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Synthesis of Adaptive Robust Output Feedback Controllers for a Class of Uncertain Linear Systems

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***Abstract*— This paper discusses a design problem of an adaptive robust output feedback controller for a class of uncertain linear systems. The uncertainties under consideration satisfy the matching condition and are bounded, but their upper bounds are unknown. The proposed adaptive robust output feedback controller consists of a fixed gain controller, an adjustable parameter and a variable gain controller. In this paper, we present a design method of an adaptive robust output feedback control system. Finally, illustrative examples are presented to show the efficiency of the proposed adaptive robust controller.**

I. INTRODUCTION

Robustness of control systems to uncertainties has always been the central issue in feedback control and therefore for linear dynamical systems with unknown parameters, a large number of design methods of robust state feedback controllers have been presented[1], [2]. In particular, for the state feedback controller of a linear system with structured uncertainties, a connection between quadratic stabilization and $\mathcal{H}^\infty$ control has been established[3].

On the other hand, since not all the state are measurable in practical systems because of technical, physical and/or economic reasons, the control scheme may be designed via observer-based controllers[4], [5] or output feedback one[6]. In Schmitendorf[4], a method for obtaining observer-based robust controllers has been presented and they have shown that using an estimate of the state reduces the magnitude of the uncertainty bounds for which stability can be guaranteed. We have also proposed an observer-based robust controller which achieves not only robust stability but also reducing the error between the plant output and the desired one for linear systems with structured uncertainties[5]. In this work, adopting 2-stage design approach, the design problem of the observer-based controller can be reduced to constrained convex optimization problem. Additionally, a design method of an output feedback guaranteed cost controller has been presented[7]. In the work of Moheimani and Petersen[7], in order to obtain output feedback gain of the guaranteed cost control, it is necessary to solve a set of cross-coupled algebraic Riccati equations and algebraic Lyapunov equations. Besides, algorithms by Geromel[8] and Iwasaki[9] use linear matrix inequalities (LMI) methods to design static output feedback controllers based on a set of Lyapunov inequalities coupled by the constraint that one Lyapunov matrix is the inverse of another. It is well known that in the case of designing control schemes solely based on an output feedback controller, it is difficult to find the output feedback gain for linear systems even without uncertainties[6].

By the way, we have proposed a robust controller with an adaptive compensation input improving transient behavior for a class of uncertain linear systems[10]. The adaptive compensation input is tuned on-line based on the information about parameter uncertainties. Additionally, for linear systems with matched uncertainties of which upper bounds are unknown, some design methods of adaptive robust controllers have been suggested[11], [12], [13]. These adaptive robust controllers consist of a fixed gain controller and a variable gain controller, and the variable gain controller is tuned by updating laws so as to reduce the effect of uncertainties. However, so far the design problem of adaptive robust output feedback controllers for linear systems with matched uncertainties of which upper bounds are unknown has little been considered as far as we know.

From this viewpoint, this paper deals with a design problem of an adaptive robust output feedback controller for linear systems with uncertainties of which upper bounds are unknown. The proposed adaptive robust output feedback controller consists of a fixed gain controller and a variable gain controller with an adjustable parameter. The fixed gain controller is determined by using the nominal system, and the adjustable parameter and the variable gain controller are designed such that the effect of uncertainties is reduced. In this paper, we show a design method of the proposed adaptive robust output feedback controller.

This paper is organized as follows. In Sec. 2, we show the notation used in this paper and introduce preparatory results. In Sec. 3, we define the class of uncertain linear systems under consideration, and introduce an adaptive robust output feedback controller. Sec. 4 contains our main results and a design method of the adaptive robust output feedback controller is presented. Finally, illustrative examples are included to illustrate the results developed in this paper.

II. PRELIMINALIES

In this section, we show notations, useful and well-known lemmas which are used in this paper.

In the sequel, we use the following notation. For a matrix $\mathcal{A}$, the transpose of a matrix $\mathcal{A}$ and the inverse of one are denoted by $\mathcal{A}^T$ and $\mathcal{A}^{-1}$ respectively and $H_e\{\mathcal{A}\}$ means $\mathcal{A}+$

