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Clustering in a network of mutually attracting agents

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Abstract— **We introduce a model of mutually attracting agents in an arbitrary network, for which the long term behavior results in the emergence of several clusters. The cluster structure is independent of the initial condition and is characterized by a set of inequalities in the parameters of the model. With varying coupling strength, transitions between different cluster structures may take place. We illustrate the relation with the Kuramoto model on interconnected oscillators and we discuss an application on opinion formation.**

I. INTRODUCTION AND MOTIVATION

The formation of several clusters, arising from attracting forces between non-identical agents, is observed in fields ranging from the exact sciences to social and life sciences; consider e.g. swarm behavior of animals or social insects [20], opinion formation [12] or the clusters in the frequency space for synchronized coupled oscillators [22] as a model for heart cells. Swarming models mostly focus on the behavior and the cohesion of a single cluster [5], [16], [11], and models for opinion formation often consider the coexistence of only two opposite opinions [23], [24], [4], although the emergence of multiple opinions has also been investigated [9]. For clustering in systems of coupled oscillators one distinguishes between *phase* clustering [18], [10] and *frequency* clustering [17], depending on whether a cluster is characterized by identical phases or identical frequencies.

We extend a previously introduced model [1], [2] that captures this phenomenon and at the same time allows a mathematical analysis, by considering an arbitrary network structure (instead of all-to-all interaction) and introducing sensitivity factors and weighting factors. The model from [1], [2] can be considered as a simplification of the Kuramoto model [14] of coupled oscillators that retains its (frequency) clustering behavior, with a greatly increased potential for analytical results. For the Kuramoto model and its extensions to arbitrary network structures analytical results have been obtained (e.g. [13], [19]), but a full analysis seems unfeasible.

The main contribution of this paper is the extension of the model from [1], [2]. We formulate *necessary and sufficient* conditions for the occurrence of a given cluster structure, and we describe how the cluster structure varies with varying coupling strength. We discuss an important difference with the results of the basic version (i.e. the version presented in [1], [2]): if there is no all-to-all coupling, clusters may split with increasing coupling strength.

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In the next section we review the basic model and its analytical results. We then introduce the extended version and its results, and we present a condensed version of the proofs. In Section V we relate the model to the Kuramoto model [14], which describes systems of coupled oscillators. We indicate the similarities between partial entrainment of oscillators and clustering as described in the proposed model, both for the time evolution and the dependence on the coupling strength. In Section VI we describe how the model applies to opinion formation, and how the different types of cluster structures relate to the possible outcomes of the opinion formation process. We compare the basic version of the clustering model with the extended version regarding their suitability for this application.

II. BASIC VERSION

A. The Dynamics

The differential equations for the basic version consisting of $N > 1$ agents are

$$\dot{x}_i(t) = b_i + \frac{K}{N}\sum_{j=1}^{N} f(x_j(t) - x_i(t)), \tag{1}$$

$\forall t \in \mathbb{R}, \forall i \in I_N \triangleq \{1, \ldots, N\}$, with $x_i(t) \in \mathbb{R}$. The parameter b_i represents the autonomous component in the behavior of agent i, the summation term represents the attraction exerted by the other agents. The function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, odd and non-decreasing, resulting in a *symmetric attraction* between any pair of agents. We assume that the interaction intensifies with separation up to a certain saturation level:

$$\exists d > 0 : \forall \xi \geq d, \qquad f(\xi) = F.$$

Assume that, for a particular solution of (1), the behavior of the agents can be characterized as follows by an *ordered* set of *clusters* $(G_1, \ldots, G_M)$ defining a partition of I_N:

- The distances between agents in the same cluster remain bounded (i.e. $|x_i(t) - x_j(t)|$ is bounded for all $i, j \in G_k$, for any $k \in I_M$, for $t \geq 0$).
- After some positive time T, the distances between agents in different clusters are at least d and grow unbounded with time.
- The agents are ordered by their membership to a cluster: $k < l \Rightarrow x_i(t) < x_j(t)$, $\forall i \in G_k$, $\forall j \in G_l$, $\forall t \geq T$.

We will refer to this behavior as *clustering behavior*.

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B. Results

For any set $G_0 \subset I_N$, with the number of elements denoted by $|G_0|$, we introduce the notation $\langle b \rangle_{G_0}$ for the average value of b_i over G_0:

$$\langle b \rangle_{G_0} \triangleq \frac{1}{|G_0|} \sum_{i \in G_0} b_i.$$

For the proofs of the results in this section we refer to [2].

