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The Vanishing Approach for the Average Continuous Control of Piecewise Deterministic Markov Processes

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Abstract— **This paper deals with the long run average continuous control problem of piecewise deterministic Markov processes (PDMP's) taking values in a general Borel space and with compact action space depending on the state variable. The control variable acts on the jump rate and transition measure of the PDMP, and the running and boundary costs are assumed to be positive but superiorly unbounded. Our main result is to obtain the existence and characterization of an ordinary optimal feedback control for the long run average cost problem using the so-called vanishing discount approach.**

I. INTRODUCTION

A general family of non-diffusion stochastic models suitable for formulating many optimization problems in several areas of operations research, namely piecewise-deterministic Markov processes (PDMP's), was introduced in [1], [2]. These processes are determined by three local characteristics; the flow ϕ, the jump rate λ and the transition measure Q. A suitable choice of the state space and the local characteristics provide stochastic models covering a great number of problems of operations research [2].

This paper deals with the long run average continuous control problem of PDMP's taking values in a general Borel space and can be seen as a continuation of the previous works [3], [4]. At each point x of the state space a control variable is chosen from a compact action set $\mathbb{U}(x)$ and is applied on the jump parameter λ and transition measure Q. The goal is to minimize the long run average cost, which is composed by a running cost and a boundary cost, both assumed to be positive but superiorly unbounded.

Our main result is to obtain conditions for the existence and characterization of an ordinary optimal feedback control for the long run average cost problem using the so-called vanishing discount approach (see [5], page 83). As far as the authors are aware of, this is the first time that this kind of approach is applied to this class of Markov processes. We take advantage of a connection for the characterization for the optimality equation, one in terms of an one-stage optimization equation with relaxed control space and the other one in terms of an integro-differential equation.

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The paper is organized in the following way. In section II we introduce some notation, basic assumptions, and the control problems to be considered. The definition of the ordinary and relaxed control spaces as well as some operators required for characterizing the optimality equation. are presented in section III. In section IV we derive conditions for the existence and characterization of an ordinary feedback optimal control based on a connection between the discrete-time optimality equation and an integro-differential equation. Section V considers the discounted optimal control problem. Our main result is presented in section VI with some conditions for the existence and characterization of an ordinary optimal feedback control for the long run average cost using the so-called vanishing discount approach (see Theorem 6.4).

II. NOTATION AND ASSUMPTIONS

In this section we present some standard notation and some basic definitions related to the motion of a PDMP $\{X(t)\}$, and the control problems we will consider throughout the paper. For further details and properties the reader is referred to [2]. The following notation will be used in this paper:

- $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers and $\mathbb{R}^d$ the d-dimensional euclidian space. The set of natural numbers is denoted by $\mathbb{N}$.
- η denotes the Lebesgue measure on $\mathbb{R}$.
- For X a metric spaces, we denote $\mathcal{B}(X)$ as the σ-algebra generated by the open sets of X. $\mathcal{M}(X)$ (respectively, $\mathcal{P}(X)$) denotes the set of all finite (respectively probability) measures on $(X, \mathcal{B}(X))$.
- Let X and Y be metric spaces. The set of all Borel measurable (respectively bounded) functions from X into Y is denoted by $\mathbb{M}(X;Y)$ (respectively $\mathbb{B}(X;Y)$). Moreover, for notational simplicity $\mathbb{M}(X)$ (respectively $\mathbb{B}(X)$, $\mathbb{M}(X)^+$, $\mathbb{B}(X)^+$) denotes $\mathbb{M}(X;\mathbb{R})$ (respectively $\mathbb{B}(X;\mathbb{R})$, $\mathbb{M}(X;\mathbb{R}_+)$, $\mathbb{B}(X;\mathbb{R}_+)$). $\mathbb{C}(X)$ denotes the set of continuous functions from X into $\mathbb{R}$.

Let E be an open subset of $\mathbb{R}^n$, ∂E its boundary, and $\overline{E}$ its closure. A PDMP is determined by its local characteristics (ϕ, λ, Q) as presented below:

• the flow $\phi(x,t)$ is a function $\phi : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ continuous in (x,t) and such that $\phi(x,t+s) = \phi(\phi(x,t),s)$. For each $x \in E$ the time the flow takes to reach the boundary starting from x is defined as $t_*(x) \doteq \inf\{t > 0 : \phi(x,t) \in \partial E\}$. For $x \in E$ such that $t_*(x) = \infty$ (that is, the flow starting from x never touches the boundary), we set $\phi(x, t_*(x)) = \Delta$, where Δ is a fixed point in ∂E.

We define the following space of functions absolutely continuous along the flow with limit towards the boundary:

$\mathbb{M}^{ac}(E) = \big\{g \in \mathbb{M}(E); g(\phi(x,t)) : [0,t_*(x)) \mapsto \mathbb{R}$ is absolutely continuous for each $x \in E$ and whenever $t_*(x) < \infty$ the limit $\lim_{t\to t_*(x)} g(\phi(x,t))$ exists$\big\}$. For $g \in \mathbb{M}^{ac}(E)$ and $z \in \partial E$ for which there exists $x \in E$ such that $z = \phi(x,t_*(x))$ where $t_*(x) < \infty$ we define $g(z) = \lim_{t\to t_*(x)} g(\phi(x,t))$ (note that the limit exists by assumption). As shown in [6], for $g \in \mathbb{M}^{ac}(E)$ there exists a function $\mathcal{X}g \in \mathbb{M}(E)$ such that for all $t \in [0,t_*(x))$ $g(\phi(x,t)) - g(x) = \int_0^t \mathcal{X}g(\phi(x,s))ds$. The local characteristics λ and Q depend on a control action $u \in \mathbb{U}$ where $\mathbb{U}$ is a compact Borel space, in the following way: $\lambda \in \mathbb{M}(\overline{E}\times\mathbb{U})^+$ and Q is a stochastic kernel on E given $\overline{E}\times\mathbb{U}$.

