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A generalised entropy of curves: an approach to the analysis of dynamical systems

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***Abstract*— A generalisation of the entropy of a plane curve to $\mathbb{R}^n$ space is provided and the generalised entropy is used to evaluate the nonlinear behaviour of dynamical systems. The entropy of a curve, first introduced within the theory of thermodynamics of plane curves, has been used to quantify the irregularity of a curve. This paper extends the concept to higher dimensions and provides an algorithmic procedure to evaluate the entropy of a curve evolving in the phase space according to the equations of a dynamical system. The thermodynamic indicator is eventually used to infer some properties about the dynamical system. In particular, according to the proposed indicator, all linear systems evolve at entropy zero, while higher entropies characterise nonlinear systems, leading to an effective criterion for their classification.**

I. INTRODUCTION

This paper provides a new approach for the analysis and eventually the classification of dynamical systems. Some classification methods are already known from literature and are now a cornerstone in modern systems theory. For instance, it is possible to classify a linear system as stable, asymptotically stable or unstable, on the basis of the eigenvalues of its transition matrix. The same approach can be extended to nonlinear systems applying a linearisation procedure with respect to some nominal evolution, although now information can only be inferred about the particular evolution. A second approach based on Lyapunov Exponents is used to classify attractors into equilibria, cycles and chaotic sets [1]. This second approach has the advantage of providing a general view on nonlinear systems, without restrictions to a particular equilibrium, but has the drawback that it can be applied only to dynamical systems that admit attractive sets. Other attempts have been made in the past to tackle the problem of nonlinear systems classification. For instance in [2] a method based on the time-frequency plane is used to distinguish between 5 possible nonlinearities. This method can only be applied to SDOF (Single Degree Of Freedom) nonconservative systems. In [3], two systems are considered equivalent if any variable of one system may be expressed as a function of the variables of the other system and of a finite number of their derivatives. This is also known as Lie-Bäcklund isomorphism. From this point of view, the class of simplest systems, also called flat systems, includes all systems equivalent to a linear controllable system of any dimension with the same number of inputs. The drawback of this approach from a mathematical point of view is the difficulty of obtaining criteria for checking flatness. Besides, a complete classification of nonlinear systems according to the Lie-Bäcklund isomorphism is still to be done.

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In this paper a novel method for comparing nonlinear systems is proposed and is based on a generalisation of the entropy of a plane curve [4]. Roughly speaking, the entropy of a curve is 0 when the curve is a straight line, and increases as the curve becomes more "wiggly". Starting from the seminal work of [4], a new theory called thermodynamics of curves was developed [5], with some analogies with thermodynamics. However, defining the entropy of a curve only for plane curves has restricted its use to a few applications, as for instance [6]. In [7] and [8] a first attempt to extend the concept to higher dimensions was made leading to a natural application in the analysis of nonlinear systems. The main idea is to study the evolution of the entropy of a line which starts from entropy 0. With the proposed method, all lines evolving according to linear dynamics keep their entropy constant and fixed to 0. Besides, if the entropy of a line increases, it is a symptom of some nonlinear behaviour. In particular, larger or smaller values of the entropy might discriminate between stronger and milder nonlinearities. This ability is potentially very useful for the analysis and control of dynamical systems. First of all, a class of well-established methods for controlling linear systems is available, but an analogous systematic approach for non linear systems is still an open problem. Therefore, it would be convenient to know how much a nonlinear system can be considered close to a linear one. Besides, nonlinear behaviours often refer to the capacity of predicting a trajectory in the phase space in the long term. If a straight segment is dynamically bent and folded and its entropy increases, then it can be expected that it is harder to make a reliable prediction.

This paper relies on the works [7] and [8] and provides a more general and systematic approach to the analysis of nonlinear systems; many new interesting properties that have been found recently are described in detail. In addition a comparison with Lyapunov Exponents is proposed, motivated by some similarities between the algorithm required to compute them and the procedure proposed here.

The paper is organised as follows: next section is dedicated to a review of the main concepts from thermodynamics of curves and the generalisation to higher dimensions. Section 3 describes the algorithmic procedure to compute the proposed indicator and lists its main properties. Section 4 compares the indicator with the Lyapunov Exponents and section 5 shows the behaviour of the index in the simulations of some well-

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known benchmark problems. Finally in the last section we summarise our results and conclude the paper.