Theorem 1: The following set of conditions is *necessary and sufficient* for clustering behavior of *all* solutions of the system (1), with the cluster structure $(G_1, \ldots, G_M)$ *independent of the initial condition*:

$$\langle b \rangle_{G_{k+1}} - \langle b \rangle_{G_k} > \frac{KF}{N} \left(|G_{k+1}| + |G_k| \right), \quad \forall k \in I_{M-1}, \tag{2a}$$

$$\langle b \rangle_{G_{k,2}} - \langle b \rangle_{G_{k,1}} \leq \frac{KF}{N} |G_k|, \quad \forall G_{k,1}, G_{k,2} \subsetneq G_k, \text{ with } G_{k,2} = G_k \setminus G_{k,1}, \quad \forall k \in I_M. \tag{2b}$$

Setting

$$v_k \triangleq \langle b \rangle_{G_k} + \frac{KF}{N} \left(\sum_{k'>k} |G_{k'}| - \sum_{k'<k} |G_{k'}| \right),$$

the clustering behavior of a solution x of (1) can be shown to be equivalent with:

$$\exists l > 0 : |x_i(t) - v_k t| \leq l,$$

$\forall i \in G_k, \forall k \in I_M, \forall t \geq 0$.

Theorem 2: For every $b \in \mathbb{R}^N$ and every $K \in \mathbb{R}^+$, $F \in \mathbb{R}_0^+$, there exists a unique ordered set partition G of I_N, satisfying (2).

In general there exist $N-1$ *bifurcation* values for the intensity of attraction K, defining N intervals for K, which correspond to N different cluster configurations. Transitions to new cluster configurations take place at these bifurcation points. This is illustrated in Fig. 1, where the long term average velocities v_k are shown with varying coupling strength K. In the next section we will extend these results to a more general system with an arbitrary network structure.

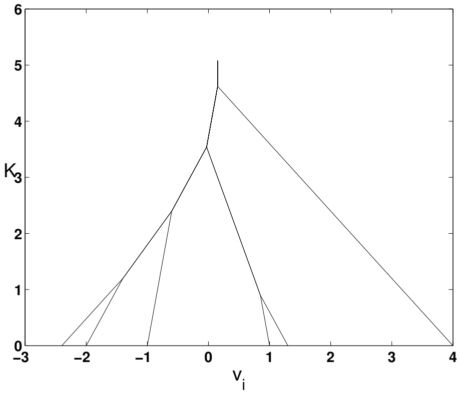

Fig. 1. Long term average velocities (horizontal axis) for varying coupling strength (vertical axis).

III. EXTENDED VERSION

A. The Model

The extended model is described by:

$$\dot{x}_i(t) = b_i + K A_i \sum_{j=1}^{N} \gamma_j f_{ij}\left(x_j(t) - x_i(t)\right), \tag{3}$$

$\forall t \in \mathbb{R}$, $\forall i \in I_N$. The functions f_{ij} are Lipschitz continuous, non-decreasing, and satisfy $f_{ji}(\xi) = -f_{ij}(-\xi)$, for all $i, j \in I_N$ and $\xi \in \mathbb{R}$, while again attaining a saturation value:

$$\exists d > 0 : \forall i, j \in I_N, \forall \xi \geq d, \qquad f_{ij}(\xi) = F_{ij},$$

for some $F_{ij} \in \mathbb{R}$. The interpretation of b_i remains unaltered. The parameters A_i and γ_i are all positive. The matrix F is assumed to be symmetric and irreducible with $F_{ij} \geq 0$ $(i, j \in I_N)$. The values of the diagonal elements F_{ii} are irrelevant and they can be set equal to zero. The interaction structure of (3) is arbitrary, as opposed to the structure of the model (1), which involves all-to-all coupling. The elements of matrix F represent levels of attraction between agent pairs (e.g. no attraction between agents i and j if $F_{ij} = 0$); the extent to which each individual agent j tends to attract other agents is denoted by γ_j. The parameter A_i reflects the sensitivity of agent i to interactions with other agents. The parameter K is the global coupling strength.