For each $x \in \overline{E}$ we define the subsets $\mathbb{U}(x)$ of $\mathbb{U}$ as the set of feasible control actions that can be taken when the state process is in $x \in \overline{E}$, that is, the control action that will be applied to λ and Q must belong to $\mathbb{U}(x)$. The following assumptions will be made throughout the paper:

A.1) For all $x \in \overline{E}$, $\mathbb{U}(x)$ is a compact subspace of $\mathbb{U}$.
A.2) The set $K = \big\{(x,a) : x \in \overline{E}, a \in \mathbb{U}(x)\big\}$ is a Borel subset of $\overline{E}\times\mathbb{U}$.
A.3) For all $x \in E$, and $t \in [0,t_*(x))$, we have that $\int_0^t \sup_{a\in\mathbb{U}(\phi(x,s))} \lambda(\phi(x,s),a)\,ds < \infty$. If $t_*(x) < \infty$ then $\int_0^{t_*(x)} \sup_{a\in\mathbb{U}(\phi(x,s))} \lambda(\phi(x,s),a)\,ds < \infty$.

We present next the definition of an open loop policy and the associated motion of the controlled process. A piecewise open loop policy U is a pair of functions $(u,u_\partial) \in \mathbb{M}(\mathbb{N}\times E\times\mathbb{R}_+;\mathbb{U})\times\mathbb{M}(\mathbb{N}\times E;\mathbb{U})$ satisfying $u(n,x,t) \in \mathbb{U}(\phi(x,t))$, and $u_\partial(n,x) \in \mathbb{U}(\phi(x,t_*(x)))$ for all $(n,x,t) \in \mathbb{N}\times E\times\mathbb{R}_+$. The class of piecewise open loop policy will be denoted by $\mathcal{U}$. Given a piecewise open loop policy $U = (u,u_\partial)$, one describes the motion of the piecewise deterministic process $X(t)$ in the following manner. Define $T_0 = 0$ and $X(0) = x$. Assume that the process $\{X(t)\}$ is located at Z_n at the n^{th} jump time T_n then select a random variable S_n having distribution $F(t) = 1 - I_{\{t<t_*(Z_n)\}} e^{-\int_0^t \lambda(\phi(Z_n,s),u(n,Z_n,s))ds}$. Define $T_{n+1} = T_n + S_n$ and for $t \in [T_n,T_{n+1})$, $X(t) = \phi(Z_n,t-T_n)$, $Z(t) = Z_n$, $\tau(t) = t - T_n$, $N(t) = n$. Now consider a random variable Z_{n+1} having distribution $Q(\phi(Z_n,S_n),u(n,Z_n,S_n));.)$ if $\phi(Z_n,S_n) \in E$, or $Q(\phi(Z_n,S_n),u_\partial(n,Z_n);.)$ if $\phi(Z_n,S_n) \in \partial E$. At time T_{n+1}, the process $\{X(t)\}$ is defined by $X(T_{n+1}) = Z_{n+1}$.

As in Davis [2], we consider the following assumption to avoid any accumulation point of the jump times:

A.4) For any $x \in E$, $U = (u,u_\partial) \in \mathcal{U}$, and $t \geq 0$, $E_x^U\left[\sum_{i=1}^{\infty} I_{\{T_i\leq t\}}\right] < \infty$

The costs of our control problem will contain 2 terms, a running cost f and a boundary cost r, satisfying the following properties:

A.5) $f \in \mathbb{M}(\overline{E}\times\mathbb{U})^+$.
A.6) $r \in \mathbb{M}(\partial E\times\mathbb{U})^+$.

The long-run average cost we want to minimize over $\mathcal{U}$ is given by:

$$\mathcal{A}(U,x) = \limsup_{t\to+\infty} \frac{1}{t} E_x^U\left[\int_0^t f\big(X(s),u(N(s),Z(s),\tau(s))\big)ds + \int_0^t r\big(X(s-),u_\partial(N(s-),Z(s-))\big)dp^*(s)\right],$$

where $p^*(t) = \sum_{i=1}^{\infty} I_{\{T_i\leq t\}} I_{\{X(T_i-)\in\partial E\}}$ counts the number of times the process touched the boundary up to time t, and we set

$$\mathcal{J}_{\mathcal{A}}(x) = \inf_{U\in\mathcal{U}} \mathcal{A}(U,x). \tag{1}$$

For the α discounted case, with $\alpha > 0$, the cost we want to minimize is given by:

$$\mathcal{D}^\alpha(U,x) = E_x^U\left[\int_0^\infty e^{-\alpha s} f\big(X(s),u(N(s),Z(s),\tau(s))\big) ds + \int_0^\infty e^{-\alpha s} r\big(X(s-),u_\partial(N(s-),Z(s-))\big)dp^*(s)\right]$$

and we set

$$\mathcal{J}_{\mathcal{D}}^\alpha(x) = \inf_{U\in\mathcal{U}} \mathcal{D}^\alpha(U,x). \tag{2}$$

We also consider a truncated version of problem (2) defined, for each $m = 0,1,\ldots$, as

$$\mathcal{D}_m^\alpha(U,x) = E_x^U\left[\int_0^{T_m} e^{-\alpha s} f\big(X(s),u(N(s),Z(s),\tau(s))\big) ds + \int_0^{T_m} e^{-\alpha s} r\big(X(s-),u_\partial(N(s-),Z(s-))\big)dp^*(s)\right] \tag{3}$$

We need the following assumption, to avoid infinite costs for the discounted case.

A.7) For all $\alpha > 0$ and all $x \in E$, $\mathcal{J}_{\mathcal{D}}^\alpha(x) < \infty$.

It is clear that for all $x \in E$, $0 \leq \inf_{U\in\mathcal{U}} \mathcal{D}_m^\alpha(U,x) \leq \mathcal{J}_{\mathcal{D}}^\alpha(x) < \infty$.