II. A GENERALISED THERMODYNAMICS OF CURVES

Here, the main properties of thermodynamics of curves are briefly recalled, while [4] and [5] provide full general theory. From a theorem by Steinhaus [9] the expected number of intersections between a plane curve Γ and a random line D intersecting it, is:

$$E\left[D\right]=\sum_{n=1}^{\infty}nP_n=\frac{2L}{C}, \tag{1}$$

where P_n is the probability for a line to intersect a plane curve in n points. Probability is intended as the number of lines which intersect the plane curve with respect to the total number of intersecting lines. The quantity L is the length of the plane curve Γ and C is the length of the boundary of its convex hull. Therefore the probability only depends on the shape of the particular plane curve Γ. In analogy with Shannon's measure of entropy [10], the entropy of a curve can be defined as $H\left(\Gamma\right)=-\sum_{n=1}^{\infty}P_n\ \log\left(P_n\right)$, where P_n has the same meaning as in equation (1). A classic computation for the maximal entropy provides the entropy of a plane curve Γ [6] as $H\left(\Gamma\right)=\log\left(\frac{2L}{C}\right)$. The definition of the temperature of a curve is generally given as the inverse of $\beta\left(\Gamma\right)$ [5], where $\beta\left(\Gamma\right)=\log\left(\frac{2L}{2L-C}\right)$. The main feature of the previous entities is that only straight segments are represented by a temperature $T=\beta^{-1}=0$, and then $H=0$. This is in accordance with Nernst's thermodynamic assumption and provides the analogy with thermodynamics as in physics.

An extension to higher dimensions of thermodynamics of curves was first proposed in [7] and [8], where the entropy of a curve was defined as

$$H\left(\Gamma\right)=\log\left(\frac{2L}{d}\right), \tag{2}$$

where d is the diameter of the smallest hypersphere covering the curve Γ. This definition circumvents the difficulty of defining the length of the convex hull perimeter C in higher dimensions, but preserves the property that the minimal entropy is associated only with straight segments.

III. ENTROPY BASED ANALYSIS OF DYNAMICAL SYSTEMS

The generalised entropy indicator (2) is applied to the analysis of dynamical systems by evaluating the entropy of a line as it evolves in the phase space. The line is approximated by a number of segments, and the evolution of the line is computed by evaluating the dynamics of the end-points of the segments. The upper bound of the indicator (2) depends on the number of the used points [7], so a normalised version can be used instead:

$$H=\frac{\log\left(\frac{L}{d}\right)}{\log\left(N-1\right)}, \tag{3}$$

where $N-1$ is the number of segments approximating the line. According to the definition (3), $H\in[0\quad 1)$, as will be proved later in Property 1. From now on, the normalised version of the generalised entropy indicator (3) will be used.

Assuming that a dynamical system evolves according to a discrete-time model

$$x\left(k+1\right)=f\left(x\left(k\right),k\right)\qquad x\in\mathbb{R}^m, \tag{4}$$

where m represents the dimension of the state vector, then the indicator (3) can be computed according to the following procedure

Numerical Algorithm:

1) **Initialisation:** $k=0$
   a) *Choose N points $x_1\left(0\right),...,x_N\left(0\right)$ ordered along a straight line*
   b) *$H\left(0\right)$ is 0*
2) **Evolution:** step k
   a) *Compute the next state for each point $x_1\left(k+1\right),...,x_N\left(k+1\right)$ according to (4)*
   b) *Consider the line that connects sequentially all the points and take its length L*
   c) *Compute the smallest hypersphere that contains all the points, and take its diameter d*
   d) *Compute $H\left(k\right)$ according to (3)*
   e) *Go to next step ($k=k+1$).*

Deterministic inputs can be included in equation (4) without significant changes, and have not been considered here for sake of simplicity. The procedure to compute the minimum covering sphere in point 2.c) is described in detail in the appendix.