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$$\begin{array}{l}\lambda_{\min}\{\mathcal{Z}\} I_n - \mathcal{Z} \leq 0 \;\Rightarrow\; \lambda_{\min}\{\mathcal{Z}\} \xi^T\xi - \gamma^T\mathcal{Z}\mathcal{W}\gamma = \gamma^T\left(\lambda_{\min}\{\mathcal{Z}\} I_n - \mathcal{Z}\right)\mathcal{F}^T\mathcal{F}\gamma \leq 0 \\ \lambda_{\max}\{\mathcal{Z}\} I_n - \mathcal{Z} \geq 0 \;\Rightarrow\; \lambda_{\max}\{\mathcal{Z}\} \xi^T\xi - \gamma^T\mathcal{Z}\mathcal{W}\gamma = \gamma^T\left(\lambda_{\max}\{\mathcal{Z}\} I_n - \mathcal{Z}\right)\mathcal{F}^T\mathcal{F}\gamma \geq 0\end{array} \tag{3}$$

$\mathcal{A}^T$. Also, I_n represents n-dimensional identity matrix and $\mathrm{vec}(\mathcal{A})$ denotes the column vector of the matrix $\mathcal{A}$, i.e. the operator "vec" vectorizes a matrix by stacking its columns. For real symmetric matrices $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} > \mathcal{B}$ (resp. $\mathcal{A} \geq \mathcal{B}$) means that $\mathcal{A} - \mathcal{B}$ is positive (resp. nonnegative) definite matrix. For a square matrix $\mathcal{S}$, $\mathrm{Trace}\{\mathcal{S}\}$ denotes its trace. For a vector $\alpha \in \Re^n$, $||\alpha||$ denotes standard Euclidian norm and for a matrix $\mathcal{A}$, $||\mathcal{A}||$ represents a its induced norm. The symbol "$\stackrel{\triangle}{=}$" means equality by definition. Besides, for a symmetric matrix $\mathcal{P}$, $\lambda_{\max}\{\mathcal{P}\}$ (resp. $\lambda_{\min}\{\mathcal{P}\}$) represent the maximal eigenvalue (resp. miminal eigenvalue)*.

Furthermore, we show the following useful lemmas used in this paper.

Lemma 1: For arbitrary vectors λ and ξ and the matrices $\mathcal{G}$ and $\mathcal{H}$ which have the appropriate dimensions, the following relation holds.

$$\begin{aligned} H_e\left\{\lambda^T\mathcal{G}\Delta(t)\mathcal{H}\xi\right\} &\leq 2\left\|\mathcal{G}^T\lambda\right\|\left\|\Delta(t)\mathcal{H}\xi\right\| \\ &\leq 2\delta^*\left\|\mathcal{G}^T\lambda\right\|\left\|\mathcal{H}\xi\right\| \end{aligned} \tag{1}$$

where $\Delta(t) \in \Re^{p\times q}$ is a time-varying matrix satisfying $\left\|\Delta(t)\right\| \leq \delta^*$.

Proof: The above relation is easily obtained by Schwartz's inequality[14]. ∎

Lemma 2: For a vector $\gamma \in \Re^n$ and a symmetric positive definite matrix $\mathcal{Z} \in \Re^{n\times n}$ and a symmetric nonnegative definite matrix $\mathcal{W} = \mathcal{F}^T\mathcal{F} \in \Re^{n\times n}$ ($\mathcal{F} \in \Re^{p\times n}$), the following relation holds.

$$\lambda_{\min}\{\mathcal{Z}\}\left\|\xi\right\|^2 \leq \gamma^T\mathcal{Z}\mathcal{W}\gamma \leq \lambda_{\max}\{\mathcal{Z}\}\left\|\xi\right\|^2 \tag{2}$$

where ξ is a vector given by $\xi = \mathcal{F}\gamma$.

Proof: The inequalities $\lambda_{\min}\{\mathcal{Z}\} I_n - \mathcal{Z} \leq 0$ and $\lambda_{\max}\{\mathcal{Z}\} I_n - \mathcal{Z} \geq 0$ are obvious. Additionally, one can see from (3) that the above relation is fulfilled. ∎

III. PROBLEM FORMULATION

Consider the uncertain linear system described by the following state equation† (see **Remark 1**).

$$\begin{aligned} \frac{d}{dt}x(t) &= \left(A + B\Delta(t)C\right)x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{4}$$

where $x(t) \in \Re^n, u(t) \in \Re^m$ and $y(t) \in \Re^l$ are the vectors of the state, the control input and the measurement output, respectively, and the matrices A, B and C denote the nominal values of the system parameters. In (4), $\Delta(t) \in \Re^{m\times l}$ represents unknown time-varying parameters. Besides, we assume that the pair (A, B) is stabilizable and introduce the following assumption.

$$B^T = \mathcal{D}C \tag{5}$$

where $\mathcal{D} \in \Re^{m\times l}$ is a known constant matrix.

