Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Symbolic models for nonlinear control systems affected by disturbances

Giordano Pola and Paulo Tabuada

Abstract— **Symbolic models are abstract descriptions of continuous systems in which symbols represent aggregates of continuous states. In the last few years there has been a growing interest in the use of symbolic models as a tool for analysis and synthesis of complex systems. In fact, symbolic models enable the use of well known algorithms in the context of supervisory control and algorithmic game theory, for controller synthesis. Since the 1990s many researchers faced the problem of identifying classes of dynamical and control systems that admit symbolic models. In this paper we make further progress along this research line by focusing on control systems affected by disturbances. Our main contribution is to show that incrementally globally asymptotically stable nonlinear control systems with disturbances admit symbolic models. When specializing these results to linear systems, we show that these symbolic models can be easily constructed.**

I. INTRODUCTION

A recent trend in the control systems community is the use of symbolic models for the analysis and control of large scale systems, with the aim of mitigating their inherent complexity. A system is called a *symbolic model* when its state space and its input space are finite sets. The use of symbolic models provides a unified language to describe physical systems as well as, software and hardware and therefore it plays an important role, when dealing with design of embedded systems (see e.g. [1]). Furthermore, the use of symbolic models allows one to leverage the rich literature on supervisory control [2] and algorithmic approaches to game theory [3], [4] for control design. The search for classes of systems admitting symbolic models goes back to the 1990's. After the pioneering work [5] of Alur and Dill that showed that timed automata admit symbolic models, many researchers faced the problem of identifying more general classes of systems admitting symbolic models; these include: multirate automata, rectangular automata and o–minimal hybrid systems (see e.g. [6]). Symbolic models for control systems were considered later in the work of [7] which showed that discrete–time controllable linear systems admit symbolic models. Most of these results are based on appropriately adapting the notion of bisimulation introduced by Milner [8] and Park [9] to the context of continuous and hybrid systems. A different approach emerged recently through the work of [10], where an approximate version of bisimulation was considered. While (exact) bisimulation requires that observations of the states are identical, the notion of approximate bisimulation relaxes this condition, by allowing observations to be close and within a desired precision. This more flexible notion of bisimulation allows the identification of more classes of systems, admitting symbolic models. Indeed, the work in [11] showed that, for the class of incrementally globally asymptotically stable nonlinear control systems, symbolic models exist which are approximately bisimulation equivalent to control systems, with a precision that can be chosen a priori, as a design parameter. Control systems in this work are not affected by exogenous disturbance inputs. However, in many realistic situations, physical processes are characterized by a certain degree of uncertainty which is often modeled by additional disturbance inputs. The aim of the present paper is to extend the results of [11] to nonlinear control systems influenced by disturbances. The presence of disturbances requires us to replace the notion of approximate bisimulation used in [11] with the notion of alternating approximate bisimulation, inspired by Alur and coworkers' alternating bisimulation [12]. This novel notion of bisimulation is a critical ingredient of our results since, as illustrated in Section III-B through a simple example, symbolic models based on the notion of approximate bisimulation cannot be used for controller synthesis of systems with exogenous inputs. Alternating approximate bisimulation solves this problem by guaranteeing that control strategies synthesized on symbolic models, based on alternating approximate bisimulations, can be readily transferred to the original model. *The main contribution of this paper is to show that incrementally globally asymptotically stable control systems affected by exogenous inputs do admit symbolic models.* Moreover we show that for linear control systems, symbolic models can be easily constructed by leveraging existing results on approximation of reachable sets (see e.g. [13] and the references therein). A notion of alternating approximate simulation (one sided version of alternating approximate bisimulation) has been used in [14] to construct abstractions of linear control systems with disturbances. Notions of bisimulation for nonlinear control systems with disturbances have also been studied in [15], albeit with a different purpose. While we are interested in the construction of bisimilar models that are finite, the work in [15] uses bisimulation to relate continuous, and thus infinite, control systems. This paper is organized as follows. Section II introduces the class of control systems that we consider and some stability notions that will be used in the subsequent developments. Section III introduces alternating transition systems and the notion of alternating approximate

This work has been partially supported the National Science Foundation CAREER award 0717188 and by the Center of Excellence DEWS, University of L'Aquila, Italy.

G. Pola is with the Department of Electrical and Information Engineering, Center of Excellence DEWS, University of L'Aquila, Poggio di Roio, 67040 L'Aquila, Italy, pola@ing.univaq.it

P. Tabuada is with the Department of Electrical Engineering, University of California at Los Angeles, Los Angeles, CA 90095-1594, USA pola,tabuada@ee.ucla.edu.

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

bisimulation upon which our results rely. In Section IV we show existence of symbolic models for incrementally globally asymptotically stable nonlinear control systems and the construction of symbolic models for linear control systems. In Section V we illustrate our results by means of a simple example and Section VI offers some concluding remarks.

