots\ A_m]$ and $E = [I_n\ 0_n\ \cdots\ 0_n]$. Then the system is μ-practically stable with

$$\mu > \frac{\gamma\lambda_{\max}(P)}{\sigma\lambda_{\min}(P)}. \tag{24}$$

and

$$T = \begin{cases} 0, & \text{if } \|\varphi\|_H \leq \frac{\gamma\lambda_{max}(P)}{\sigma\sqrt{\lambda_{\min}(P)[\lambda_{\max}(P)+\sum_{i=1}^m h_i\lambda_{\max}(Q_i)]}}; \\ \frac{1}{\sigma}\ln\left(\frac{\sigma\sqrt{\lambda_{\min}(P)[\lambda_{\max}(P)+\sum_{i=1}^m h_i\lambda_{\max}(Q_i)]}\|\varphi\|_H - \gamma\lambda_{max}(P)}{\sigma\lambda_{\min}(P)\mu - \gamma\lambda_{max}(P)}\right), & \text{elsewhere.} \end{cases} \tag{25}$$

Furthermore,

$$\|x(t)\| \leq \frac{\sqrt{\lambda_{\max}(P)+\sum_{i=1}^m h_i\lambda_{\max}(Q_i)}}{\sqrt{\lambda_{\min}(P)}} e^{-\sigma t}\|\varphi\|_H + \frac{\gamma\lambda_{\max}(P)}{\sigma\lambda_{\min}(P)}(1-e^{-\sigma t}). \tag{26}$$

Proof: The result follows by using the same arguments of the proof of Theorem 3 with the Lyapunov-Krasovskii functional

$$v(x_t) = x^T(t)Px(t) + \sum_{i=1}^{m}\int_{-h_i}^{0} x^T(t+\theta)e^{2\sigma\theta}Q_i x(t+\theta)d\theta \tag{27}$$

and the following inequalities

$$\|x(t)\| \leq \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}e^{-\sigma t}\|\varphi\|_h + \frac{\kappa}{2\sigma\sqrt{\alpha_1}}(1-e^{-\sigma t}). \tag{28}$$

where $\alpha_1 = \lambda_{\min}(P)$, and $\alpha_2 = \lambda_{\max}(P) + \sum_{i=1}^m h_i\lambda_{\max}(Q_i)$. ■

Corollary 6: Consider the linear time delay system of the form

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t-h_i) + Bu(t) + n(t), \tag{29}$$

where B, $A_i \in \mathbb{R}^{n\times n}$, $i = 0,1,\dots,m$, $0 = h_0 < h_1 < \cdots < h_m = H$ are delay, and $n(t)$ is an external signal such that $\|n(t)\| \leq \gamma, t \geq 0$. If there exist positive definite matrices $R_i \in R^{n\times n}$, $i = 0,\dots,m$, a positive constant σ and a matrix $Y \in R^{n\times n}$ such that

$$\mathscr{M}(R_0,R_1,\dots,R_m) + 2\sigma\mathscr{N}(R_0) < 0 \tag{30}$$

holds, where

$$\mathscr{M}(R_0,R_1,\dots,R_m) = \text{diag}\Big\{\sum_{i=0}^{m} R_i + Y^TB + BY, -e^{-2\sigma h_1}R_1, \dots, -e^{-2\sigma h_m}R_m\Big\} + RA^TE + E^TAR,$$

$$\mathscr{N}(R_0) = \text{diag}\{R_0, 0_n, \dots, 0_n\},$$

with $A = [A_0\ A_1\ \cdots\ A_m]$, $E = [I_n\ 0_n\ \cdots\ 0_n]$ and $R = \text{diag}\{R_0,\dots,R_0\}$, then the feedback control law

$$u(t) = YR_0^{-1}x(t) \tag{31}$$

μ-practically stabilizes the system (29) with

$$\mu > \frac{\gamma\lambda_{\max}(R_0^{-1})}{\sigma\lambda_{\min}(R_0^{-1})}, \tag{32}$$

and

$$T = \begin{cases} 0, & \text{if } \|\varphi\|_H \leq c_1 \\ \frac{1}{\sigma}\ln(c_2), & \text{elsewhere,} \end{cases} \tag{33}$$

where

$$c_1 = \frac{\gamma\lambda_{max}(R_0^{-1})}{\sigma\sqrt{\lambda_{\min}(R_0^{-1})[\lambda_{\max}(R_0^{-1}) + \sum_{i=1}^m h_i\lambda_{\max}(R_0^{-1}R_iR_0^{-1})]}}$$

and

$$c_2 = \frac{\sigma\sqrt{\lambda_{\min}(R_0^{-1})[\lambda_{\max}(R_0^{-1})+\sum_{i=1}^m h_i\lambda_{\max}(R_0^{-1}R_iR_0^{-1})]}\|\varphi\|_H - \gamma\lambda_{max}(R_0^{-1})}{\sigma\lambda_{\min}(R_0^{-1})\mu - \gamma\lambda_{max}(R_0^{-1})}.$$

Furthermore,

$$\|x(t)\| \leq \frac{\sqrt{\lambda_{\max}(R_0^{-1}) + \sum_{i=1}^m h_i\lambda_{\max}(R_0^{-1}R_iR_0^{-1})}}{\sqrt{\lambda_{\min}(R_0^{-1})}}e^{-\sigma t}\|\varphi\|_h + \frac{\gamma\lambda_{\max}(R_0^{-1})}{\sigma\lambda_{\min}(R_0^{-1})}(1-e^{-\sigma t}). \tag{34}$$

V. EXAMPLE

In this section we present two examples of practical stability analysis. In orden to guarantee a small $\lambda_{\max}(P)/\lambda_{\min}(P)$, where unpack in the conservativeness of the results is evident, an additional restriction $\alpha I < P < \beta I$, β/α is introduced.