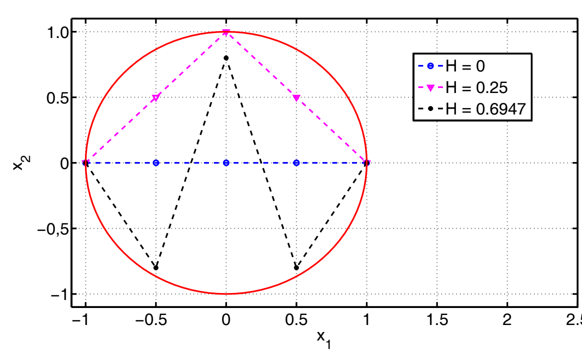

Fig. 1. Three plane lines covered by the same circle: more irregular lines have higher entropies.

The rationale of indicator (3) is that it describes the regularity of a line: if all points are aligned sequentially along a straight line, then the segment has entropy 0. On the other hand, more tortuous lines have higher entropies, as illustrated in the example in figure 1. A second property is that the generalised entropy takes into account the ordering of the points in the sequence. This is described in Figure 2. At the beginning points P_1, P_2 and P_3 are chosen ordered along a straight line according to the first step of the previous algorithm. In this case $L=d=\overline{P_1P_3}$ and thus $H\left(0\right)=0$. If at the next step the system dynamics exchange P_2 and

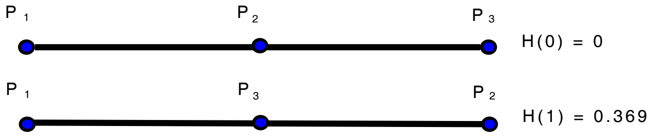

Fig. 2. A straight line turns into another straight line, but if the ordering of the points changes then the entropy changes as well.

P_3, then $L = \overline{P_1P_2} + \overline{P_2P_3}$ while $d = \overline{P_1P_2}$. Therefore $H(1) > 0$. By taking into account the ordering of the points, the generalised entropy realises if the system dynamics are folding the initial segment over itself, which is clearly a symptom of nonlinearity. The remainder of this section is an overview of the properties of the proposed indicator.

A. Properties of the generalised entropy

The generalised entropy indicator has several properties that make it interesting for the analysis of nonlinear dynamical systems. A list of them is reported in the following:

Theorem 1: *The entropy of a curve as defined in (3) is* 0 *if and only if the curve is straight and the points are ordered sequentially along it.*
This is the main property of the entropy of a plane curve and it is still valid for (3) in higher dimensions. Proof is provided in the appendix.

Property 1: *The entropy range of a line as defined in (3) is always between* 0 *and* 1
As a consequence of the proof of theorem 1, the entropy is bounded from below by 0. Besides, the maximum distance between two points within an hypersphere can not exceed the diameter. Thus, $L \leq (N-1) \cdot d$ so $H < 1$.
The introduction of a normalising constant in the index proposed in [6] might not seem a proper choice because it affects the computation of the entropy of a curve. For instance, two identical curves described by a different number of points have now different entropies. However, from the point of view of dynamical systems analysis, it is convenient to have a fixed full scale range, although some care is required in providing the value of the entropy always accompanied with the information about the number of used points. Moreover, the normalisation step paves the way to a probabilistic approach.

An important consequence of property 1 is that the entropy of a line is well defined even when the dynamical system is unstable and the length of the line tends to infinity.

Theorem 2: *A curve starting from entropy equal to zero has constant* 0 *entropy according to (3) if the dynamical system is linear*
This holds for any line and for any linear dynamical system (time-variant systems as well). A proof is provided in [7].

Property 2: *The entropy of a line as defined in (3) is insensitive to changes of scale, rotations and translations*
The entropy is defined as a ratio between lengths. Therefore, if all the point vectors that define the line are scaled, rotated or translated, the entropy remains the same.

Property 3: *A curve starting from entropy equal to zero has constant* 0 *entropy according to (3) if the dynamical system is one-dimensional and the state function is monotonic*
The case when the state space dimension is 1 is a degenerate situation in the sense that all the points remain necessarily aligned along the only available dimension. Therefore the entropy is larger than 0 only if the points change their ordering, as in figure 2. The sequence does not change if $x_1(k) < x_2(k)$ implies $x_1(k+1) < x_2(k+1)$ for each choice of the points $x_1(k)$ and $x_2(k)$. This is equivalent to the notion of monotonic function.