II. Control systems and stability notions

A. Notation

The symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of integers, positive integers, reals, positive and nonnegative reals, respectively. Given $x \in \mathbb{R}^n$ the symbol x' denotes the transpose of x and x_i the i–th element of x; furthermore $\|x\|$ denotes the infinity norm of x and $\|M\|$ the infinity norm of some matrix M. Given a set $A \subseteq \mathbb{R}^n$, the symbol $\overline{A}$ denotes the topological closure of A. The symbol $\mathcal{B}_\varepsilon(x)$ denotes the closed ball centered at $x \in \mathbb{R}^n$ with radius $\varepsilon \in \mathbb{R}_0^+$, i.e. $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^n : \|x - y\| \leq \varepsilon\}$. For any $A \subseteq \mathbb{R}^n$ and $\mu \in \mathbb{R}$ define $[A]_\mu := \{a \in A \mid a_i = k_i\mu, \ k_i \in \mathbb{Z}, i = 1, ..., n\}$. Given a metric space $(X, \mathbf{d})$, we denote by $\mathbf{d}_h$ the Hausdorff pseudo–metric induced by $\mathbf{d}$ on 2^X, i.e. for any $X_1, X_2 \subseteq X$, $\mathbf{d}_h(X_1, X_2) := \max\{\vec{\mathbf{d}}_h(X_1, X_2), \vec{\mathbf{d}}_h(X_2, X_1)\}$, where $\vec{\mathbf{d}}_h(X_1, X_2) = \sup_{x_1 \in X_1} \inf_{x_2 \in X_2} \mathbf{d}(x_1, x_2)$. We recall that $\mathbf{d}_h$ satisfies the following properties for any $X_1, X_2, X_3 \subseteq X$: (i) $X_1 = X_2$ implies $\mathbf{d}_h(X_1, X_2) = 0$; (ii) $\mathbf{d}_h(X_1, X_2) = \mathbf{d}_h(X_2, X_1)$; (iii) $\mathbf{d}_h(X_1, X_3) \leq \mathbf{d}_h(X_1, X_2) + \mathbf{d}_h(X_2, X_3)$. The identity map on a set A is denoted by 1_A. Given two sets A and B, if $A \subseteq B$ we denote by $\imath_A : A \hookrightarrow B$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. Given a function $f : A \to B$ the symbol $f(A)$ denotes the image of A through f, i.e. $f(A) := \{b \in B : \exists a \in A \text{ s.t. } b = f(a)\}$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ if and only if $(a, b) \in R$. Given $R \subseteq A \times B$, R^{-1} denotes the inverse relation of R, i.e. $R^{-1} := \{(b, a) \in B \times A : (a, b) \in R\}$.

B. Control Systems

The class of systems that we consider in this paper is formalized in the following definition.

Definition 1: A *control system* is a quadruple $\Sigma = (\mathbb{R}^n, W, \mathcal{W}, f)$, where:

- $\mathbb{R}^n$ is the state space;
- $W = U \times V$ is the input space, where $U \subseteq \mathbb{R}^m$ is the control input space and $V \subseteq \mathbb{R}^s$ is the disturbance input space;
- $\mathcal{W} = \mathcal{U} \times \mathcal{V}$ is a subset of the set of all measurable and locally essentially bounded functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to W with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times W \to \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $K \subset \mathbb{R}^n$, there exists a constant $\kappa > 0$ such that

$$\|f(x, w) - f(y, w)\| \leq \kappa \|x - y\|,$$

for all $x, y \in K$ and all $w \in W$.

A locally absolutely continuous curve $\mathbf{x} :]a, b[\to \mathbb{R}^n$ is a *trajectory* of Σ if there exists $\mathbf{w} \in \mathcal{W}$ satisfying $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{w}(t))$, for almost all $t \in]a, b[$.

Although we have defined trajectories over open domains, we shall refer to trajectories $\mathbf{x} : [0, \tau] \to \mathbb{R}^n$ defined on closed domains $[0, \tau]$, $\tau \in \mathbb{R}^+$ with the understanding of the existence of a trajectory $\mathbf{z} :]a, b[\to \mathbb{R}^n$ such that $\mathbf{x} = \mathbf{z}|_{[0,\tau]}$. We will also write $\mathbf{x}(\tau, x, \mathbf{w})$ to denote the point reached at time $\tau \in]a, b[$ under the input $\mathbf{w}$ from initial condition x; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories. Whenever we need to distinguish between $\mathbf{u}$ and $\mathbf{v}$ in an input signal $(\mathbf{u}, \mathbf{v}) \in \mathcal{W}$, we write $\mathbf{x}(\tau, x, \mathbf{u}, \mathbf{v})$ instead of $\mathbf{x}(\tau, x, (\mathbf{u}, \mathbf{v}))$. A control system Σ is *forward complete* if every trajectory is defined on an interval of the form $]a, \infty[$. The results presented in this paper will rely upon the following stability notion:

Definition 2: [16] A control system Σ is *incrementally globally asymptotically stable* (δ–GAS) if it is forward complete and there exist a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $x_1, x_2 \in \mathbb{R}^n$ and any input signal $\mathbf{w} \in \mathcal{W}$ the following condition is satisfied:

$$\|\mathbf{x}(t, x_1, \mathbf{w}) - \mathbf{x}(t, x_2, \mathbf{w})\| \leq \beta(\|x_1 - x_2\|, t). \quad (1)$$

The above definition can be thought of as an incremental version of the classical notion of global asymptotic stability (GAS). Sufficient and necessary conditions for a control system to be δ–GAS can be found in [16].

III. Symbolic models and approximate equivalence notions

A. Alternating transition systems

In this paper we will use the class of alternating transition systems as abstract models of control systems.

Definition 3: An (alternating) transition system is a tuple:

$$T = (Q, L, \longrightarrow, O, H),$$

consisting of:

- A set of states Q;
- A set of labels $L = A \times B$, where A is the set of control labels and B is the set of disturbance labels;
- A transition relation $\longrightarrow \subseteq Q \times L \times Q$;
- An output set O;
- An output function $H : Q \to O$.