In this paper, the unknown parameter $\Delta(t) \in \Re^{m\times l}$ is supposed to be bounded, but its upper bound is unknown. Namely, although there exists a positive constant $\delta^* \in \Re^1$ satisfying the following relation, δ^* is unknown.

$$\left\|\Delta(t)\right\| \leq \delta^* \tag{6}$$

Besides, the nominal system, ignoring the unknown parameters in (4), is given by

$$\begin{aligned} \frac{d}{dt}\overline{x}(t) &= A\overline{x}(t) + B\overline{u}(t) \\ \overline{y}(t) &= C\overline{x}(t) \end{aligned} \tag{7}$$

In this paper, the nominal system (7) is supposed to be stabilizable via static output feedback control. Therefore, there exist an output feedback control $\overline{u}(t) = K\overline{y}(t)$ (i.e. a fixed gain matrix $K \in \Re^{m\times l}$) such that the following closed-loop system is asymptotically stable.

$$\frac{d}{dt}\overline{x}(t) = A_K\overline{x}(t) \tag{8}$$

where A_K is a stable matrix given by $A_K \stackrel{\triangle}{=} A + BKC$‡.

Now on the basis of the work of Oya and Hagino[10], we introduce the error vectors $e(t) \stackrel{\triangle}{=} x(t) - \overline{x}(t)$ and $e_y(t) \stackrel{\triangle}{=} y(t) - \overline{y}(t)$. Beside, using the fixed gain matrix $K \in \Re^{m\times l}$, we consider the following control input for the uncertain linear system (4).

$$u(t) \stackrel{\triangle}{=} Ky(t) + \psi(y, e_y, \theta, \mathcal{L}, t) \tag{9}$$

where $\psi(y, e_y, \theta, \mathcal{L}, t)$ is a adaptive compensation input[10] for the purpose of reducing the effect of uncertainties and has the following form.

$$\psi(y, e_y, \theta, \mathcal{L}, t) \stackrel{\triangle}{=} \theta(t)\mathcal{L}(y, e_y, t)y(t) \tag{10}$$

In (10), $\theta(t) \in \Re^1$ and $\mathcal{L}(y, e_y, t) \in \Re^{m\times l}$ are an adjustable parameter and a variable gain matrix, respectively, and are determined such that the effect of uncertainties is reduced. From (4) and (7) – (10), we have the following error system.

$$\begin{aligned} \frac{d}{dt}e(t) &= A_K e(t) + B\Delta(t)Cx(t) + \theta(t)\mathcal{L}(y, e_y, t)y(t) \\ e_y(t) &= Ce(t) \end{aligned} \tag{11}$$

Note that the adjustable parameter $\theta(t) \in \Re^1$ is not an estimate of the unknown upper bound $\delta^* \in \Re^1$ for uncertainties, and if asymptotical stability of the uncertain error system (11) is ensured, then asymptotical stability of the uncertain linear system (4) is also guaranteed, because the nominal system (8) is asymptotically stable.

*For a symmetric matrix $\mathcal{P}$, eigenvalues of $\mathcal{P}$ are real number[14].

†i.e. uncertainties satisfy the matching condition[15].

‡Note that the feedback gain matrix $K \in \Re^{m\times l}$ is designed by using the existing result, e.g. Kucera and Souza[6] or Benton et al[16].

$$\mathcal{L}(y,e_y,t) = \begin{cases} -\dfrac{\left\|\mathcal{D}e_y(t)\right\|\mathcal{D}}{\lambda_{\min}(\mathcal{S})\left\|y(t)\right\|\left\|\mathcal{D}e_y(t)\right\|^2}\ e_y(t)y^T(t) & \text{if} \quad \mathcal{D}e_y(t) \neq 0 \quad \text{and} \quad y(t) \neq 0 \\ -\dfrac{\left\|\mathcal{D}e_y(t^-)\right\|\mathcal{D}}{\lambda_{\min}(\mathcal{S})\left\|y(t^-)\right\|\left\|\mathcal{D}e_y(t^-)\right\|^2}\ e_y(t^-)y^T(t^-) & \text{if} \quad \mathcal{D}e_y(t) \equiv 0 \quad \text{or} \quad y(t) \equiv 0 \end{cases} \tag{13}$$

$$\frac{d}{dt}\theta(t) = \frac{1}{\vartheta}\left\|\mathcal{D}e_y(t)\right\|\left\|y(t)\right\|$$

$$\frac{d}{dt}\mathcal{V}(e,\theta,t) = e^T(t)\left[H_e\left\{A_K^T\mathcal{S}\right\}\right]e(t) + 2e^T(t)\mathcal{S}\left\{B\Delta(t)Cx(t) + \theta(t)B\mathcal{L}(y,e_y,t)y(t)\right\} - 2\vartheta\left(\xi-\theta\right)\frac{d}{dt}\theta(t) \tag{15}$$