Remark: Even if the state function is not monotonic, the indicator might still be 0 if the initial segment is chosen in a region of the state space where the state function is monotonic. For instance, if an initial segment of ordered positive numbers is chosen and the state function is $x(k+1) = x(k)^2$, the non-monotonic behaviour of this system is missed.

The value of the proposed indicator can be computed either theoretically exploiting the previous properties and theorems, or by the algorithmic procedure introduced at the beginning of the section. The theoretical approach can be used if the dynamical system is linear to infer that the entropy will be constantly zero, or if the dynamical system has symmetry properties that lead to a geometrical solution. For instance, if the nonlinear system is $x(k+1) = x(k)^2$, then an initial line symmetric with respect to the origin will fold up on itself so that (3) will be $\log(2)/\log(N-1)$. If the dynamical system does not have particular symmetries or properties, it will be generally very hard to predict the behaviour of the indicator and it is necessary to use the algorithmic procedure. On the other hand, a drawback of the algorithmic approach is that it might be numerically unstable if all the points converge to an equilibrium or diverge. In the first case, all distances among the points go to zero, while in the second case they go to infinity. For this reason, the algorithmic procedure gives the best results in the case that the norms of the state points remain bounded and do not converge to one only value. A possible application is therefore the analysis of chaotic systems, where the states evolve within attractors. For this reason, next paragraph is dedicated to a comparison between the proposed indicator and Lyapunov Exponents, which are one of the most used methods to evaluate chaotic behaviours.

IV. A COMPARISON WITH LYAPUNOV EXPONENTS

The generalised entropy indicator introduced in this work can be computed in an algorithmic way which is particularly convenient for chaotic systems. Several ways are known from the literature for the evaluation of chaotic behaviours, such for instance [11] and [12], but the most wide spread method is the computation of the Lyapunov Exponents (LEs) [13]. The maximum LE gives a quantitative characterization of the exponential divergence of initially nearby trajectories. Also in this case, it is not generally possible to compute LEs in an analytic way, but one has to resort to numerical methods. LEs are computed evaluating the evolution of some points in the phase space, in a way that is similar to the approach proposed here. Two recent references for computing the LEs for continuous time and discrete time systems can be found in [14] and [15] respectively. A drawback of LEs is that two dynamical systems having the same LEs can behave in a very different way [16]. In particular, if a chaotic system in $\mathbb{R}^n$ is characterised by LEs $\lambda_1, \lambda_2, ..., \lambda_n$, then a linear continuous dynamical system with state transition matrix $A = diag\left[\lambda_1, \lambda_2, ..., \lambda_n\right]$ has the same LEs. Although it is still true that two trajectories arbitrarily close diverge in the long term if the dynamical system is linear and unstable, yet the behaviour is far from chaotic. This is a major difference with the proposed generalised entropy indicator. A second drawback of LEs is that it is not clear how to compute them for dynamical systems defined over a discrete state space. In this case, it is not straightforward how to define the distance between two states and whether two trajectories converge or not. On the other hand, the proposed generalised indicator provides a nice extension for the investigation of unusual dynamical systems defined on a discrete state space, as for instance the Kaprekar routine [17] addressed in the next section.

V. SIMULATION EXAMPLES

Extensive simulations have been performed to study the behaviour of the proposed index, and many benchmark problems have been investigated and compared. In the following some of the most significant results that have been obtained are presented.

An interesting example is obtained in the case when the state is either 0 or 1, according to the outcome of a Bernoulli experiment. The generalised entropy computed in this situation is larger when the two probabilities are closer, as shown in figure 3. This is in analogy with the entropy

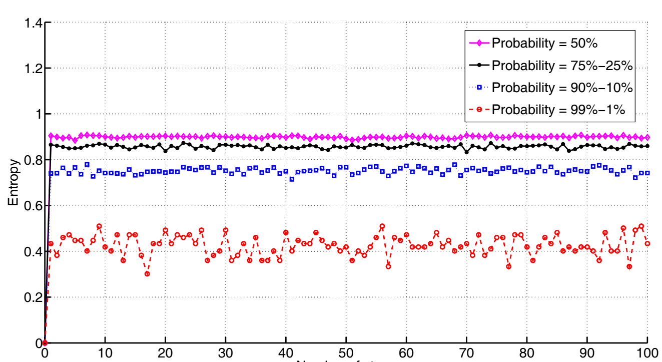

Fig. 3. Comparison of the entropy of a dynamical system with two possible outcomes on the basis of the probability of each outcome

as in information theory, [10] and [18], and emphasises that the proposed indicator takes into account the predictability of a dynamical system. Indeed, a probability of 50% for each outcome represents the most unpredictable situation.