The model is overparameterized. Inspection of (3) reveals that the parameters A_i can be omitted after replacing f_{ij} by $f_{ij}/(A_i A_j)$ and γ_j by $A_j \gamma_j$. The formulation (3) may be useful for applications, since the parameters may be given physical relevance. For the mathematical results we will set $A_i = 1$:

$$\dot{x}_i(t) = b_i + K \sum_{j=1}^{N} \gamma_j f_{ij}\left(x_j(t) - x_i(t)\right), \tag{4}$$

$\forall t \in \mathbb{R}$, $\forall i \in I_N$. The results in the following section concern (4); the results are valid also for (3), with an adapted formulation.

B. Results

Redefine the averaging operator $\langle \cdot \rangle$ by taking into account the weights γ_i:

$$\langle b \rangle_{G_0} \triangleq \frac{\sum_{i \in G_0} \gamma_i b_i}{\sum_{i \in G_0} \gamma_i}.$$

Define the set function $\widetilde{v}$ by

$$\widetilde{v}(G_-, G_0, G_+) \triangleq \langle b \rangle_{G_0} + \frac{K}{\sum_{i \in G_0} \gamma_i} \sum_{i \in G_0} \gamma_i \left(\sum_{j \in G_+} \gamma_j F_{ij} - \sum_{j \in G_-} \gamma_j F_{ij} \right),$$

for all $G_-, G_0, G_+ \subset I_N$ with G_0 non-empty. The value of $\widetilde{v}(G_-, G_0, G_+)$ represents the average velocity $\langle \dot{x}(t) \rangle_{G_0}$ over a group G_0 of agents at time instances t for which the other agents have $x(t)$-values that are either at least d smaller than (for the agents in G_-) or at least d larger than (for the agents in G_+) the $x(t)$-values of the agents in G_0, as will explained in the first paragraph of Section IV-A.

Let $G_k^<$ be a shorthand notation for $\bigcup_{k'<k} G_{k'}$, and set $G_k^> \triangleq \bigcup_{k'>k} G_{k'}$.

For the model (4) we are able to formulate the following theorem.

Theorem 3: The following set of conditions is *necessary and sufficient* for clustering behavior of *all* solutions of (4), with the cluster structure $(G_1,\ldots,G_M)$ *independent of the initial condition*:

$$\widetilde{v}(G_k^<, G_k, G_k^>) < \widetilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>), \quad \forall k \in I_{M-1}, \tag{5a}$$

$$\widetilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>) \leq \widetilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}), \quad \forall G_{k,1}, G_{k,2} \subsetneq G_k, \text{ with } G_{k,2} = G_k \setminus G_{k,1}, \quad \forall k \in I_M. \tag{5b}$$

Under the conditions of Theorem 3, the average velocity $\langle \dot{x}(t) \rangle_{G_k}$ over cluster G_k will be constant after some time T:

$$\langle \dot{x}(t) \rangle_{G_k} = \widetilde{v}(G_k^<, G_k, G_k^>),$$

for all $t \geq T$, for some $T \geq 0$. Denoting the right hand side by v_k we can again derive that

$$\exists l > 0 : |x_i(t) - v_k t| \leq l,$$

$\forall i \in G_k, \forall k \in I_M, \forall t \geq 0$.

Remark 1: Notice that for a given set of parameters, the conditions (5) cannot be satisfied for two different cluster structures, as by Theorem 3 all solutions of (4) are characterized by a single cluster structure.

Theorem 4: For every $b \in \mathbb{R}^N$, every $K \in \mathbb{R}^+$, and every symmetric and irreducible matrix $F \in (\mathbb{R}^+)^{N\times N}$ there exists a unique ordered set partition G of I_N, satisfying (5).

Concerning the dependence of the cluster structure on the model parameters, the extended version is different from the basic version. When K is varied, transitions between different cluster structures may take place. However, contrary to the all-to-all case, the number of possible cluster configurations with varying K may be larger than N. This follows from the fact that clusters may split with increasing coupling strength, a phenomenon that cannot occur in the all-to-all coupled case described by (1), and which is illustrated in Fig. 2, where the long term average velocities of the agents are shown for varying coupling strength.

Remark 2: It is easily verified that for $\gamma_i = \frac{1}{N}$ and $f_{ij} = f$ $(i, j \in I_N)$ the results of this section are equivalent to the results of the previous section.