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(e,\theta,t) \le\ & e^T(t)\left[H_e\left\{A_K^T\mathcal{S}\right\}\right]e(t) + 2\delta^*\left\|\mathcal{D}C\mathcal{S}e(t)\right\|\left\|Cx(t)\right\| + 2\theta(t)e^T(t)\mathcal{S}C^T\mathcal{D}^T\mathcal{L}(y,e_y,t)y(t) \\ & - 2\vartheta\left(\xi-\theta(t)\right)\frac{d}{dt}\theta(t)\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(e,\theta,t) \le\ & e^T(t)\left[H_e\left\{A_K^T\mathcal{S}\right\}\right]e(t) + 2\xi\left\|\mathcal{D}e_y(t)\right\|\left\|y(t)\right\| + 2\theta(t)e^T(t)\mathcal{S}C^T\mathcal{D}^T\mathcal{L}(y,e_y,t)y(t) \\ & - 2\vartheta\left(\xi-\theta(t)\right)\frac{d}{dt}\theta(t)\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(e,\theta,t) \le\ & e^T(t)\left[H_e\left\{A_K^T\mathcal{S}\right\}\right]e(t) + 2\xi\left\|\mathcal{D}e_y(t)\right\|\left\|y(t)\right\| - 2\vartheta\left(\xi-\theta(t)\right)\frac{d}{dt}\theta(t) \\ & + 2\theta(t)e^T(t)\mathcal{S}C^T\mathcal{D}^T\left(-\frac{\left\|\mathcal{D}e_y(t)\right\|\mathcal{D}}{\lambda_{\min}(\mathcal{S})\left\|y(t)\right\|\left\|\mathcal{D}e_y(t)\right\|^2}\ e_y(t)y^T(t)\right)y(t) \\ \le\ & e^T(t)\left[H_e\left\{A_K^T\mathcal{S}\right\}\right]e(t) + 2\left(\xi-\theta(t)\right)\left\|\mathcal{D}e_y(t)\right\|\left\|y(t)\right\| - 2\vartheta\left(\xi-\theta(t)\right)\frac{d}{dt}\theta(t)\end{aligned} \tag{19}$$

From the above, our control objective in this paper is to design the adaptive compensation input $\psi(y,e_y,\theta,\mathcal{L},t) \in \Re^m$ which stabilizes the uncertain error system (11). That is to design the updating law of the adjustable parameter $\theta(t) \in \Re^1$ and the variable gain matrix $\mathcal{L}(y,e_y,t) \in \Re^{m\times l}$.

Remark 1: In this paper, we consider the uncertain dynamical system (4) which has uncertainties in the state matrix only. The proposed design scheme of the adaptive robust controller derived in next section can also be applied to the case that the uncertainties are included in both the system matrix and the input matrix. By introducing additional actuator dynamics and constituting an augmented system, uncertainties in the input matrix are embedded in the system matrix of the augmented system[17]. Therefore the same design procedure can be applied.

IV. Main Results

In this section, we show a design method of the variable matrix $\mathcal{L}(y,e_y,t) \in \Re^{m\times l}$ and the updating law of the adjustable parameter $\theta(t) \in \Re^1$ such that the uncertain error system (11) is asymptotically stable.

Now, we consider the nominal closed-loop system matrix $A_K \in \Re^{n\times n}$. Since the matrix A_K is asymptotically stable, there exist a symmetric positive definite matrix $\mathcal{S} \in \Re^{n\times n}$ which satisfies the following Lyapunov equation.

$$A_K^T\mathcal{S} + \mathcal{S}A_K = -\mathcal{Q} \tag{12}$$

where $\mathcal{Q} \in \Re^{n\times n}$ is a symmetric positive definite matrix.

The following theorem gives a design method of the variable gain matrix $\mathcal{L}(y,e_y,t) \in \Re^{m\times l}$ and the updating law of the adjustable parameter $\theta(t) \in \Re^1$.

Theorem 1: Consider the uncertain error system (11) and the variable gain matrix $\mathcal{L}(y,e_y,t) \in \Re^{m\times l}$ and the updating law of the adjustable parameter $\theta(t) \in \Re^1$.