III. Discrete-time ordinary and relaxed controls

In this section we first present the definitions of the discrete-time ordinary and relaxed control sets used in the formulation of the optimality equation of the discrete-time Markov control problem as well as the characterization of some topological properties of these sets. In particular, by using a result of the theory of multifunctions (see the book by Castaing and Valadier [7]), it is shown that the set of relaxed controls is compact. In the sequel we present some important operators associated to the optimality equation of the discrete-time problem as well as some mesurability properties.

A. Relaxed and ordinary control

We present in this sub-section the set of discrete-time relaxed controls and the sub-set of ordinary controls. Consider the Banach spaces $L^1(\mathbb{R}_+;\mathbb{C}(\mathbb{U}))$ and $L^\infty(\mathbb{R}_+;\mathcal{M}(\mathbb{U}))$ where $\mathbb{C}(\mathbb{U})$ is equipped with the topology of uniform convergence and $\mathcal{M}(\mathbb{U})$ is equipped with the weak* topology $\sigma(\mathcal{M}(\mathbb{U}),\mathbb{C}(\mathbb{U}))$. Let $\mathcal{V}^r$ (respectively $\mathcal{V}^r(x)$ for $x\in E$) be the set of all η-measurable functions μ defined on $\mathbb{R}_+$ with value in $\mathcal{P}(\mathbb{U})$ such that $\mu(t,\mathbb{U})=1$ η-a.e. (respectively $\mu(t,\mathbb{U}(\phi(x,t)))=1$ η-a.e.). From Theorem V-2 in [7], it follows that $\mathcal{V}^r$ (respectively $\mathcal{V}^r(x)$ for $x\in E$) are compact sets with respect to the weak* topology $\sigma(L^\infty(\mathbb{R}_+;\mathcal{M}(\mathbb{U})),L^1(\mathbb{R}_+;\mathbb{C}(\mathbb{U})))$. Moreover, from Bishop's Theorem (see Theorem I.3.11 in [8]), there is a metric such for all $x\in E$, $\mathcal{V}^r(x)$ is a compact set of the Borel set $\mathcal{V}^r$. Note that a sequence $\big(\mu_n\big)_{n\in\mathbb{N}}$ in $\mathcal{V}^r(x)$ converges to μ if and only if

$$\lim_{n\to\infty}\int_{\mathbb{R}_+}\int_{\mathbb{U}(\phi(x,t))} g(t,u)\mu_n(t,du)dt = \int_{\mathbb{R}_+}\int_{\mathbb{U}(\phi(x,t))} g(t,u)\mu(t,du)dt, \tag{4}$$

for all $g\in L^1(\mathbb{R}_+;\mathbb{C}(\mathbb{U}))$.

Therefore the set of relaxed controls are defined as follows. For $x\in E$, $\mathbb{V}^r(x)=\mathcal{V}^r(x)\times\mathcal{P}\big(\mathbb{U}(\phi(x,t_*(x)))\big)$, $\mathbb{V}^r=\mathcal{V}^r\times\mathcal{P}\big(\mathbb{U}\big)$. The set of ordinary controls, denoted by $\mathbb{V}$ (respectively $\mathbb{V}(x)$ for $x\in E$), is defined as above except that it is composed by deterministic functions instead of probability measures and thus, the set of ordinary controls is a subset of the set of relaxed control $\mathbb{V}^r$ (respectively $\mathbb{V}^r(x)$ for $x\in E$) by identifying any control action $u\in\mathbb{U}$ with the Dirac measure concentrated on u. Therefore we can write that $\mathbb{V}\subset\mathbb{V}^r$ (respectively $\mathbb{V}(x)\subset\mathbb{V}^r(x)$ for $x\in E$) and from now on we will consider that $\mathbb{V}$ (respectively $\mathbb{V}(x)$ for $x\in E$) will be endowed the topology generated by $\mathbb{V}^r$.

The necessity to introduce the class of relaxed control $\mathbb{V}^r$ is justified by the fact that in general there does not exist a topology for which $\mathbb{V}$ and $\mathbb{V}(x)$ are compact sets. However from the previous construction, it follows that $\mathbb{V}^r$ and $\mathbb{V}^r(x)$ are compact sets.

As in [5], page 14, we need that the set of feasible state/relaxed-control pairs is a measurable sub-set of $\mathcal{B}(E)\times\mathcal{B}(\mathbb{V}^r)$, that is, we need the following assumption.

A.8) $\mathcal{K}\doteq\big\{(x,\Theta):\Theta\in\mathbb{V}^r(x),x\in E\big\}\in\mathcal{B}(E)\times\mathcal{B}(\mathbb{V}^r)$.

We present a sufficient condition, based on the continuity of the sets $\mathbb{U}(x)$, to ensure that assumption A.8) holds. The proof is presented in [6].

Proposition 3.1: Assumption A.8) is satisfied if for all convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in E $\mathbb{U}(x)=\cap_{n\in\mathbb{N}}\cup_{m\geq n}\mathbb{U}(x_m)=\cup_{n\in\mathbb{N}}\cap_{m\geq n}\mathbb{U}(x_m)$, where $\lim_{n\to\infty}x_n=x$.

B. Discrete-time operators and measurability properties

In this sub-section we present some important operators associated to the discrete-time optimality equation as well as some measurability properties. We consider the following notation for $x\in\overline{E}$, $y\in E$ and $\mu\in\mathcal{P}\big(\mathbb{U}\big)$, $g\in\mathbb{M}(E)$ bounded from below, and $w\in\mathbb{M}(\overline{E}\times\mathbb{U})$:

$$Qg(x,\mu)=\int_{\mathbb{U}}\int_E g(z)Q(x,u;dz)\mu(du),$$
$$\lambda Qg(x,\mu)=\int_{\mathbb{U}}\lambda(x,u)\int_E g(z)Q(x,u;dz)\mu(du),$$
$$\Lambda^\mu(y,t)=\int_0^t\int_{\mathbb{U}}\lambda(\phi(y,s),u)\mu(du)ds,$$
$$w(x,\mu)=\int_{\mathbb{U}}w(x,u)\mu(du).$$