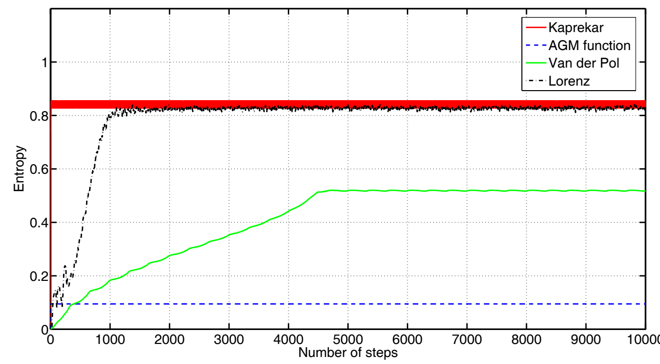

Fig. 4. Comparison of the entropy of several dynamical systems

In figure 4 a comparison among several dynamical systems is shown. In particular, the comparison involves a chaotic system described by Lorenz equations, the Van Der Pol oscillator (i.e. a dynamical system that admits a limit cycle), Kaprekar routine on 6-digit numbers, and a nonlinear system based on the arithmetic-geometric mean (AGM) equations. Lorenz equations are described by

$$\left\{\begin{array}{l} \dot{x_1} = -\frac{8}{3}\ x_1 + x_2 x_3 \\ \dot{x_2} = -10\ x_2 + 10\ x_3 \\ \dot{x_3} = -x_1 x_2 + 28\ x_2 - x_3 \end{array}\right. . \quad (5)$$

Van Der Pol oscillator is described by

$$\left\{\begin{array}{l} \dot{x_1} = x_2 \\ \dot{x_2} = \left(1 - x_1^2\right) x_2 - x_1 \end{array}\right. . \quad (6)$$

Both the previous problems have been solved discretising the continuous system with a sampling time of $0.01\ s$. The AGM system is described by

$$\left\{\begin{array}{l} x_1\left(k+1\right) = \left(x_1\left(k\right) + x_2\left(k\right)\right)/2 \\ x_2\left(k+1\right) = \sqrt{\left(x_1\left(k\right) \cdot x_2\left(k\right)\right)} \end{array}\right. , \quad (7)$$

where the initial conditions have to be chosen inside the positive quadrant. Finally, the Kaprekar routine, see for example [17], consists of an algorithm that can be applied to integer numbers and works performing alternately the following steps:

Step 1 Rearrange the digits of a number $a\left(k\right)$ in ascending and descending order so to obtain two new numbers $\underline{a}\left(k\right)$ and $\overline{a}\left(k\right)$ respectively.

Step 2 Compute $a\left(k+1\right) = \overline{a}\left(k\right) - \underline{a}\left(k\right)$. Increase time index k and go back to *step 1*

In the example, Kaprekar routine has been applied to six digit numbers. In this case the solution is known to converge to a fixed number, (e.g. when the starting number has all equal digits) or to a limit cycle. Although all the four dynamical systems introduced previously have very different characteristics, it is still possible to compare them according to the proposed index. In particular, the highest entropies were obtained for Kaprekar routine, then Lorenz equations, Van Der Pol oscillator and the AGM system respectively. All entropies have been computed using 1000 points.