A transition system T is *metric*, if the output set O is equipped with a metric $\mathbf{d} : O \times O \to \mathbb{R}_0^+$; *countable*, if Q and L are countable sets; *finite*, if Q and L are finite sets.

We will follow standard practice and denote by $q \xrightarrow{a,b} p$, a transition from q to p labeled by a and b. Transition systems capture dynamics through the transition relation. For any states $q, p \in Q$, $q \xrightarrow{a,b} p$ simply means that it is possible to evolve or jump from state q to state p under the action labeled by a and b. We will use transition systems as an abstract representation of control systems. There are several different ways in which we can transform control systems into transition systems. We now describe one of these which has the property of capturing all the information contained in a control system Σ. Given $\Sigma = (\mathbb{R}^n, U \times V, \mathcal{U} \times \mathcal{V}, f)$ define the transition system:

$$T(\Sigma) := (Q, L, \longrightarrow, O, H),$$

where $Q = \mathbb{R}^n$, $L = A \times B$ with $A = \mathcal{U}$ and $B = \mathcal{V}$, $q \xrightarrow{\mathbf{u},\mathbf{v}} p$, if $\mathbf{x}(\tau, q, \mathbf{u}, \mathbf{v}) = p$ for some $\tau \in \mathbb{R}^+$, $O = \mathbb{R}^n$, and $H = 1_{\mathbb{R}^n}$. In the subsequent developments we will work with a sub–transition system of $T(\Sigma)$ obtained by selecting those transitions from $T(\Sigma)$ describing trajectories of duration τ for some chosen $\tau \in \mathbb{R}^+$. This can be seen as a time discretization or sampling process.

Definition 4: Given a control system $\Sigma = (\mathbb{R}^n, U \times V, \mathcal{U} \times \mathcal{V}, f)$ and a parameter $\tau \in \mathbb{R}^+$ define the transition system:

$$T_\tau(\Sigma) := (Q_\tau, L_\tau, \underset{\tau}{\longrightarrow}, O_\tau, H_\tau),$$

where:

- $Q_\tau = \mathbb{R}^n$;
- $L_\tau = A_\tau \times B_\tau$ where

$$A_\tau = \{\mathbf{u} \in \mathcal{U} \mid \text{the domain of } \mathbf{u} \text{ is } [0, \tau]\},$$
$$B_\tau = \{\mathbf{v} \in \mathcal{V} \mid \text{the domain of } \mathbf{v} \text{ is } [0, \tau]\};$$

- $q \xrightarrow[\tau]{\mathbf{u},\mathbf{v}} p$, if $\mathbf{x}(\tau, q, \mathbf{u}, \mathbf{v}) = p$;
- $O_\tau = \mathbb{R}^n$;
- $H_\tau = 1_{\mathbb{R}^n}$.

Note that $T_\tau(\Sigma)$ is a metric transition system when we regard O_τ as being equipped with the metric $\mathbf{d}(p, q) = \|p - q\|$.

B. Alternating and approximate bisimulations

In this section we introduce a notion of approximate equivalence upon which all the results in this paper rely. The following definition has been introduced in [10] and in a slightly different formulation in [1].

Definition 5: Given two metric transition systems $T_i = (Q_i, L_i, \underset{i}{\longrightarrow}, O, H_i), i = 1, 2$ with the same output set and metric $\mathbf{d}$, and given a precision $\varepsilon \in \mathbb{R}_0^+$, a relation $R \subseteq Q_1 \times Q_2$ is an ε–approximate bisimulation relation between T_1 and T_2, if for any $(q_1, q_2) \in R$:

(i) $\mathbf{d}(H_1(q_1), H_2(q_2)) \leq \varepsilon$;
(ii) $q_1 \xrightarrow[1]{l_1} p_1$ implies existence of $q_2 \xrightarrow[2]{l_2} p_2$ such that $(p_1, p_2) \in R$;
(iii) $q_2 \xrightarrow[2]{l_2} p_2$ implies existence of $q_1 \xrightarrow[1]{l_1} p_1$ such that $(p_1, p_2) \in R$.

Moreover T_1 is ε–approximately bisimilar to T_2 if there exists an ε–approximate bisimulation relation R between T_1 and T_2 such that $R(Q_1) = Q_2$ and $R^{-1}(Q_2) = Q_1$.

The work in [11] showed existence of symbolic models that are approximately bisimilar to δ–GAS control systems (with no disturbance). However, the notion in Definition 5 employed in [11], does not capture the different role of control and disturbance inputs in control systems, as the following example shows.

Example 1: Consider the control system $\Sigma = (\mathbb{R}, U \times V, \mathcal{U} \times \mathcal{V}, f)$, where $U = [1, 2] \subset \mathbb{R}$, $V = [0.4, 1] \subset \mathbb{R}$, $\mathcal{U} \times \mathcal{V}$ is the class of all measurable and locally essentially bounded functions taking values in $U \times V$, and $f : \mathbb{R} \times U \times V \to \mathbb{R}$ is defined by $f(x, (u, v)) = -2x + uv$. We work in the compact state space $X = [0, 2]$. Consider the transition system $T = (Q, L, \longrightarrow, O, H)$, where $Q = \{q_1, q_2, q_3\}$, $L = \{l_1, l_2, l_3\}$, $q \xrightarrow{l} p$ is depicted in Figure 1, $O = \mathbb{R}$, $H : O \to \mathbb{R}$ is defined by $H(q_1) = 0$, $H(q_2) = 1$, and $H(q_3) = 2$. Given the desired precision $\varepsilon = 0.6$ and $\tau = 1$, by using the results in [11], it is possible to show that the relation $R \subset Q_\tau \times Q$ defined by:

$$R = R_1 \times \{q_1\} \cup R_2 \times \{q_2\} \cup R_3 \times \{q_3\}, \qquad (2)$$

where $R_1 = [0, 0.6]$, $R_2 = [0.4, 1.6]$ and $R_3 = [1.4, 2]$, is a 0.6–approximate bisimulation relation between $T_\tau(\Sigma)$ and T. Furthermore, since $R(Q_\tau) = Q$ and $R^{-1}(Q) = Q_\tau$, transition systems $T_\tau(\Sigma)$ and T are 0.6–approximately bisimilar. Suppose now that the goal is to find a control strategy on T such that, starting from state q_1 it is possible to reach the set $\{q_2, q_3\}$ in one step. By Figure 1, $q_1 \xrightarrow{l_2} q_2$ and $q_1 \xrightarrow{l_3} q_3$ and hence both labels l_2 and l_3 solve that problem. Since $(0, q_1) \in R$, the notion of approximate bisimulation (see condition (iii) of Definition 5) guarantees that starting from $0 \in R_1$ there exists a pair of labels (a_2, b_2), $(a_3, b_3) \in A_\tau \times B_\tau$ so that $0 \xrightarrow[\tau]{a_2,b_2} x_2 \in R_2$ and $0 \xrightarrow[\tau]{a_3,b_3} x_3 \in R_3$ in transition system $T_\tau(\Sigma)$. Indeed, by choosing constant curves $(a_2(t), b_2(t)) = (1, 1)$ and $(a_3(t), b_3(t)) = (2, 1)$, $t \in [0, 1]$ we have:

$$0 \xrightarrow[\tau]{a_2,b_2} 0.86 \in R_2, \qquad 0 \xrightarrow[\tau]{a_3,b_3} 1.73 \in R_3. \qquad (3)$$

However, if the constant disturbance label $b(t) = 0.4, t \in [0, 1]$ occurs instead of $b_2 = b_3$, we obtain:

$$0 \xrightarrow[\tau]{a_2,b} 0.35 \in R_1, \qquad 0 \xrightarrow[\tau]{a_3,b} 0.69 \in R_2, \qquad (4)$$

thus showing that the control strategy in (3) does not produce the desired result on the transition system $T_\tau(\Sigma)$. Although T is not adequate to solve this problem, a solution does exist. Since $0 \xrightarrow[\tau]{a_3,b} 0.69 \in R_2$ and the set X is invariant for Σ, it is easy to see that for any $\hat{b} \in B_\tau$, $0 \xrightarrow[\tau]{a_3,\hat{b}} x$ with $x \geq 0.69$ and hence $x \in R_2 \cup R_3$. Therefore, control label a_3 guarantees that state $0 \in R_1$ reaches $R_2 \cup R_3$, robustly with respect to the disturbance labels action, whereas control label a_2 does not.

The above example motivates us to propose the following definition that combines the notion of [10] with the notion of alternating bisimulation, introduced by Alur and coworkers in [12].

Definition 6: Given two metric transition systems $T_i = (Q_i, A_i \times B_i, \underset{i}{\longrightarrow}, O, H_i), i = 1, 2$ with the same observation set and the same metric $\mathbf{d}$ and given a precision $\varepsilon \in \mathbb{R}_0^+$, a relation $R \subseteq Q_1 \times Q_2$ is an alternating

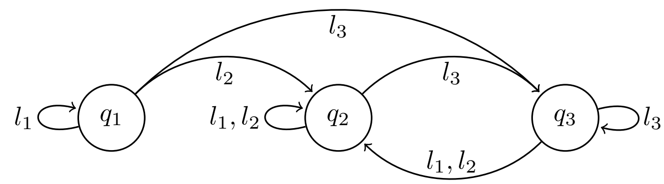

Fig. 1. Transition system T associated with control system Σ of Example 1.

ε–approximate ($A\varepsilon A$) bisimulation relation between T_1 and T_2 if for any $(q_1, q_2) \in R$:

(i) $\mathbf{d}(H_1(q_1), H_2(q_2)) \leq \varepsilon$;

(ii) $\forall a_1 \in A_1\ \exists a_2 \in A_2\ \forall b_2 \in B_2\ \exists b_1 \in B_1$ such that $q_1 \xrightarrow[1]{a_1,b_1} p_1$ and $q_2 \xrightarrow[2]{a_2,b_2} p_2$ with $(p_1, p_2) \in R$;

(iii) $\forall a_2 \in A_2\ \exists a_1 \in A_1\ \forall b_1 \in B_1\ \exists b_2 \in B_2$ such that $q_1 \xrightarrow[1]{a_1,b_1} p_1$ and $q_2 \xrightarrow[2]{a_2,b_2} p_2$ with $(p_1, p_2) \in R$.

Moreover, T_1 is $A\varepsilon A$ bisimilar to T_2 if there exists an $A\varepsilon A$ bisimulation relation R between T_1 and T_2 such that $R(Q_1) = Q_2$ and $R^{-1}(Q_2) = Q_1$.

It is easy to see that Definition 5 can be recovered as a special case of Definition 6, when the cardinality of each of the sets B_1 and B_2 in transition systems T_1 and T_2 is one.