The following operators will be associated to the optimality equations of the discrete-time problems that will be presented in the next sections. For $x\in E$, $\Theta=\big(\mu,\mu_\partial\big)\in\mathbb{V}^r$, $g\in\mathbb{M}(E)$ bounded from below, $w_1\in\mathbb{M}(E\times\mathbb{U})$ bounded from below, $w_2\in\mathbb{M}(\partial E\times\mathbb{U})$, and $\alpha\geq 0$, define:

$$G_\alpha g(x,\Theta)\doteq\int_0^{t_*(x)}e^{-\alpha s-\Lambda^\mu(x,s)}\lambda Qg(\phi(x,s),\mu(s))ds + e^{-\alpha t_*(x)-\Lambda^\mu(x,t_*(x))}Qg(\phi(x,t_*(x)),\mu_\partial), \tag{5}$$

$$L_\alpha w_1(x,\Theta)\doteq\int_0^{t_*(x)}e^{-\alpha s-\Lambda^\mu(x,s)}w_1(\phi(x,s),\mu(s))ds, \tag{6}$$

$$H_\alpha w_2(x,\Theta)\doteq e^{-\alpha t_*(x)-\Lambda^\mu(x,t_*(x))}w_2(\phi(x,t_*(x)),\mu_\partial). \tag{7}$$

It will be useful in the sequel to define the function $\mathcal{L}_\alpha(x,\Theta)$ as follows: $\mathcal{L}_\alpha(x,\Theta)\doteq L_\alpha I_{E\times\mathbb{U}}(x,\Theta)$. In particular for $\alpha=0$ we write for simplicity $G_0=G$, $L_0=L$, $H_0=H$, $\mathcal{L}_0=\mathcal{L}$.

The next proposition presents some important measurability properties of the operators (5), (6), (7) and its proof can be found in [6].

Proposition 3.2: Let $\alpha\in\mathbb{R}_+$, $g\in\mathbb{M}(E)$ be bounded from below, $w_1\in\mathbb{M}(E\times\mathbb{U})$ be bounded from below, and $w_2\in\mathbb{M}(\partial E\times\mathbb{U})$. Then the mappings $G_\alpha g(x,\Theta)$, $L_\alpha w_1(x,\Theta)$, and $H_\alpha w_2(x,\Theta)$ defined on $E\times\mathbb{V}^r$ with values in $\mathbb{R}$ are $\mathcal{B}(E\times\mathbb{V}^r)$-measurable.

For $\alpha\geq 0$, $h\in\mathbb{M}(E)$ bounded from below and $\rho\in\mathbb{R}$ we present now the (ordinary) one-stage optimization operator $\mathcal{T}_\alpha(\rho,h)$ and the relaxed one-stage optimization operator $\mathcal{R}_\alpha(\rho,h)$ as follows. For $x\in E$ set

$$\mathcal{T}_\alpha(\rho,h)(x)=\inf_{\Theta\in\mathbb{V}(x)}\Big\{-\rho\mathcal{L}_\alpha(x,\Theta)+ L_\alpha f(x,\Theta)+H_\alpha r(x,\Theta)+G_\alpha h(x,\Theta)\Big\}, \tag{8}$$

$$\mathcal{R}_\alpha(\rho,h)(x)=\inf_{\Theta\in\mathbb{V}^r(x)}\Big\{-\rho\mathcal{L}_\alpha(x,\Theta)+ L_\alpha f(x,\Theta)+H_\alpha r(x,\Theta)+G_\alpha h(x,\Theta)\Big\}. \tag{9}$$

In particular for $\alpha=0$ we write for simplicity $\mathcal{T}_0=\mathcal{T}$, $\mathcal{R}_0=\mathcal{R}$. We have the following measurable selector sets:

$$\mathcal{S}_{\mathbb{U}} = \Big\{ u \in \mathbb{M}(\overline{E}, \mathbb{U}) : (\forall x \in \overline{E}), u(x) \in \mathbb{U}(x) \Big\},$$
$$\mathcal{S}_{\mathbb{V}} = \Big\{ (\nu, \nu_\partial) \in \mathbb{M}(E, \mathbb{V}) : (\forall x \in E), \big(\nu(x),\ \nu_\partial(x)\big) \in \mathbb{V}(x) \Big\},$$
$$\mathcal{S}_{\mathbb{V}^r} = \Big\{ (\mu, \mu_\partial) \in \mathbb{M}(E, \mathbb{V}^r) : (\forall x \in E), \big(\mu(x),\ \mu_\partial(x)\big) \in \mathbb{V}^r(x) \Big\}.$$

Remark 3.3: Notice that $u \in \mathcal{S}_{\mathbb{U}}$ characterizes what we call an ordinary feedback control since the control law $u(x)$ only depends on the value of the state variable x. On the other hand $(\nu, \nu_\partial) \in \mathcal{S}_{\mathbb{V}}$ characterizes a deterministic (also called ordinary) control law for the one stage problem since that, starting from x, it defines the control law for all $t \in [0, t_*(x))$ through the function $\nu(x,t)$ and at $t = t_*(x)$ (if $t_*(x) < \infty$) through ν_∂. Finally $(\mu, \mu_\partial) \in \mathcal{S}_{\mathbb{V}^r}$ characterizes a relaxed control law for the one stage problem since that, starting from x, it defines a probability over the feasible control actions for all $t \in [0, t_*(x))$ through the probability measure $\mu(x,t)$ and at $t = t_*(x)$ (if $t_*(x) < \infty$) through the probability measure μ_∂.

We have the following result showing that there exists a natural mapping from $\mathcal{S}_{\mathbb{U}}$ into $\mathcal{S}_{\mathbb{V}}$.

Proposition 3.4: If $\widehat{u} \in \mathcal{S}_{\mathbb{U}}$ then the mapping $\widehat{u}_\phi : x \to \big(\widehat{u}(\phi(x,.)), \widehat{u}(\phi(x, t_*(x)))\big)$ of the space E into $\mathbb{V}$ belongs to $\mathcal{S}_{\mathbb{V}}$.