Remark 3: Given the model (4) and faced with the question of describing the cluster structure which emerges eventually, our analysis offers two options: one may check the inequalities (5) or one may simply run a simulation of the model: the mathematical analysis guarantees convergence to a cluster structure, irrespective of the initial condition.

IV. Proofs

We present condensed versions of the proofs of Theorems 3 and 4. We refer to [7] for full details.

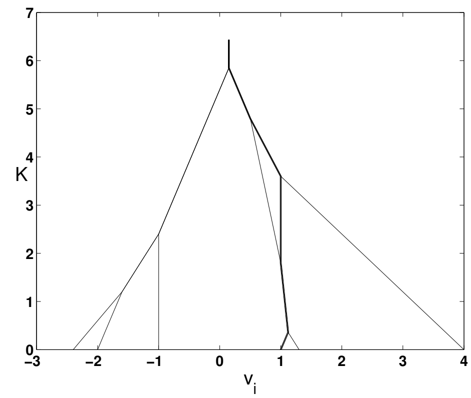

Fig. 2. Long term average velocities (horizontal axis) for varying coupling strength K (vertical axis). One of the agents is represented in bold.

A. Theorem 3

The characteristics of the interaction play a key role in the proof of Theorem 3. Because of the anti-symmetry properties of the functions f_{ij}, all internal interactions (i.e. interactions between agents in the same cluster) cancel when calculating the velocity of the 'center of mass' (weighted with the parameters γ_i) of a cluster, similar to the cancellation of internal interactions in mechanics. The saturation of the interaction functions implies that the interactions between agents from different clusters reduce to constants whenever agents from different clusters are separated over at least a distance d.

These properties lead to the aforementioned conclusion that, with $\{G_-, G_0, G_+\}$ partitioning I_N,

$$\langle \dot{x}(t) \rangle_{G_0} = \widetilde{v}(G_-, G_0, G_+) \tag{6}$$

for all time instances t for which the agents in G_- (resp. G_+) have $x(t)$-values that are at least d smaller (resp. larger) than the $x(t)$-values of all agents in G_0.

Applying (6) under the assumption of clustering behavior leads to

$$\langle \dot{x}(t) \rangle_{G_{k+1}} - \langle \dot{x}(t) \rangle_{G_k} = \widetilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>) - \widetilde{v}(G_k^<, G_k, G_k^>),$$

for t sufficiently large, and the ordering of the agents and distances growing unbounded with time for agents in different clusters then implies the conditions (5a). Since the functions f_{ij} are non-decreasing, one similarly derives that

$$\langle \dot{x}(t) \rangle_{G_{k,2}} - \langle \dot{x}(t) \rangle_{G_{k,1}} \geq \widetilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>) - \widetilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}),$$

for any two subsets $G_{k,1}$ and $G_{k,2}$ partitioning G_k. Since distances between agents from the same cluster remain bounded, the conditions (5b) follow. This implies the necessity of the inequalities (5) for the existence of a solution of (4) satisfying clustering behavior.

For the proof of sufficiency of the conditions (5), the main idea is to show that the region R, defined below, is a trapping region for solutions of (4). Setting $\gamma_{\min} \triangleq \min_{i\in I_N} \gamma_i$, $R \subset \mathbb{R}^N$ is defined by:

$$y \in R \Leftrightarrow \begin{cases} \langle y\rangle_{G_{k+1}} - \langle y\rangle_{G_k} \geq \dfrac{d\sum_{i\in G_k\cup G_{k+1}}\gamma_i}{2\gamma_{\min}}, \\ \qquad \forall k \in I_{M-1}, \\ \langle y\rangle_{G_{k,2}} - \langle y\rangle_{G_{k,1}} \leq \dfrac{d\sum_{i\in G_k}\gamma_i}{2\gamma_{\min}}, \\ \qquad \forall G_{k,1}, G_{k,2} \subsetneq G_k, \text{ with } G_{k,2} = G_k \setminus G_{k,1}, \\ \qquad \forall k \in I_M. \end{cases}$$

To show that R is a trapping region, one proceeds as follows. First one derives that for $x(t) \in R$, agents from different sets G_k are separated over at least a distance d, allowing us to apply (6) to each of the sets G_k. It easily follows that $x(t)$ cannot leave R by violation of an inequality in the first set of inequalities defining R, since, as long as $x(t) \in R$,

$$\langle \dot{x}(t)\rangle_{G_{k+1}} - \langle \dot{x}(t)\rangle_{G_k} = \widetilde{v}(G_{k+1}^{<}, G_{k+1}, G_{k+1}^{>}) - \widetilde{v}(G_k^{<}, G_k, G_k^{>}),$$

which is positive (by (5a)).