IV. Main result

We start by stating the main result of this paper:

Theorem 1: Consider a control system $\Sigma = (\mathbb{R}^n, U \times V, \mathcal{U} \times \mathcal{V}, f)$. If Σ is δ–GAS and $U \times V$ is compact, then for any desired precision $\varepsilon \in \mathbb{R}^+$ there exist $\tau \in \mathbb{R}^+$ and a countable transition system T that is $A\varepsilon A$ bisimilar to $T_\tau(\Sigma)$.

The proof of the above result can be found in [17]. This result is important because it shows existence of symbolic models for nonlinear control systems *in the presence of disturbances*. Bisimulation theory for nonlinear control systems in presence of disturbances has been also considered in [15]. While the focus in [15] was the reduction of continuous systems to continuous systems with lower dimension in the state space, the focus of the present paper is the reduction of continuous systems to symbolic models. The symbolic model T constructed in the proof of Theorem 1 relies upon the knowledge of reachable sets of nonlinear control systems and this is a difficult task in general.

We now focus on the class of linear control systems and we show that in this case symbolic models can be constructed. A *linear* control system is a control system $\Sigma = (\mathbb{R}^n, U \times V, \mathcal{U} \times \mathcal{V}, f)$ where the function f is linear, i.e. for any $x \in \mathbb{R}^n$, $u \in U$ and $v \in V$,

$$f(x, (u, v)) = \mathbf{A}x + \mathbf{B}u + \mathbf{G}v,$$

for some matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{G}$ of appropriate dimensions. With a slight abuse of notation we say that a linear control system Σ is asymptotically stable, when Σ with $U \times V = \{0\} \times \{0\}$ is so. For any given $\tau \in \mathbb{R}^+$, consider the following sets:

$$\mathcal{R}_{A_\tau} := \left\{ p \in Q_\tau : 0 \xrightarrow[\tau]{\mathbf{a},\mathbf{0}} p,\ \mathbf{a} \in A_\tau \right\},$$
$$\mathcal{R}_{B_\tau} := \left\{ p \in Q_\tau : 0 \xrightarrow[\tau]{\mathbf{0},\mathbf{b}} p,\ \mathbf{b} \in B_\tau \right\}, \tag{5}$$

of reachable states of $T_\tau(\Sigma)$ from the origin 0 by means of any control label $\mathbf{a} \in A_\tau$ and identically null disturbance label $\mathbf{0}$ and, respectively, by means of any disturbance label $\mathbf{b} \in B_\tau$ and identically null control label $\mathbf{0}$. We can now propose the following symbolic models for linear systems.

Definition 7: Given a linear control system $\Sigma = (\mathbb{R}^n, U \times V, \mathcal{U} \times \mathcal{V}, f)$ and any $\tau \in \mathbb{R}^+$, $\eta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^+$, define the following transition system:

$$T_{\tau,\eta,\mu}(\Sigma) := (Q, A \times B, \longrightarrow, O, H), \tag{6}$$

where:

- $Q = [\mathbb{R}^n]_\eta$;
- A is a subset of $[\mathbb{R}^n]_\mu$ for which $\mathbf{d}_h(A, \mathcal{R}_{A_\tau}) \leq \mu/2$;
- B is a subset of $[\mathbb{R}^n]_\mu$ for which $\mathbf{d}_h(B, \mathcal{R}_{B_\tau}) \leq \mu/2$;
- $q \xrightarrow{a,b} p$, if the following inequality is satisfied
$$\|\mathbf{x}(\tau, q, 0, 0) + a + b - p\| \leq \eta/2; \tag{7}$$
- $O = \mathbb{R}^n$;
- $H = \imath : Q \hookrightarrow O$.

Since set of states Q and sets of labels A and B are *countable*, transition system $T_{\tau,\eta,\mu}(\Sigma)$ is countable, as well. Furthermore, transition system $T_{\tau,\eta,\mu}(\Sigma)$ of (6) can be easily constructed. The construction of $T_{\tau,\eta,\mu}(\Sigma)$ relies on the computation of the reachable sets in (5). The exact computation of those sets is, in general, hard. However, there are several results available in the literature, that propose approximations of reachable sets for linear control systems with arbitrarily small approximation error (e.g. [13] and the references therein). We can now give the following result.

Theorem 2: Consider a linear control system $\Sigma = (\mathbb{R}^n, U \times V, \mathcal{U} \times \mathcal{V}, f)$ and any desired precision $\varepsilon \in \mathbb{R}^+$. If Σ is asymptotically stable then for any $\tau \in \mathbb{R}^+$, $\mu \in \mathbb{R}^+$ and $\eta \in \mathbb{R}^+$ satisfying the following condition:

$$\left\| e^{\mathbf{A}\tau} \right\| \varepsilon + \mu + \eta/2 < \varepsilon, \tag{8}$$

the corresponding transition system $T_{\tau,\eta,\mu}(\Sigma)$ is $A\varepsilon A$ bisimilar to $T_\tau(\Sigma)$.