The next example is dedicated to the comparison of the entropy of a dynamical system where a parameter a is used to tune the amount of nonlinearity. The investigated dynamical system evolves according to the equations:

$$\begin{cases} x_1(k+1) = a \cdot x_1(k) \\ x_2(k+1) = 3.2 \cdot x_2(k) \cdot (1 - x_2(k)) \end{cases} . \quad (8)$$

The first component of the system (8) evolves with linear dynamics, so its expected entropy is 0. The second component evolves according to a logistic equation, which is a function that can be used to represent demographic models, where the multiplicative parameter is a positive number representing a combined rate for reproduction and starvation [1]. In the example here the value of the parameter is 3.2, and the equation has two equilibrium points. In the example, when $|a| < 1$, the linear component goes to zero and the dynamical system reduces to the nonlinear part. On the contrary, when $|a| > 1$, the linear component overrides the nonlinear one. Finally when $|a| = 1$ the two components have comparable values. The entropy of the system reflects this situation, and either assumes the value of the dominant component or an intermediate value in the last case. The simulation results are shown in figure

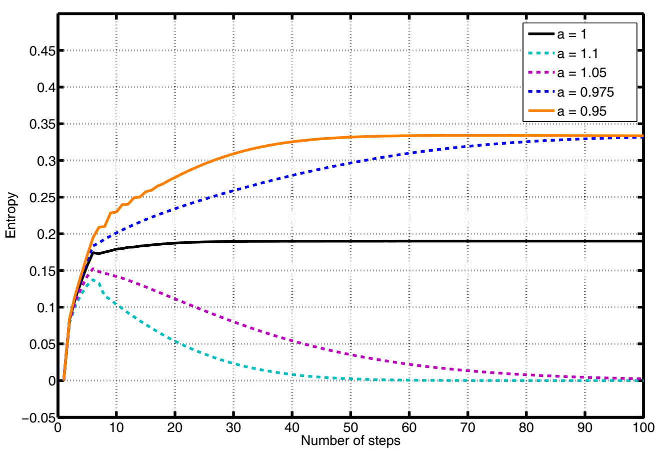

Fig. 5. Comparison of the entropy of a dynamical system as a function of a parameter that weighs the contribute of the linear and nonlinear component

5, where the higher entropies are reached when a has values 0.95 and 0.975 (nonlinear dominance). The curves converging to 0 entropy are due to values of a equal to 1.1 and 1.05 (linear dominance). The curve in the centre is obtained when $a = 1$. In addition, it can be noted that the value of a affects the time that is necessary before the linear component becomes dominant or can be neglected, therefore affecting the transient behaviour of the entropy.

It was remarked when discussing Property 3 that the choice of the initial conditions (i.e. the first straight line) can affect the generalised entropy indicator and its final value. This can be unavoidable if the behaviour of the nonlinear system depends on the initial state space region. In the examples proposed so far, extensive simulations showed that the behaviour of the indicator is not affected by the initial conditions, provided that the initial line is chosen within the region of interest (e.g. inside the attractor set in chaotic systems). In other situations special care is required in the choice of the initial conditions, and Monte Carlo simulations might be necessary to distinguish among different behaviours of the same nonlinear system. As a final example the logistic equation is considered in the case that the state is extended to include the fixed rate parameter:

$$\begin{cases} x_1(k+1) = x_2(k) \cdot x_1(k)(1 - x_1(k)) \\ x_2(k+1) = x_2(k) \end{cases} . \quad (9)$$

It is known from literature [1] that the logistic function presents both stable and chaotic behaviours depending on the value of x_2. Therefore it is not very significant to study the overall entropy of the system, but it is more significant to investigate the entropy as a function of the second component which only depends on its initial condition. The graph of the entropy so obtained is shown in figure 6 and reproduces realistically the known behaviour of the logistic function. In fact system (9) has one equilibrium when the parameter is smaller than 3, it oscillates for values between 3 and 3.57 (approximately), it shows a chaotic behaviour for values greater than 3.57 and smaller than 4 and there is a so-called "island of stability" for values around 3.8. Figure 6 interpolates the value of the entropies computed for possible values of the parameter from 0.1 to 4, computed with a 0.1 interval step.