Using the symmetric positive definite matrix $\mathcal{S} \in \Re^{n\times n}$ satisfiyng the Lyapunov equation (12), the variable gain matrix $\mathcal{L}(y,e_y,t) \in \Re^{m\times l}$ and the updating law of the adjustable parameter $\theta(t) \in \Re^1$ are determined by (13), and then the asymptotical stability of the uncertain error system (11) is guaranteed. In (13), $t^- = \lim_{\epsilon>0,\epsilon\to 0}(t-\epsilon)$[7] and $\vartheta \in \Re^1$ is a positive scalar, and $\vartheta \in \Re^1$ and the initial value of the time-varying parameter $\theta(t) \in \Re^1$, denoted by $\theta(0)$, are chosen by the designer.

Proof: Using symmetric positive definite matrix $\mathcal{S} \in \Re^{n\times n}$, we introduce the following quadratic function as a Lyapunov function candidate.

$$\mathcal{V}(e,\theta,t) \stackrel{\triangle}{=} e^T(t)\mathcal{S}e(t) + \vartheta\left(\xi-\theta(t)\right)^2 \tag{14}$$

where ξ is an unknown constant stated below. The time derivative of the function $\mathcal{V}(e,\theta,t)$ along the trajectory of the uncertain error system (11) is given by (15).

In the case of $\mathcal{D}e_y(t) \neq 0$ and $y(t) \neq 0$, using the assumptions (5) and (6), we have the inequality (16). Besides, for the upper bound of the term $\left\|\mathcal{D}C\mathcal{S}x(t)\right\|$ in (16), one can see that the following inequality holds.

$$\begin{aligned}\left\|\mathcal{D}C\mathcal{S}e(t)\right\| &\le \lambda_{\max}\left\{\mathcal{S}\right\}\left\|\mathcal{D}Ce(t)\right\| \\ &= \lambda_{\max}\left\{\mathcal{S}\right\}\left\|\mathcal{D}e_y(t)\right\|\end{aligned} \tag{17}$$

Thus, we obtain the inequality (18) for the time derivative of the quadratic function $\mathcal{V}(e,\theta,t)$ where ξ is unknown positive constant given by $\xi = \delta^* \times \lambda_{\max}\left\{\mathcal{S}\right\}$. Additionally,

$$\frac{d}{dt}\mathcal{V}(e,\theta,t) \le e^T(t)\left[H_e\left\{A_K^T\mathcal{S}\right\}\right]e(t) + 2(\xi-\theta(t))\left\|\mathcal{D}e_y(t)\right\|\left\|y(t)\right\| - 2\vartheta(\xi-\theta(t))\left(\frac{1}{\vartheta}\left\|\mathcal{D}e_y(t)\right\|\left\|y(t)\right\|\right)$$
$$= e^T(t)\left[H_e\left\{A_K^T\mathcal{S}\right\}\right]e(t) \tag{20}$$

$$\frac{d}{dt}x(t) = \begin{pmatrix} -2 & 0 & -6 \\ 0 & 0 & 1 \\ 3 & 0 & -7 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Delta(t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t) \tag{24}$$

using **Lemma 2** and the variable gain matrix (13) and after some trivial algebraic manipulations, from the inequality (18) we can further obtain the inequality (19) at the top of the previous page. In addition, substituting the updating law (13) of the adjustable parameter $\theta(t)$, we have the inequality (20).

Therefore we see from (12) and (20) that the following relation for the quadratic function $\mathcal{V}(e,\theta,t)$ is obtained.

$$\frac{d}{dt}\mathcal{V}(e,\theta,t) \le -e^T(t)\mathcal{Q}e(t) < 0 \quad \text{for} \quad \forall e(t) \neq 0 \tag{21}$$

In the above, we have proved the case of $\mathcal{D}e_y(t) \neq 0$ and $y(t) \neq 0$. Next, we consider the case of $\mathcal{D}e_y(t) \equiv 0$ or $y(t) \equiv 0$. In this case, one can see from (17) and (18) that the inequality (21) also holds.

Obviously, from the above discussion, the uncertain error system (11) is ensured to be stable, because the quadratic function $\mathcal{V}(e,\theta,t)$ becomes a Lyapunov function of the error system (11). Namely, asymptotical stability of the uncertain linear system (4) is also guaranteed.

It follows that the result of the theorem is true. Thus the proof of **Theorem 1** is completed. ■

Remark 2: In this paper for the uncertain linear system (4) satisfying the assumption (5) and (6), we have proposed an adaptive robust output controller. The adjustable parameters $\mathcal{L}(y,e_y,t) \in \Re^{m\times l}$ and $\theta(t) \in \Re^1$ are introduced so as to reduce the effect of uncertainties. Although the adjustable parameter $\theta(t) \in \Re^1$ cannot be an accurate estimate of an unknown upper bound $\delta^* \in \Re^1$ for uncertainties $\Delta(t) \in \Re^{m\times l}$ except for some particular cases, it might be utilized for information about the upper bound on unknown parameters. Besides, the adaptive action for the adjustable parameter $\theta(t) \in \Re^1$ is kept small by making the weighting parameter $\vartheta \in \Re^1$ in (13) sufficiently large.