Example 7: Consider the model of a mechanical system with self-excited oscillations a possessing a retarded action [6]

$$\ddot{\theta}(t)+2\delta\omega\dot{\theta}(t)+\rho\dot{\theta}(t-h)+\omega^2\theta(t)=M\sin kt, \quad (35)$$

where δ, ρ and ω are well-known physical constants, θ is the angle of oscillation, $\dot{\theta}(t-h)$ is the retarded velocity, h is the time lag which we shall assume to be constant, and $M\sin kt$ is the external moment.

By introducing the new variable $x(t)=[x_1(t)\ x_2(t)]^T=[\theta(t)\ \dot{\theta}(t)]^T$, system (35) can be rewritten as:

$$\dot{x}(t)=\begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\delta\omega \end{pmatrix}x(t)+\begin{pmatrix} 0 & 0 \\ \rho & 0 \end{pmatrix}x(t-h) + \begin{pmatrix} 0 \\ M\sin kt \end{pmatrix}. \quad (36)$$

For $\omega=$3.1321, $\delta=$1.6762, $\rho=$0.32, $h=0.5$, $M=$0.1, and $k=$1 we obtain $\|n(t)\|\le 0.1$ and the LMI conditions (7) is feasible for $\sigma=0.94$ and

$$P=\begin{pmatrix} 0.4917 & 0.1833 \\ 0.1833 & 0.1983 \end{pmatrix}, \quad Q=\begin{pmatrix} 1.1160 & 1.2871 \\ 1.2871 & 1.4942 \end{pmatrix}.$$

By Theorem 3, we derive from (12) that the solution of system (36) satisfies:

$$\|x(t)\|\le 4.134e^{-0.9t}\|\varphi\|_{0.5}+0.585(1-e^{-0.9t})$$

Furthermore, from (10) and (11) we have

$$\|x(t)\|\le 0.585, \quad \forall \quad t>10.9.$$

Now, we will consider parameter values from which the system (36) is unstable and we apply the results of Theorem 5 to obtain a control law that μ-practical stabilizes.

Example 8: Consider a system of the form

$$\dot{x}(t)=\begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\delta\omega \end{pmatrix}x(t)+\begin{pmatrix} 0 & 0 \\ \rho & 0 \end{pmatrix}x(t-h) + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}u+\begin{pmatrix} 0 \\ M\sin kt \end{pmatrix}. \quad (37)$$

First, we observe that for $\omega=$3.1321, $\delta=$0.006, $\rho=$0.9, $h=$1.5, $M=$0.1, and $k=$1 we obtain $\|n(t)\|\le 0.1$ the system (36) is unstable (see, Figure 1).
Now, applying the results of Theorem 3, we have that the LMI conditions (7) is feasible for σ=1.1, and

$$Q_0=\begin{pmatrix} 0.5482 & 0 \\ 0 & 0.55 \end{pmatrix}, \quad Q=\begin{pmatrix} 0.5126 & 0 \\ 0 & 0.5147 \end{pmatrix}$$
$$Y=\begin{pmatrix} -5.3811 & 1.207 \\ 1.207 & -5.8427 \end{pmatrix}.$$

It follows from (17) that the feedback control law

$$u(t)=\begin{pmatrix} -9.8159 & 2.1945 \\ 2.2017 & -10.6232 \end{pmatrix}x(t) \quad (38)$$

μ-practically stabilizes the system (37) with $\mu=0.0909$, and $T=5.7896$.
Furthermore, we derive from (12) that the solution of the system satisfy:

$$\|x(t)\|\le 1.4e^{-1.1t}\|\varphi\|_1+0.09(1-e^{-1.1t}), \quad \forall\ t>0.$$

The practical stabilization of the system (37) with the control law (38) is show on Figure 2. The phase diagram of the system is showed on Figure 3.

VI. CONCLUDING REMARKS

A constructive approach for the determination of the practical stability of time delay system is presented. The sufficient conditions we obtain for the stability analysis and stabilizability problem are derived by using Lyapunov-Krasovskii functionals. The characterization is given in terms of LMI conditions.

Notice that our conditions imply a "practically exponential convergence" of the solutions of the time delay system (see, Figure 2).

Our current researches include the analysis of cases that do not meet the conditions for a slider mode strategy.