For $x(t) \in R$ with one of the inequalities in the second set of inequalities defining R becoming an equality, one can show that agents in the corresponding subsets $G_{k,1}$ and $G_{k,2}$ are also separated over at least d, again allowing us to apply (6), and therefore

$$\langle \dot{x}(t)\rangle_{G_{k,2}} - \langle \dot{x}(t)\rangle_{G_{k,1}} = \widetilde{v}(G_k^{<} \cup G_{k,1}, G_{k,2}, G_k^{>}) - \widetilde{v}(G_k^{<}, G_{k,1}, G_k^{>} \cup G_{k,2}),$$

which is non-positive (by (5b)), and it follows that $x(t)$ cannot leave R.

Since R is non-empty, there exists a solution x of (4) with $x(t) \in R$, for all $t \geq 0$, and it is easily shown that this solution exhibits clustering behavior (with $T = 0$, and the clusters equal to the sets G_k).

Any other solution $\hat{x}$ of (4) will exhibit the same clustering behavior (i.e. identical clusters, possibly a different value for T). This follows by observing that we can introduce a modified square distance in the state space $\mathbb{R}^N$ between x and $\hat{x}$ that is non-increasing in time, due to the monotonicity of the functions f_{ij}:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} \gamma_i (x_i(t) - \hat{x}_i(t))^2\right) \leq 0.$$

It follows that $x_i(t) - \hat{x}_i(t)$ remains bounded for all i in I_N, and therefore x and $\hat{x}$ exhibit the same clustering behavior.

B. Theorem 4

The existence of a cluster structure for arbitrary choices of the parameters is shown as follows. For K sufficiently large the conditions (5) are satisfied for the cluster structure $G = (I_N)$ because of the irreducibility of the matrix F. When K decreases, transitions will take place each time one of the inequalities in (5) becomes an equality. At a transition value K_t for K, a new cluster structure can be constructed that satisfies (5) for values of K in some interval $\mathscr{I}$ with upper end point K_t. (Each end point may or may not be included in the interval, dependent on whether the inequality becoming an equality at the corresponding end point is in the set (5a) or in the set (5b).) If one of the inequalities in (5a) becomes an equality at K_t, then the corresponding clusters G_k and G_{k+1} will merge and form a new cluster. If one of the inequalities in (5b) becomes an equality at K_t, then the cluster G_k will split in two new clusters $G_{k,1}$ and $G_{k,2}$ corresponding to the two subsets involved in the equality. (The calculations showing that this new cluster structure satisfies (5) for K in $\mathscr{I}$ may be tedious, but they are quite straightforward.) This procedure can be repeated until K becomes zero. The uniqueness of the cluster structure satisfying (5) follows from Remark 1.

V. THE KURAMOTO MODEL

The Kuramoto model [14] was introduced to describe synchronization in systems of coupled oscillators. We refer to [22] for its relation to flashing fireflies, pacemaker cells, Josephson junctions. A finite-dimensional version of the Kuramoto model is described by

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)),$$

$\forall i \in I_N$, $\forall t \in \mathbb{R}$. The natural frequencies ω_i are drawn randomly from a distribution g. Kuramoto showed [14] that, for $N \to \infty$, and with g unimodal and even about a value Ω, there is a critical value for the coupling strength K above which a solution exists exhibiting partial synchronization. For this solution a group of oscillators is moving at the same frequency Ω, while the remaining oscillators are moving with different (average) frequencies. Details can be found in [21].

For finite N, simulations indicate the following. For a fixed $K > 0$ the oscillator population can be partitioned into different subsets of which the members have bounded phase differences and have the same long term average frequency: the system exhibits partial entrainment [6]. The partition and the associated average frequencies are independent of the initial condition for most choices of the natural frequencies and the coupling strength. When the coupling strength is increased the average frequencies of the different entrained subsets move towards each other, and when a critical value for K is passed and two long term average frequencies coincide, the corresponding entrained subsets merge into a new entrained subset. This scenario is repeated until there is full entrainment for K sufficiently large, i.e. all phase differences are bounded. For more information on the stability properties of this latter solution, see [3].