Proof: From Lemma A.3 in [9] and item (i) of Lemma 3 in [10] it follows that the mapping $x \to \widehat{u}(\phi(x,.))$ of the space E into $\mathbb{M}(\mathbb{R}_+, \mathbb{U})$ is measurable. Moreover, for all $(x,t) \in E \times \mathbb{R}_+$, $\widehat{u}(\phi(x,t)) \in \mathbb{U}(\phi(x,t))$. Therefore, $\widehat{u}_\phi$ belongs to $\mathcal{S}_{\mathbb{V}}$. ∎

IV. Ordinary optimal feedback control

In this section we will study the existence and characterization of and ordinary optimal feedback measurable selector for the relaxed one-stage optimization operator. For $\alpha \geq 0$, $h \in \mathbb{M}(E)$ bounded from below, $\rho \in \mathbb{R}$ and $x \in E$ set

$$\overline{w}(x) = \mathcal{R}_\alpha(\rho, h)(x). \tag{10}$$

We present next the assumptions and results that will guarantee some convergence and lower semicontinuity properties with respect to the weak* topology of the operators (5), (6), (7) that appear in the one-stage optimization problem (10). From now on we will consider the following assumptions.

A.9) For each $x \in E$, $\lambda(x,.) : \mathbb{U}(x) \mapsto \mathbb{R}_+$ is continuous.

A.10) There exists a sequence of measurable functions $(f_j)_{j \in \mathbb{N}}$ in $\mathbb{M}(\overline{E} \times \mathbb{U})^+$ such that for all $y \in \overline{E}$, $f_j(y,.) \uparrow f(y,.)$ as $j \to \infty$ and $f_j(y,.) \in \mathbb{C}(\mathbb{U}(y))$.

A.11) There exists a sequence of measurable functions $(r_j)_{j \in \mathbb{N}}$ in $\mathbb{M}(\partial E \times \mathbb{U})^+$ such that for all $z \in \partial E$, $r_j(z,.) \uparrow r(z,.)$ as $j \to \infty$ and $r_j(z,.) \in \mathbb{C}(\mathbb{U}(z))$.

A.12) For all $x \in \overline{E}$ and $g \in \mathbb{B}(E)$, $Qg(x,.) : \mathbb{U}(x) \mapsto \mathbb{R}$ is continuous.

A.13) There exists $\xi \in \mathbb{M}(E)^+$, $\xi \geq 0$, such that

a) $\lambda(y,a) \geq \xi(y)$ for all $y \in E$ and $a \in \mathbb{U}(y)$,

b) $\displaystyle\int_0^{t_*(x)} e^{-\int_0^t \xi(\phi(x,s))ds} dt < \infty$,

c) $e^{-\int_0^{t_*(x)} \xi(\phi(x,s))ds} = 0$ whenever $t_*(x) = \infty$,

d) $\displaystyle\int_0^{t_*(x)} e^{-\int_0^t \xi(\phi(x,s))ds} \times \sup_{a \in \mathbb{U}(\phi(x,t))} f(\phi(x,t),a) dt < \infty$.

The next proposition presents some important convergence results of the operators (5), (6), (7) with respect to the weak* topology (recall (4) for the convergence in this topology). The proof of the proposition is in [6].

Proposition 4.1: Consider $\alpha \in \mathbb{R}_+$ and a non increasing sequence of positive numbers $\{\alpha_k\}$, $\alpha_k \downarrow \alpha$, a sequence of functions $h_{\alpha_k} \in \mathbb{M}(E)$ uniformly bounded from below by a positive constant K_h (that is, $h_{\alpha_k}(y) \geq -K_h$ for all $y \in E$). Set $h = \varliminf_{k \to \infty} h_{\alpha_k}$. For $x \in E$, consider $\Theta_n = \big(\mu_n, \mu_{\partial,n}\big) \in \mathbb{V}^r(x)$ and $\Theta = \big(\mu, \mu_\partial\big) \in \mathbb{V}^r(x)$ such that $\Theta_n \to \Theta$. We have the following results:

a) $\lim_{n \to \infty} \mathcal{L}_{\alpha_n}(x, \Theta_n) = \mathcal{L}_\alpha(x, \Theta)$.

b) $\varliminf_{n \to \infty} L_{\alpha_n} f(x, \Theta_n) \geq L_\alpha f(x, \Theta)$.

c) $\varliminf_{n \to \infty} H_{\alpha_n} r(x, \Theta_n) \geq H_\alpha r(x, \Theta)$.

d) $\varliminf_{n \to \infty} G_{\alpha_n} h_{\alpha_n}(x, \Theta_n) \geq G_\alpha h(x, \Theta)$.

The lower semicontinuity properties mentioned at the beginning of this section follow easily from this proposition as stated in the next corollary.

Corollary 4.2: Consider $h \in \mathcal{M}(E)$ bounded from below. We have the following results:

a) $\mathcal{L}_\alpha(x, \Theta)$ is continuous on $\mathbb{V}^r(x)$.

b) $L_\alpha f(x, \Theta)$ is lower semicontinuous on $\mathbb{V}^r(x)$.

c) $H_\alpha r(x, \Theta)$ is lower semicontinuous on $\mathbb{V}^r(x)$.

d) $G_\alpha h(x, \Theta)$ is lower semicontinuous on $\mathbb{V}^r(x)$.

Proof: By taking $\alpha_k = \alpha \geq 0$, $h_{\alpha_k} = h$ in Proposition 4.1 the results follow. ∎

We show next that under assumptions A.9)-A.13) there exists an ordinary feedback optimal control (see Remark 3.3) for the one-stage optimization equation (10). First we need the following definition.