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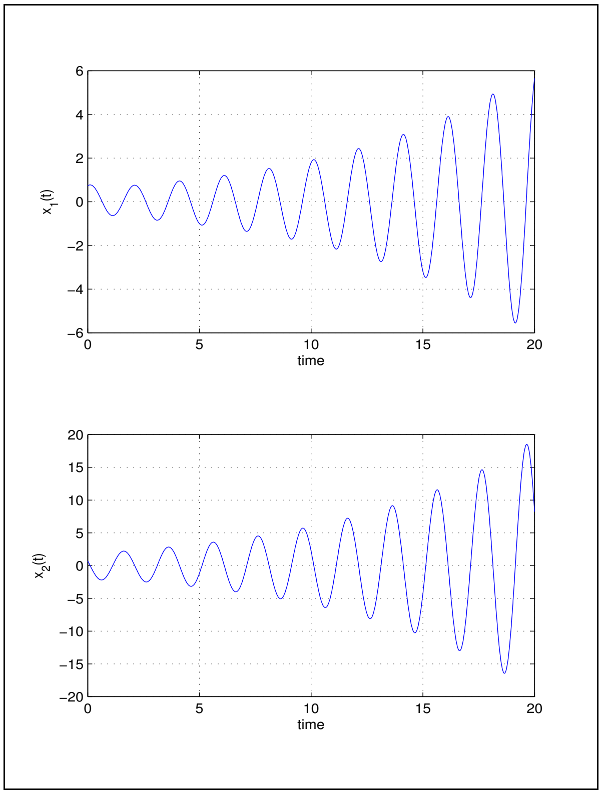

Fig. 1. Instability

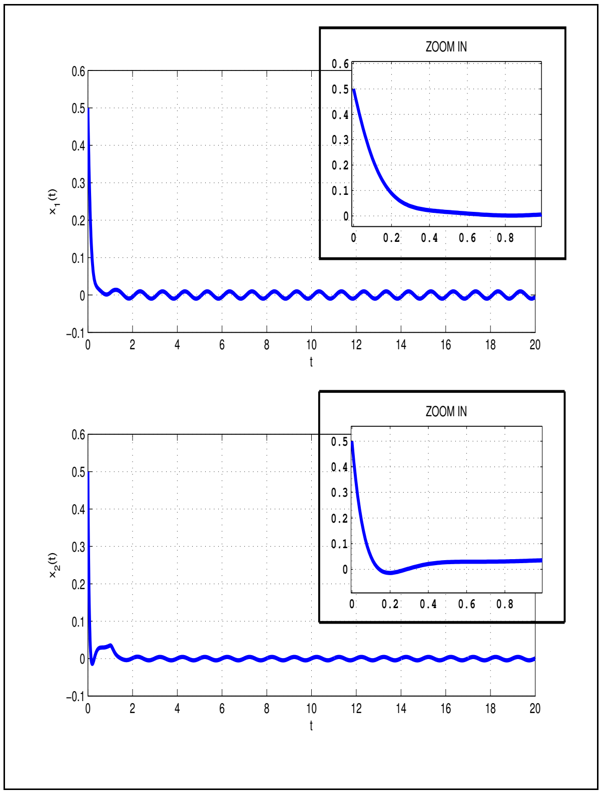

Fig. 2. μ-practical stabilization of system (37)

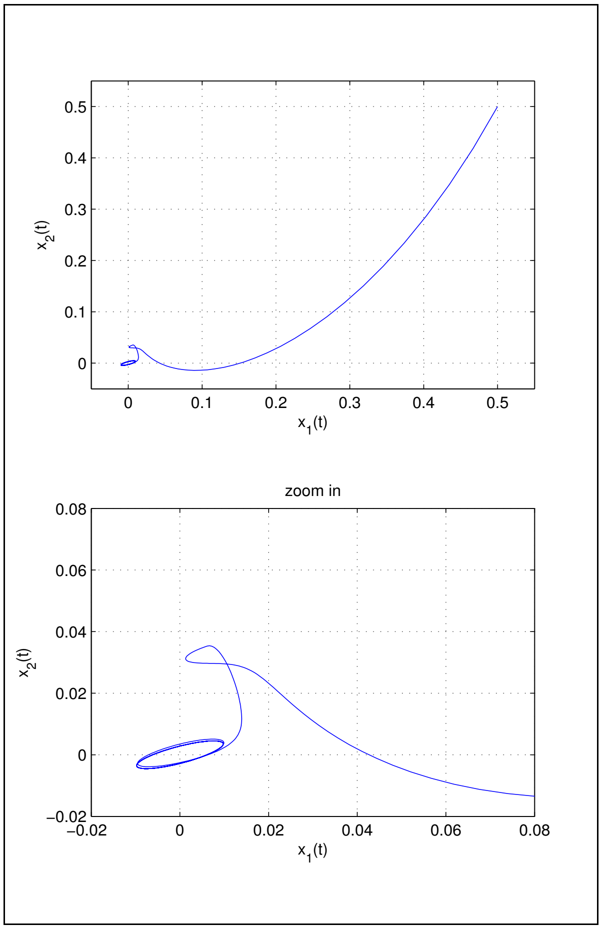

Fig. 3. Phase diagram of system (37)