Proof: Consider the relation $R \subseteq Q_\tau \times Q$ defined by $(x, q) \in R$ if and only if $\|x - q\| \leq \varepsilon$. By construction $R^{-1}(Q) = Q_\tau$; by geometrical considerations on the infinity norm, $Q_\tau \subseteq \bigcup_{p \in [\mathbb{R}^n]_\eta} \mathcal{B}_{\eta/2}(p)$ and therefore, since by (8) $\eta/2 < \varepsilon$, we have that $R(Q_\tau) = Q$. Consider any $(x, q) \in R$. Condition (i) in Definition 6 is satisfied by the definition of R and of the involved metric transition systems. We now show that R satisfies also conditions (ii) and (iii). Consider any $(x, q) \in R$, any $\mathbf{a}_1 \in A_\tau$ and choose $a_2 \in A$ such that:

$$\|a_2 - \mathbf{x}(\tau, 0, \mathbf{a}_1, \mathbf{0})\| \leq \mu/2. \tag{9}$$

Consider any $b_2 \in B$. By the definition of B and of $\mathbf{d}_h$, there exists $b_3 \in \overline{\mathcal{R}_{B_\tau}}$ such that:

$$\mathbf{d}(b_2, b_3) = \|b_2 - b_3\| \leq \mu/2. \quad (10)$$

The vector b_3 can be either in $\mathcal{R}_{B_\tau}$ or in $\overline{\mathcal{R}_{B_\tau}} \setminus \mathcal{R}_{B_\tau}$; in both cases for any $\sigma \in \mathbb{R}^+$ there exists $b_4 \in \mathcal{R}_{B_\tau}$ such that:

$$\|b_3 - b_4\| \leq \sigma. \quad (11)$$

Choose $\mathbf{b}_1 \in B_\tau$ such that $b_4 = \mathbf{x}(\tau, 0, \mathbf{0}, \mathbf{b}_1)$ and consider the transition $x \xrightarrow[\tau]{\mathbf{a}_1, \mathbf{b}_1} y$ in $T_\tau(\Sigma)$. Set $z = \mathbf{x}(\tau, q, \mathbf{0}, \mathbf{0}) + a_2 + b_2 \in Q_\tau$; since $Q_\tau \subseteq \bigcup_{q' \in [\mathbb{R}^n]_\eta} \mathcal{B}_{\eta/2}(q')$, there exists $p \in Q = [\mathbb{R}^n]_\eta$ such that:

$$\|z - p\| \leq \eta/2. \quad (12)$$

Thus $q \xrightarrow{a_2, b_2} p$ in $T_{\tau,\eta,\mu}(\Sigma)$. By inequalities (12), (9), (11) and (8), the following chain of inequalities holds:

$$\begin{aligned}
\|y - p\| &= \|y - z + z - p\| \leq \|y - z\| + \|z - p\| \\
&\leq \|\mathbf{x}(\tau, x, \mathbf{a}_1, \mathbf{b}_1) - (\mathbf{x}(\tau, q, \mathbf{0}, \mathbf{0}) + a_2 + b_2)\| + \eta/2 \\
&= \|\mathbf{x}(\tau, x - q, \mathbf{0}, \mathbf{0}) + \mathbf{x}(\tau, 0, \mathbf{a}_1, \mathbf{0}) - a_2 \\
&\quad + \mathbf{x}(\tau, 0, \mathbf{0}, \mathbf{b}_1) - b_2 + b_3 - b_3\| + \eta/2 \\
&\leq \|e^{\mathbf{A}\tau}(x - q)\| + \|\mathbf{x}(\tau, 0, \mathbf{a}_1, \mathbf{0}) - a_2\| \\
&\quad + \|b_4 - b_3\| + \|b_3 - b_2\| + \eta/2 \\
&\leq \|e^{\mathbf{A}\tau}\|\varepsilon + \mu/2 + \sigma + \mu/2 + \eta/2.
\end{aligned}$$

By inequality (8), there exists a sufficiently small value of $\sigma \in \mathbb{R}^+$ such that $\beta(\varepsilon, \tau) + \sigma + \mu + \eta/2 \leq \varepsilon$, and hence $(y, p) \in R$ and condition (ii) in Definition 6 holds. Condition (iii) can be shown by using same arguments of condition (ii) and therefore it is omitted. ∎

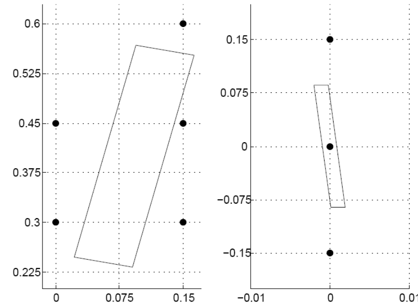

Fig. 2. Left panel: Outer approximation $P_{e_{A_\tau}}(\mathcal{R}_{A_\tau})$ of reachable set $\mathcal{R}_{A_\tau}$ and control labels set A (black dots). Right panel: Outer approximation $P_{e_{B_\tau}}(\mathcal{R}_{B_\tau})$ of reachable set $\mathcal{R}_{B_\tau}$ and disturbance labels set B (black dots).

$q \xrightarrow{a,b} p$	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
a_1, b_1	q_6	q_3	q_3	q_3	q_3	q_3	–	–	–
a_1, b_2	q_6	q_3	q_3	q_3	q_3	q_3	q_3	q_3	q_3
a_1, b_3	q_6	q_3	q_3	q_3	q_3	q_3	q_3	q_3	q_3
a_2, b_1	q_3	q_3	q_3	q_3	q_3	q_3	q_3	q_3	q_3
a_2, b_2	q_3	q_3	q_3	q_3	q_3	q_3	q_3	q_3	q_3
a_2, b_3	q_2	q_2	q_2	q_3	q_3	q_3	q_3	q_3	q_3
a_3, b_1	q_2	q_3	q_3	q_3	q_3	q_3	q_3	q_3	q_3
a_3, b_2	q_2	q_2	q_2	q_3	q_3	q_3	q_3	q_3	q_3
a_3, b_3	q_2	q_2	q_2	q_2	q_2	q_2	q_3	q_3	q_3
a_4, b_1	q_2	q_2	q_2	q_3	q_3	q_3	q_3	q_3	q_3
a_4, b_2	q_2	q_2	q_2	q_2	q_2	q_2	q_3	q_3	q_3
a_4, b_3	q_2	q_2	q_2	q_2	q_2	q_2	q_2	q_2	q_2