Remark 3: For the uncertain linear system (4), we introduce the assumption (5) for the input matrix $B \in \Re^{n\times m}$ and the measurement one $C \in \Re^{l\times n}$. The assumption (5) means that the state variables which are affected by the control input can be measured. For example, if the input matrix B and the measurement matrix C are given by

$$B^T = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{22}$$

then the matrix $\mathcal{D} \in \Re^{1\times 2}$ can be written as

$$\mathcal{D} = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{23}$$

V. Illustrative Examples

In order to demonstrate the efficiency of the proposed control scheme, we have run a simple example. The control problem considered here is not necessary practical. However, the simulation results stated below illustrate the distinct feature of the proposed adaptive robust controller.

Consider the uncertain linear system with unknown parameter described by (24), i.e. $\mathcal{D} = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

Firstly, adopting the LMI-based algorithm based on the work of Benton et al.[16] (see Appendix), we design an output feedback gain matrix $K \in \Re^{1\times 2}$ for the nominal system. We select the design parameter α such as $\alpha = 1.0$, then by solving the LMI problem (A.2), we obtain the following solution and the gain matrix $K_{sf} \in \Re^{1\times 3}$.

$$\mathcal{X} = \begin{pmatrix} 156.09119 & -25.61906 & 39.53988 \\ -25.61906 & 8.92329 & -11.16948 \\ 39.53988 & -11.16948 & 21.33103 \end{pmatrix}$$
$$\mathcal{Y} = \begin{pmatrix} 381.78499 & -107.34119 & -61.09239 \end{pmatrix}$$
$$K_{sf} = \begin{pmatrix} 3.633509 & -39.507585 & -30.286408 \end{pmatrix} \tag{25}$$

Besides, solving the LMI feasibility problem (A.3), we have

$$\mathcal{P} = \begin{pmatrix} 193.92806 & 200.97939 & -242.79069 \\ 200.97939 & 12181.43241 & 1956.35399 \\ -242.79069 & 1956.35399 & 2345.6122 \end{pmatrix}$$
$$\sigma = 1.64710 \times 10^4 \tag{26}$$

Finally, in order to obtain the output feedback gain matrix $K \in \Re^{1\times 2}$, we solve the LMI minimization problem (A.4). By solving the LMI minimization problem (A.4), the following gain matrix can be obtained.

$$K = \begin{pmatrix} -9.38536 & -8.18176 \end{pmatrix} \tag{27}$$

And, $\lambda \in \Re^1$ in (A.4) can be computed as $\lambda = 155.02631$.

In this example, we consider the following three cases for the unknown parameter $\Delta(t) \in \Re^{1\times 2}$ and its unknown upper bound δ^* in (4).

- Case 1) : $\delta^* = 1.0$
  $\Delta(t) = \delta^* \times \begin{pmatrix} 5.09624 & -8.60397 \end{pmatrix} \times 10^{-1}$
- Case 2) : $\delta^* = 5.0$
  $\Delta(t) = \delta^* \times \begin{pmatrix} -\sin(2\pi t) & \cos(5\pi t) \end{pmatrix}$
- Case 3) : $\delta^* = 10.0$
  $$\Delta(t) = \begin{cases} \delta^* \times \begin{pmatrix} -1 & 0 \end{pmatrix} & \text{for} \quad 0 \le t \le 1.0 \\ \delta^* \times \begin{pmatrix} 1 & 0 \end{pmatrix} & \text{for} \quad 1.0 \le t \le 3.0 \\ \delta^* \times \begin{pmatrix} 1 & -1 \end{pmatrix} & \text{for} \quad t > 3.0 \end{cases}$$