Definition 4.3: Consider $w \in \mathbb{M}(E)$ and $h \in \mathbb{M}(E)$ bounded from below. Denote by $\widehat{u}(w,h) \in \mathcal{S}_{\mathbb{U}}$ the measurable selector satisfying

$$\inf_{a \in \mathbb{U}(x)} \{ f(x,a) - \lambda(x,a) \Big[w(x) - Qh(x,a) \Big] \} = f(x, \widehat{u}(w,h)(x)) - \lambda(x, \widehat{u}(w,h)(x)) \Big[w(x) - Qh(x, \widehat{u}(w,h)(x)) \Big],$$
$$\inf_{a \in \mathbb{U}(z)} \{ r(z,a) + Qh(z,a) \} = r(z, \widehat{u}(w,h)(z)) + Qh(z, \widehat{u}(w,h)(z)),$$

and $\widehat{u}_\phi(w,h) \in \mathcal{S}_{\mathbb{V}}$ the measurable selector derived from $\widehat{u}(w,h)$ as in Proposition 3.4.

The existence of $\widehat{u}(w,h)$ follows from assumptions A.9)-A.12) and Theorem 3.3.5 in [5].

The next theorem was proved in [6]. This proof follows similar lines as in [3], [4], where the connection between $\overline{w}$ with $\alpha = 0$ and the control problem (1) was also established.

Theorem 4.4: Let $\alpha \geq 0$, $\rho \in \mathbb{R}_+$ and $h \in \mathbb{M}(E)$ be bounded from below. Suppose that the function $\overline{w}$ defined in equation (10) is such that $\overline{w}(x) \in \mathbb{R}$ for all $x \in E$. Then $\overline{w} \in \mathbb{M}^{ac}(E)$ and the feedback measurable selector $\widehat{u}_\phi(\overline{w}, h) \in \mathcal{S}_{\mathbb{V}}$ (see Definition 4.3) satisfies the following one-stage optimization problems:

$$\begin{aligned}\mathcal{R}_\alpha(\rho,h)(x) = \mathcal{T}_\alpha(\rho,h)(x) = &-\rho\mathcal{L}_\alpha(x,\widehat{u}_\phi(\overline{w},h)(x))\\ &+ L_\alpha f(x,\widehat{u}_\phi(\overline{w},h)(x)) + H_\alpha r(x,\widehat{u}_\phi(\overline{w},h)(x))\\ &+ G_\alpha h(x,\widehat{u}_\phi(\overline{w},h)(x)).\end{aligned}$$

Note that in particular we have from the previous theorem that $\mathcal{R}_\alpha(\rho,h)(x) = \mathcal{T}_\alpha(\rho,h)(x)$.

V. Optimality equation for the discounted case

In this section we derive under the assumptions made in the previous sections an optimality equation for the discounted optimal control problem (2). As usual in this kind of problem we characterize first the optimality equation for the truncated on the jump times T_m problems (3) and then take the limit as $m \to \infty$. Throughout this section we consider $\alpha > 0$ fixed. For any $g \in \mathbb{M}(E)^+$, we set $\mathcal{W}g$ as the function on E defined as $\mathcal{W}g(x) = \mathcal{R}_\alpha(0,g)(x)$ for $x \in E$. The following proposition is an immediate consequence of the results derived in the previous section.

Proposition 5.1: We have that $\mathcal{W}$ maps $\mathbb{M}(E)^+$ into itself and there exists $\widehat{u} \in \mathcal{S}_{\mathbb{U}}$ such that $\widehat{u}_\phi \in \mathcal{S}_{\mathbb{V}}$ and satisfies

$$\begin{aligned}\mathcal{W}g(x) =& L_\alpha f(x,\widehat{u}_\phi(x)) + H_\alpha r(x,\widehat{u}_\phi(x)) +\\ & G_\alpha g(x,\widehat{u}_\phi(x)).\end{aligned} \tag{11}$$

Proof: From Theorem 4.4, we obtain the first equality and the existence and characterization of $\widehat{u} \in \mathcal{S}_{\mathbb{U}}$ such that $\widehat{u}_\phi \in \mathcal{S}_{\mathbb{V}}$ satisfies equation (11). Now applying Proposition 3.2 and by using the fact that $f \in \mathbb{M}(\overline{E} \times \mathbb{U})^+$ and $r \in \mathbb{M}(\partial E \times \mathbb{U})^+$, we obtain that $\mathcal{W}g \in \mathbb{M}(E)^+$. ∎

By using the fact that $\mathcal{W}$ is a monotone operator, the sequence of functions $(v_m)_{m\in\mathbb{N}} \in \mathbb{M}(E)^+$ defined as $v_{m+1} = \mathcal{W}v_m$, $v_0 = 0$, is monotone non-decreasing. We have the following proposition, proved in [6]. Similar results can be found in [3], [4].

Proposition 5.2: For all $x \in E$ and $m \in \mathbb{N}$ we have that $v_m(x) = \inf_{U\in\mathcal{U}} \mathcal{D}_m^\alpha(U,x)$.

Since the functions $v_m \in \mathbb{M}(E)^+$ are non-decreasing, there exists $v \in \mathbb{M}(E)^+$ such that $v_m \uparrow v$, and recalling from Proposition 5.2 that $v_m(x) = \inf_{U\in\mathcal{U}} \mathcal{D}_m^\alpha(U,x) \leq \mathcal{J}_{\mathcal{D}}^\alpha(x)$, it follows that $v \leq \mathcal{J}_{\mathcal{D}}^\alpha$. We need the following propositions:

Proposition 5.3: If $h \in \mathbb{M}(E)^+$ is such that $h(x) \geq \mathcal{W}h(x)$ then $h(x) \geq \mathcal{J}_{\mathcal{D}}^\alpha(x)$.