TABLE I

V. A SIMPLE EXAMPLE

Consider the simplified model of a direct current motor:

$$\Sigma = \begin{cases} \dot{x} = \mathbf{A}x + \mathbf{B}u + \mathbf{G}v, \\ x \in X, u \in U, v \in V, \end{cases} \quad (13)$$

where $x = (x_1 \ x_2)'$, x_1 is the current, x_2 is the angular velocity, u is the applied voltage, v is the load torque disturbance and:

$$\mathbf{A} = \begin{bmatrix} -R/L & -k_b k_m/L \\ 1/J & -k_f/J \end{bmatrix}; \mathbf{B} = \begin{bmatrix} k_m/L \\ 0 \end{bmatrix}; \mathbf{G} = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

where $R = 2$, $L = 0.5$, $k_b = 0.1$, $k_m = 0.1$, $k_f = 0.2$ and $J = 0.4$. All variables and constants appearing in system Σ are expressed in the International System. The control problem that we focus on is the one of *disturbance attenuation* and it consists in finding a (memoryless) control strategy $\mathbf{u}$, so that for any initial condition $x \in X$ and any disturbance $\mathbf{v} \in \mathcal{V}$, the corresponding angular velocity x_2 at time $t = 5$ is above 0.1, or equivalently:

$$\mathbf{x}(5, x, \mathbf{u}, \mathbf{v}) \in X^* := [0, 0.6] \times [0.1, 0.6]. \quad (14)$$

We solve this problem by using results of Section IV. Since system Σ is asymptotically stable, we can apply Theorem 2. Set the precision $\varepsilon = 0.5$ and $\tau = 5$. By choosing $\eta = 0.3$ and $\mu = 0.15$, inequality (8) is satisfied and therefore the transition system $T_{5,0.3,0.15}(\Sigma)$ defined in (6), is $A\varepsilon A$ bisimilar to $T_5(\Sigma)$ with $\varepsilon = 0.5$. The construction of $T_{5,0.3,0.15}(\Sigma)$ requires the computation of the reachable sets $\mathcal{R}_{A_\tau}$ and $\mathcal{R}_{B_\tau}$, as defined in (5). By using results in [13] it is possible to compute polytopic outer approximations of $\mathcal{R}_{A_\tau}$ and $\mathcal{R}_{B_\tau}$, resulting in the polytopes $P_{e_{A_\tau}}(\mathcal{R}_{A_\tau})$ and $P_{e_{B_\tau}}(\mathcal{R}_{B_\tau})$ shown in Figure 2. Numerical errors e_{A_τ} and e_{B_τ} for the sets $P_{e_{A_\tau}}(\mathcal{R}_{A_\tau})$ and $P_{e_{B_\tau}}(\mathcal{R}_{B_\tau})$, can be evaluated by using Lemma 1 of [13], resulting in $e_{A_\tau} = 3.0453 \cdot 10^{-6}$ and $e_{B_\tau} = 3.8067 \cdot 10^{-5}$. Since $\varepsilon >> \max\{e_{A_\tau}, e_{B_\tau}\}$ we will neglect errors e_{A_τ} and e_{B_τ} in the following developments. On the basis of the sets $P_{e_{A_\tau}}(\mathcal{R}_{A_\tau})$ and $P_{e_{B_\tau}}(\mathcal{R}_{B_\tau})$ we can compute the sets of labels A and B of transition system (6), as shown in Figure 2. The resulting symbolic model:

$$T_{5,0.3,0.15}(\Sigma) := (Q, A \times B, \longrightarrow, O, H), \quad (15)$$

is given by:

- $Q = \{q_i, i = 1, ..., 9\}$, where $q_1 = (0, 0)'$, $q_2 = (0, \eta)'$, $q_3 = (0, 2\eta)'$, $q_4 = (\eta, 0)'$, $q_5 = (\eta, \eta)'$, $q_6 = (\eta, 2\eta)'$, $q_7 = (2\eta, 0)'$, $q_8 = (2\eta, \eta)'$, $q_9 = (2\eta, 2\eta)'$;
- $A = \{a_i, i = 1, ..., 5\}$, where $a_1 = (0, 2\mu)'$, $a_2 = (0, 3\mu)'$, $a_3 = (\mu, 2\mu)'$, $a_4 = (\mu, 3\mu)'$ and $a_5 = (\mu, 4\mu)'$;

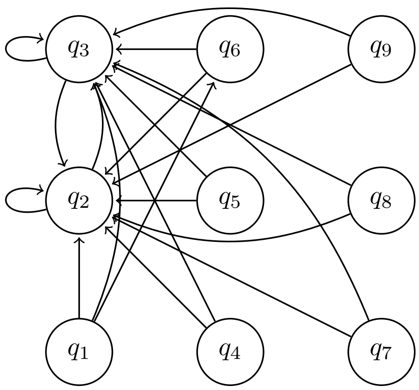

Fig. 3. Symbolic model $T_{5,0.3,0.15}(\Sigma)$ associated with the linear control system Σ. An arrow from a state q to a state p means that there exists *at least* a pair $(a,b) \in A \times B$ so that $\mathbf{x}(5,q,0,0)+a+b$ is in the closed ball $\mathcal{B}_{0.30/2}(p)$.