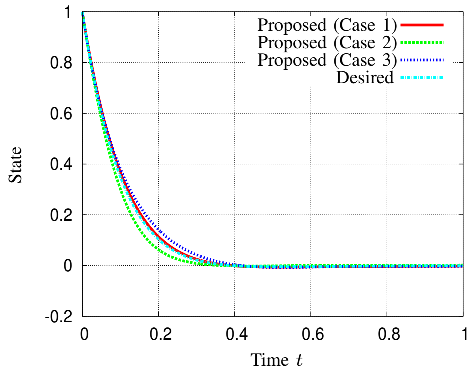

Fig. 1. Time histories of the state $x_1(t)$

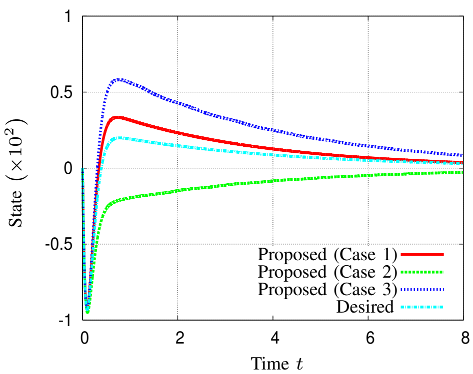

Fig. 2. Time histories of the state $x_2(t)$

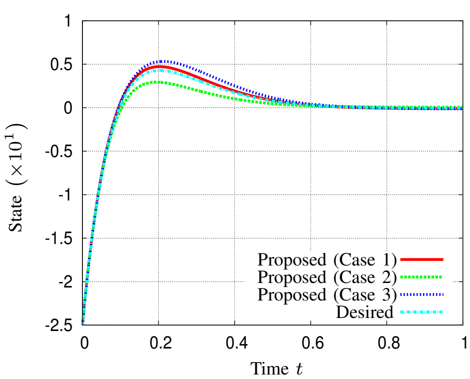

Fig. 3. Time histories of the state $x_3(t)$

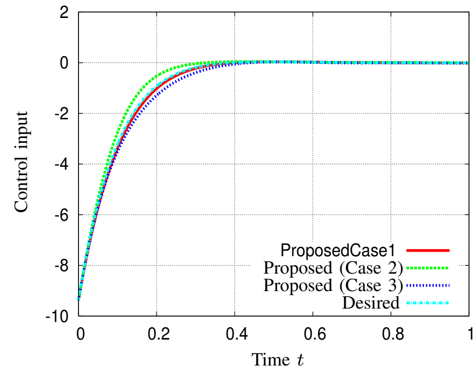

Fig. 4. Time histories of the control input $u(t)$

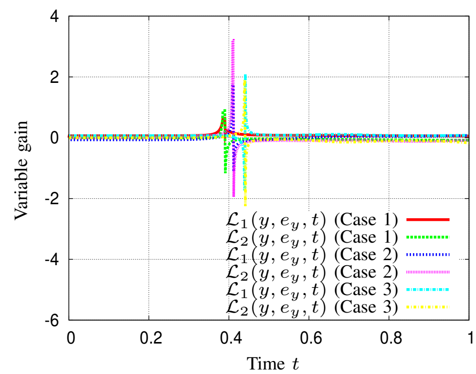

Fig. 5. Time histories of the variable gain matrix $\mathcal{L}(y, e_y, t)$

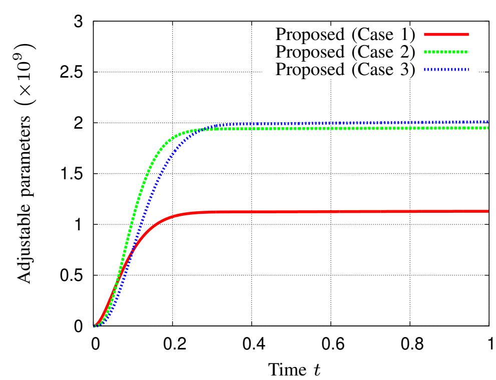

Fig. 6. Time histories of the adjustable parameter $\theta(t)$

Furthermore, the design parameter $\vartheta \in \Re^1$ is chosen as $\vartheta = 1.0 \times 10^6$ and initial values for the uncertain system (24) and the adjustable parameter $\theta(t) \in \Re^1$ are selected as $x(0) = \begin{pmatrix} 1.00 & 0.00 & -0.25 \end{pmatrix}^T$ and $\theta(0) = 0.0$, respectively.

The results of the simulation of this example are depicted in Fig.1–6. In these figures, the time-histories of the state variables $x_1(t), x_2(t)$ and $x_3(t)$, the control input $u(t)$, elements of the variable gain matrix $\mathcal{L}(y, e_y, t) \in \Re^{1\times 2}$ and the adjustable parameter $\theta(t) \in \Re^1$ are shown. From Figs. 1–3, we find that the proposed adaptive robust output feedback controller stabilize the uncertain system (24) in spite of plant uncertainties. Furthermore, we see from Fig. 5 that the variable gain matrix $\mathcal{L}(y, e_y, t) \in \Re^{1\times 2}$ is tuned by the output $y(t)$ and the output error $e_y(t)$. In addition, one can see from Fig.6 that the adjustable parameter $\theta(t)$ is not an estimate of unknown upper bound $\delta^* \in \Re^1$ for the unknown parameter $\Delta(t) \in \Re^{1\times 2}$. Magnitude of adaptive action on the adjustable parameter $\theta(t) \in \Re^1$ is affected by choosing the design parameter $\vartheta \in \Re^1$. In this example, since the design parameter ϑ is sufficiently large, magnitude of adaptive action on the parameter $\theta(t)$ is selected small.