Proof: By using Theorem 4.4 with $\rho = 0$, we obtain that there exists $\widehat{u}_\phi \in \mathcal{S}_{\mathbb{V}}$ such that

$$\begin{aligned}h(x) &\geq \inf_{\Upsilon\in\mathbb{V}(x)}\Big\{L_\alpha f(x,\Upsilon) + H_\alpha r(x,\Upsilon) + G_\alpha h(x,\Upsilon)\Big\}\\ &= L_\alpha f(x,\widehat{u}_\phi(x)) + H_\alpha r(x,\widehat{u}_\phi(x)) + G_\alpha h(x,\widehat{u}_\phi(x)).\end{aligned}$$

Define $w(x) = L_\alpha f(x,\widehat{u}_\phi(x)) + H_\alpha r(x,\widehat{u}_\phi(x)) + G_\alpha h(x,\widehat{u}_\phi(x))$. Clearly $w(x) \geq 0$. Following the same arguments as in [3], [4] (see also [6]), we have that that there exists $\widehat{U} \in \mathcal{U}$ such that for all $m \in \mathbb{N}$,

$$\begin{aligned}\mathcal{D}_m^\alpha(\widehat{U},x) &\leq -E_x^{\widehat{U}}\Big[e^{-\alpha t\wedge T_m} w\big(X(t\wedge T_m)\big)\Big] + h(x)\\ &\leq h(x).\end{aligned}$$

From assumption A.4) (which implies that $T_m \to \infty$ $P^{\widehat{U}}$ a.s.), we have that

$$\begin{aligned}E_x^{\widehat{U}}\bigg[&\int_0^t e^{-\alpha s}\Big[f\big(X(s),\hat{u}(N(s),Z(s),\tau(s))\big)\Big]ds\\ &+\int_0^t e^{-\alpha s} r\big(X(s-),\hat{u}_\partial(N(s),X(s-))\big)dp^*(s)\bigg] \leq h(x),\end{aligned}$$

and taking the limit as $t \to \infty$ we obtain that $h(x) \geq \mathcal{J}_{\mathcal{D}}^\alpha(x)$. ∎

Proposition 5.4: We have that $v(x) = \mathcal{W}v(x)$.

Proof: Let us show first that $v(x) \leq \mathcal{W}v(x)$. By using the definition of $\mathcal{W}$ we have for any $\Upsilon \in \mathbb{V}^r(x)$ that

$$v_{m+1}(x) \leq L_\alpha f(x,\Upsilon) + H_\alpha r(x,\Upsilon) + G_\alpha v_m(x,\Upsilon).$$

Taking the limit as $m \uparrow \infty$ and from the monotone convergence theorem we get that $v(x) = \lim_{m\to\infty} v_{m+1}(x) \leq L_\alpha f(x,\Upsilon)+H_\alpha r(x,\Upsilon)+G_\alpha v(x,\Upsilon)$ showing that $v(x) \leq \mathcal{W}v(x)$. From Proposition 5.1, there exists for any $m \in \mathbb{N}$, $u^m \in \mathcal{S}_{\mathbb{U}}$ such that

$$\begin{aligned}\mathcal{W}v_m(x) =& L_\alpha f(x,u_\phi^m(x)) + H_\alpha r(x,u_\phi^m(x)) +\\ & G_\alpha v_m(x,u_\phi^m(x)).\end{aligned}$$

Fix $x \in E$. Since $u_\phi^m(x) \in \mathbb{V}(x) \subset \mathbb{V}^r(x)$ and $\mathbb{V}^r(x)$ is compact we can find a further subsequence, still written as $u_\phi^m(x)$ for notational simplicity, such that $u_\phi^m(x) \to \hat{\Theta} \in \mathbb{V}^r(x)$. From Proposition 4.1,

$$\begin{aligned}v(x) &= \lim_{m\to\infty} v_{m+1}(x)\\ &= \varliminf_{m\to\infty}\Big\{L_\alpha f(x,u_\phi^m(x)) + H_\alpha r(x,u_\phi^m(x)) +\\ &\quad G_\alpha v_m(x,u_\phi^m(x))\Big\}\\ &\geq L_\alpha f(x,\hat{\Theta}) + H_\alpha r(x,\hat{\Theta}) + G_\alpha v(x,\hat{\Theta})\\ &\geq \inf_{\Theta\in\mathbb{V}^r(x)}\Big\{L_\alpha f(x,\Theta) + H_\alpha r(x,\Theta) + G_\alpha v(x,\Theta)\Big\}\\ &= \mathcal{W}v(x),\end{aligned}$$

giving the result. ∎

We have the following theorem characterizing the optimality equation for the discounted optimal control problem (2) and showing the convergence of the truncated problems.

Theorem 5.5: We have that $v_n \uparrow \mathcal{J}_{\mathcal{D}}^\alpha$ and $\mathcal{J}_{\mathcal{D}}^\alpha(x) = \mathcal{W}\mathcal{J}_{\mathcal{D}}^\alpha(x)$.

Proof: All we need to show is that $\mathcal{J}_{\mathcal{D}}^\alpha(x) \leq v(x)$. But this is immediate from Propositions 5.4 and 5.3. ∎

VI. THE VANISHING APPROACH

Our final main result presents conditions for the existence and characterization of an ordinary optimal feedback control for the long run average cost using the so-called vanishing discount approach (see Theorem 6.4). First we have the following result, which traces a parallel with the Abelian Theorem.

Proposition 6.1: We have that

$$\limsup_{\alpha\downarrow 0} \alpha\mathcal{J}^{\alpha}_{\mathcal{D}}(x) \leq \mathcal{J}_{\mathcal{A}}(x).$$

Proof: See Theorem 1, chapter 5 in [11]. ■

We shall add the following assumptions for the discounted problems:

A.14) There exists a state $x_0 \in E$, numbers $\beta > 0$, $C \geq 0$, $K_h \geq 0$, and a nonnegative function $b(.)$ such that for all $x \in E$ and $\alpha \in (0, \beta]$, $\rho_\alpha \leq C$, where $\rho_\alpha = \alpha\mathcal{J}^{\alpha}_{\mathcal{D}}(x_0)$ and $-K_h \leq h_\alpha(x) \leq b(x)$ where $h_\alpha(x) = \mathcal{J}^{\alpha}_{\mathcal{D}}(x) - \mathcal{J}^{\alpha}_{\mathcal{D}}(x_0)$.