- $B = \{b_1, b_2, b_3\}$, where $b_1 = (0, -\mu)'$, $b_2 = (0,0)'$ and $b_3 = (0, \mu)'$;
- $q \xrightarrow{a,b} p$ is shown in Table[1] I;
- $O = \mathbb{R}^2$;
- $H = \iota : Q \hookrightarrow O$,

and depicted in Figure 3. By Theorem 2 transition systems $T_{5,0.3,0.15}(\Sigma)$ and $T_5(\Sigma)$ are $A\varepsilon A$ bisimilar with $\varepsilon = 0.5$; furthermore it is easy to see that $X^* = \mathcal{B}_{0.5}(q_3) \cup \mathcal{B}_{0.5}(q_6) \cup \mathcal{B}_{0.5}(q_9)$. Hence, the disturbance attenuation problem can be solved on the symbolic model in (15), by finding for any state $q \in Q$, the set $U^*(q)$ of all control labels $a \in A$ so that $q \xrightarrow{a,b} p \in \{q_3, q_6, q_9\}$ for any disturbance label $b \in B$. A simple inspection of Table I provides the following solution:

$$U^*(q_1) = U^*(q_2) = U^*(q_3) = \{a_1\};$$
$$U^*(q_4) = U^*(q_5) = U^*(q_6) = \{a_1, a_2\};$$
$$U^*(q_7) = U^*(q_8) = U^*(q_9) = \{a_2, a_3\}.$$

VI. Discussion

In this paper we showed that (incrementally globally) asymptotically stable nonlinear control systems admit alternating approximate bisimilar symbolic models, with a precision that can be chosen a priori, as a design parameter. For the class of linear control systems we showed that the proposed symbolic models can be easily constructed. Future work will focus on constructive techniques to obtain the symbolic models whose existence was shown in this paper.

References

[1] P. Tabuada, "Approximate simulation relations and finite abstractions of quantized control systems," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, A. Bemporad, A. Bicchi, and G. Buttazzo, Eds. Berlin: Springer Verlag, 2007, vol. 4416, pp. 529–542.

[2] P. Ramadge and W. Wonham, "Supervisory control of a class of discrete event systems," *SIAM Journal on Control and Optimization*, vol. 25, no. 1, pp. 206–230, 1987.

[3] W. Zielonka, "Infinite games on finitely coloured graphs with applications to automata on infinite trees," *Theoretical Computer Science*, vol. 200, pp. 135–183, 1998.

[4] A. Arnold, A. Vincent, and I. Walukiewicz, "Games for synthesis of controllers with partial observation," *Theoretical Computer Science*, vol. 28, no. 1, pp. 7–34, 2003.

[5] R. Alur and D. Dill, "Automata for modeling real-time systems," in *17th International Colloquium on Automata, Languages and Programming*, ser. Lecture Notes in Computer Science, vol. 443. Springer, 1990, pp. 322–335.

[6] R. Alur, T. A. Henzinger, G. Lafferriere, and G. J. Pappas, "Discrete abstractions of hybrid systems," *Proceedings of the IEEE*, vol. 88, pp. 971–984, 2000.

[7] P. Tabuada and G. Pappas, "Linear Time Logic control of discrete-time linear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 12, pp. 1862–1877, 2006.

[8] R. Milner, *Communication and Concurrency*. Prentice Hall, 1989.

[9] D. Park, "Concurrency and automata on infinite sequences," ser. Lecture Notes in Computer Science, Springer-Verlag, Ed., vol. 104, 1981, pp. 167–183.

[10] A. Girard and G. Pappas, "Approximation metrics for discrete and continuous systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 782–798, 2007.

[11] G. Pola, A. Girard, and P. Tabuada, "Approximately bisimilar symbolic models for nonlinear control systems," in *46th IEEE Conference on Decision and Control*, New Orleans, LA, December 2007, pp. 432–437.

[12] R. Alur, T. Henzinger, O. Kupferman, and M. Vardi, "Alternating refinement relations," in *Proceedings of the 8th International Conference on Concurrence Theory*, ser. Lecture Notes in Computer Science, no. 1466. Springer, 1998, pp. 163–178.

[13] A. Girard, "Reachability of uncertain linear systems using zonotopes," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, M. Morari, L. Thiele, and F. Rossi, Eds. Berlin: Springer Verlag, 2005, vol. 3414, pp. 291–305.

[14] G. Pola and P. Tabuada, "Symbolic models for linear control systems with disturbances," in *46th IEEE Conference on Decision and Control*, New Orleans, LA, December 2007, pp. 4643–4647.

[15] A. van der Schaft, "Equivalence of dynamical systems by bisimulation," *IEEE Transactions on Automatic Control*, vol. 49, no. 12, pp. 2160–2172, 2004.

[16] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 410–421, 2002.

[17] G. Pola and P. Tabuada, "Symbolic models for nonlinear control systems: Alternating approximate bisimulations," 2007, submitted for publication. Available at arXiv:0707.4205 [math.OC].

[1] Table I reads as follows. Entry q_6 corresponding to the second row and second column means that there is a transition from q_1 to q_6 labeled by a_1, b_1. Entries "–" correspond to transitions that end up outside the set X.