VI. Conclusions

In this paper, a design method of an adaptive robust output feedback controller has been presented for a class of linear systems with uncertainties of which upper bounds are unknown. The uncertainties under consideration satisfy the matching condition.

A crucial feature of the proposed controller design is that the information is not required on the upper bound of matched uncertainties, and it is easy to design a robust output feedback controller, which guarantees robust stability, by deriving only the fixed gain controller for the nominal system. Besides, the proposed adaptive robust controller is adaptable when some assumptions are satisfied, and in the case where only the output of the uncertain system are available on control scheme, the proposed design method can be used. Besides, it is obvious that the proposed adaptive robust control scheme is more effective for linear systems with unknown and/or large uncertainties.

The future research subject is an extension of the proposed adaptive robust output feedback controller to such a broad class of systems as uncertain large-scale systems, uncertain time-delay systems and so on. Furthermore in future work, we will examine the assumption (4) and the application to practical systems. Additionally, we will consider the decision method for the initial value of the time-varying parameter $\theta(t) \in \Re^1$ and the design parameter $\vartheta \in \Re^1$, because the time response of a controlled system will change substantially according to the these parameters.

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APPENDIX

In this appendix, we show a LMI-based design algorithm of the static output feedback controller for the nominal system based on the work of Benton et al.[16].

Consider the following linear dynamical system.

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{A.1}$$

where $x(t) \in \Re^n, u(t) \in \Re^m$ and $y(t) \in \Re^l$ are the vectors of the state, the control input and the measurement output, respectively, and the matrices A, B and C denote the nominal values of system parameters.

For the linear system (A.1), we consider the static output feedback control $u(t) = Ky(t)$. The following LMI-based algorithm to derive the static output feedback gain matrix $K \in \Re^{m\times l}$, has been developed by the existing result of Benton et al.[16].

- **A LMI-based algorithm**

Step 1) Define $A_\alpha = A + \alpha I_n$ where $\alpha \in \Re^1$ is the desired prescribed degree of stability, as described in Anderson and Moore[18].

Step 2) Solve the following LMI problem

$$\begin{aligned} &\underset{\mathcal{X},\mathcal{Y}}{\text{Minimize}}\left[\text{Trace}\left\{\mathcal{X}\right\}\right] \\ &\text{subject to } \ \mathcal{X} - I_n > 0 \\ &\text{and } \ H_e\left\{A_\alpha \mathcal{X} + B\mathcal{Y}\right\} < 0 \end{aligned} \tag{A.2}$$

Step 3) By using the matices $\mathcal{X} \in \Re^{n\times n}$ and $\mathcal{Y} \in \Re^{m\times n}$, set the state feedback gain matrix such as $K_{sf} = \mathcal{Y}\mathcal{X}^{-1}$.

Step 4) Solve the following LMI feasibility problem.

$$\begin{aligned} &\text{Find } \sigma \text{ and } \mathcal{P} \ \text{ such that} \\ &\mathcal{P} > I_n, \quad H_e\left\{\mathcal{P}\left(A_\alpha + BK_{sf}\right)\right\} < 0 \\ &H_e\left\{\mathcal{P}A_\alpha\right\} - \sigma C^T C < 0 \ \text{ and } \ \sigma > 0 \end{aligned} \tag{A.3}$$

Step 5) In order to derive an output feedback gain $K \in \Re^{m\times l}$, fix the matrix $\mathcal{P} \in \Re^{n\times n}$ and solve the following LMI minimization problem.

$$\begin{aligned} &\underset{\lambda,K}{\text{Minimize}}\left[\lambda\right] \\ &\text{subject to } \ \begin{pmatrix} \lambda & \mathcal{V}_K^T \\ \mathcal{V}_K & I_{m\times l} \end{pmatrix} > 0 \\ &\text{and } \ H_e\left\{\mathcal{P}\left(A_\alpha + BKC\right)\right\} < 0 \end{aligned} \tag{A.4}$$

where $\mathcal{V}_K = \text{vec}\left(K\right)$.