We have the following propositions:

Proposition 6.2: There exists a decreasing sequence of positive numbers $\alpha_k \downarrow 0$ such that $\rho_{\alpha_k} \to \rho$ and for all $x \in E$, $\lim_{k\to\infty} \alpha_k \mathcal{J}^{\alpha_k}(x) = \rho$.

Proof: See Lemma in [5], page 88. ■

Proposition 6.3: Set $h = \varliminf_{k\to\infty} h_{\alpha_k}$. Then for all $x \in E$, $h(x) \geq -K_h$ and $h(x) \geq \mathcal{T}(\rho, h)(x)$.

Proof: From Proposition 5.1 and Theorem 5.5 we have that the following equation is satisfied for each $\alpha > 0$ and $x \in E$:

$$h_\alpha(x) = \mathcal{T}_\alpha(\rho_\alpha, h_\alpha)(x) = -\rho_\alpha \mathcal{L}_\alpha(x, u^\alpha_\phi(x)) + L_\alpha f(x, u^\alpha_\phi(x)) + H_\alpha r(x, u^\alpha_\phi(x)) + G_\alpha h_\alpha(x, u^\alpha_\phi(x)), \quad (12)$$

for $u^\alpha_\phi \in \mathcal{S}_{\mathbb{V}}$. For $x \in E$ fixed and for all $k \in \mathbb{N}$, $u^{\alpha_k}_\phi(x) \in \mathbb{V}(x) \subset \mathbb{V}^r(x)$ and since $\mathbb{V}^r(x)$ is compact we can find a further subsequence, still written as $u^{\alpha_k}_\phi(x)$ for notational simplicity, such that $u^{\alpha_k}_\phi(x) \to \hat{\Theta} \in \mathbb{V}^r(x)$. Combining Proposition 4.1 and equation (12),

$$\begin{aligned} h(x) = \varliminf_{k\to\infty} h_{\alpha_k}(x) = \varliminf_{k\to\infty} \Big\{ -\rho_{\alpha_k} \mathcal{L}_{\alpha_k}(x, u^{\alpha_k}_\phi(x)) + \\ L_{\alpha_k} f(x, u^{\alpha_k}_\phi(x)) H_{\alpha_k} r(x, u^{\alpha_k}_\phi(x)) + G_{\alpha_k} h_{\alpha_k}(x, u^{\alpha_k}_\phi(x)) \Big\} \\ \geq -\rho\mathcal{L}(x, \hat{\Theta}) + Lf(x, \hat{\Theta}) + Hr(x, \hat{\Theta}) + Gh(x, \hat{\Theta}). \quad (13) \end{aligned}$$

Therefore, from Theorem 4.4, it follows $h(x) \geq \mathcal{R}(\rho, h)(x) = \mathcal{T}(\rho, h)(x)$, showing the result ■

Our final result establishes the existence and characterization of an optimal control strategy for the long run average cost problem. We need to reinforce assumption A.13) by the following hypothesis:

A.15) The mapping $\xi \in \mathbb{M}(E)^+$ defined in assumption A.13) satisfies $\int_0^{t_*(x)} e^{-\int_0^t \xi(\phi(x,s))ds} dt < K_\varrho$, for all $x \in E$ where $K_\varrho \in \mathbb{R}$.

Theorem 6.4: Suppose that assumptions A.1)-A.15) hold. Then there exists $h \in \mathbb{E}$ bounded from below such that $h(x) \geq \mathcal{T}(\rho, h)(x)$. Set $w = \mathcal{T}(\rho, h)$ and $\widehat{\gamma} = \widehat{u}(w, h) \in \mathcal{S}_{\mathbb{U}}$ be an optimal measurable selector as defined in Definition 4.3. Let $\widehat{\Gamma}_\phi = (\widehat{\gamma}_\phi, \widehat{\gamma}_{\phi,\partial})$ be as in Proposition 3.4 and define the control $\widehat{U} = (\hat{u}, \hat{u}_\partial)$ by $\hat{u}(n, x, t) = \widehat{\gamma}_\phi(x, t)$, $\hat{u}_\partial(n, x) = \widehat{\gamma}_{\phi,\partial}(x)$. Then $\widehat{U} \in \mathcal{U}$ is optimal. Moreover, $\rho = \mathcal{J}_{\mathcal{A}}(x) = \mathcal{A}(\widehat{U}, x)$.

Proof: Combining Theorem 4.4 and Proposition 6.3 and along the same lines as in [3], [4] (see also [6]), it follows that $\widehat{U} \in \mathcal{U}$ defined as above satisfies

$$\begin{aligned} E^{\widehat{U}}_x \bigg[\int_0^{t\wedge T_m} \Big[f\big(X(s), \hat{u}(N(s), Z(s), \tau(s))\big) \Big] ds + \\ \int_0^{t\wedge T_m} r\big(X(s-), \hat{u}_\partial(N(s), X(s-))\big) dp^*(s) \bigg] \\ \leq E^{\widehat{U}}_{(x,k)} \Big[\rho[t \wedge T_m] - w\big(X(t \wedge T_m)\big) \Big] + w(x). \end{aligned}$$

Combining Proposition 6.3 and assumption A.15), we obtain that $w(x) \geq -\rho K_\varrho - K_h$. Moreover, we have, from assumption A.4) that $T_m \to \infty$ $P^{\widehat{U}}$ a.s. . Consequently,

$$\begin{aligned} E^{\widehat{U}}_x \bigg[\int_0^t \Big[f\big(X(s), \hat{u}(N(s), Z(s), \tau(s))\big) \Big] ds + \\ \int_0^t r\big(X(s-), \hat{u}_\partial(N(s), X(s-))\big) dp^*(s) \bigg] \\ \leq \rho\, t + \rho K_\varrho + K_h + w(x), \end{aligned}$$

showing that $\rho \geq \mathcal{A}(\widehat{U}, x)$. From Proposition 6.1 and Proposition 6.2, we have $\rho \leq \mathcal{J}_{\mathcal{A}}(x)$ completing the proof. ■

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