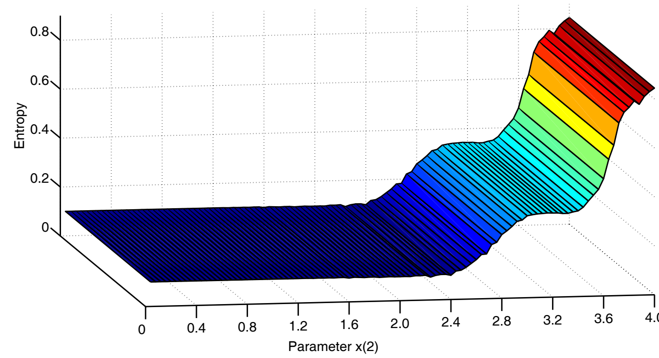

Fig. 6. Entropy of the logistic equation as a function of x_2

VI. CONCLUSIONS

This paper generalises the entropy of plane curves to $\mathbb{R}^n$ space. Moreover, it provides an algorithmic procedure to evaluate an entropy indicator that is used to infer some properties of dynamical systems. Several nonlinear systems, having completely different characteristics, have been compared and classified according to the entropy indicator. The generalised entropy is used also to analyse chaotic systems, as an alternative choice to Lyapunov Exponents, extending the application field to the investigation of dynamical systems defined on a discrete state space. Simulation results show that the proposed indicator has many useful properties and provides very promising results with a wide generality.

An intrinsic difficulty in the analysis of nonlinear systems is that very different behaviours (stability, instability, limit cycles, attractors..) can occur in different regions of the state space for the same dynamical system. In such situations it is possible to evaluate the entropy of the system as a function of the initial conditions, as described in the last example provided.

VII. ACKNOWLEDGMENTS

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APPENDIX

Minimum covering sphere

The problem of finding the smallest hypersphere containing all the points, introduced in the algorithmic procedure to compute (3) at step 2.c), requires further considerations. This problem is known from literature as the "Minimum Covering Sphere" (MCS) problem [19]. The mathematical formulation of MCS for a finite set of points $\{x_i\}$ is the *minimax* problem

$$\min_c \max_i \|x_i - c\| \,, \qquad (10)$$

where c is the unknown centre of the hypersphere. The MCS problem is known to have always one unique solution [19]. The algorithmic procedure to compute the proposed indicator $H(k)$ requires the solution of an MCS problem (10) at each step. In this work, the algorithm proposed by Hopp and Reeve [20] was used to solve problem (10), because it is in general faster than the classic solution of a convex quadratic programming problem. Algorithm [20] exploits the geometric nature of the problem and computes iteratively an outer and an inner spherical bounding of the MCS. When the two spheres eventually converge, the solution is the unique MCS. The algorithm is proved to converge always to the solution in a finite number of steps. Moreover, if n is the number of total points and d is the dimension of the state space, then the algorithm has complexity $O(n^{1.1}d^2)$ if points are distributed uniformly inside a sphere, and $O(n^1 d^{2.3})$ if points are all in the proximity of the surface of the sphere (which represents the worst case) [20]. In our simulations (performed on an AMD 64 bit processor with 2GHz clock frequency) we observed that solution was never found in more than $2s$ for a set of 1000 points in $\mathbb{R}^3$. Particular distributions of points are claimed to be potentially critical for the algorithm [20] in case of numeric noise, but they never occurred in our practice.

Proof of theorem 1

Proof: The sufficient condition, i.e. a straight line has $H = 0$, according to (3), is easily verified. The necessary condition is that if $H = 0$, and thus the diameter is equal to the length of the line, then the only possibility is that the line is straight. To prove the necessary condition it can be noted that the curve is always contained in a sphere A centered in its mid-point, with diameter L. Therefore, being the MCS minimal, its diameter is always smaller or equal than L. In addition, as a consequence of the fact that the minimum covering sphere is unique, if $d = L$, then A must be the MCS. Moreover, the line has at least two distinct points on the sphere surface, otherwise a smaller hypersphere can always be found. To sum up, $d = L$ implies that the line must have a point in the centre of the MCS and at least two on the surface. Now, since the overall length of the line is d, then it must be formed by the two segments connecting the centre with the two points on the surface, otherwise it would be longer than just d by triangle inequality. In this case, the only possibility is that the line is the diameter itself, otherwise the centre would not be a convex combination of the two points on the sphere surface, which is a necessary condition for the minimum covering sphere [19]. Therefore the line corresponds to the diameter and is straight. ■

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