This behavior is very similar to that of the models (1) and (4), where the clusters are also independent of the initial condition and the transitions between the different clusters for varying K are similar. For comparison Fig. 3 shows the time evolution for a particular configuration for the Kuramoto model (Fig. 3(a)) and the analogue for the model (1) (Fig. 3(b)), as well as the evolution of the long term

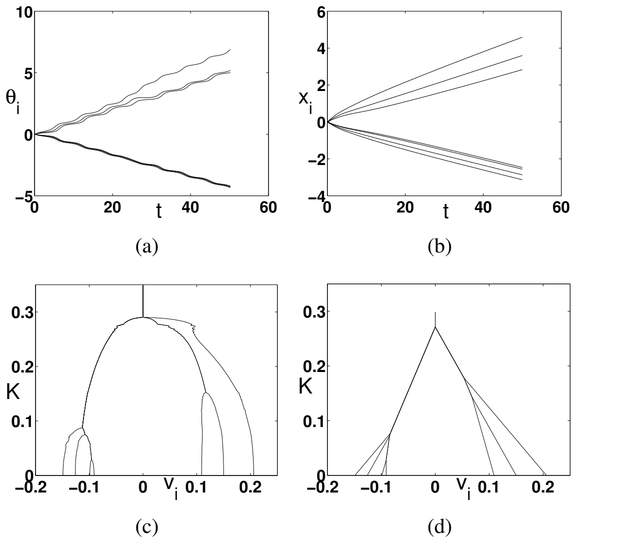

Fig. 3. Comparison of the Kuramoto model (left column) with the model (1) (right column): The first row shows the time evolution of the oscillators/agents. The second row shows the evolution of the average frequencies/velocities for varying coupling strength. The parameters are: $F = 1$ (the internal structure of f is not important for the clustering behavior), $b = \omega = (-0.149,\ -0.126,\ -0.099,\ -0.091,\ 0.109,\ 0.150,\ 0.206)$, $K = 0.2$ (Fig. (a)) and $K = 0.15$ (Fig. (b)).

average frequencies/velocities for both models in terms of the coupling strength (Fig. 3(c) and 3(d)). There is a remarkable (qualitative) correspondence between both models.

Besides entrainment/clustering behavior, the Kuramoto model also exhibits some phenomena such as frequency locking [15] or induction of clusters by resonances [8], which are not present in the model (1). However, this richer behavior of the Kuramoto model has its price: in spite of the simplicity of its formulation, the model is complicated and hard to analyze. The models (1) and (4) allow a focus on clustering behavior, while admitting a full analysis.

VI. OPINION FORMATION

We represent opinions on a particular issue by real numbers, with zero corresponding to a neutral position. We consider N individuals taking part in a meeting; each individual has his own opinion on the issue on the agenda, which may evolve in time due to discussion with the other members. Since opinions cannot grow unbounded, x_i in (3) is not an appropriate quantity to represent an opinion. Instead we will take the derivatives $y_i = \dot{x}_i$ as a measure of someone's opinion. The equations for y_i can be written as (assuming $x_i(0) = 0$, $\forall i \in I_N$, without loss of generality regarding the long term behavior)

$$y_i(t) = b_i + \frac{KA_i}{\sum_{j=1}^N \gamma_j F_{ij}} \sum_{j=1}^N \gamma_j f_{ij}\left(\int_0^t (y_j(t') - y_i(t'))\mathrm{d}t'\right), \quad (7)$$

$\forall i \in I_N$, where we have redefined the sensitivity factors A_i to explicitly include a normalization of the interaction, such that each agent deviates at most KA_i from its a priori opinion b_i (corresponding to no discussion). With $y_i(t)$ representing the opinion of agent i at time t, the absolute value of the integral $\int_0^t (y_j(t') - y_i(t'))\mathrm{d}t'$ may reflect the level of disagreement accumulated over time, or the amount of discussions taking place between agents i and j, proportional with time and with difference in opinion.

In general, everyone starts with his own opinion b_i while with time and through interaction, different groups are formed, each group characterized by a final opinion v_i obtained through discussion. The pressure to reach a decision, or the tendency to adapt one's opinion by paying attention to each other's arguments, is reflected by the coupling strength K.

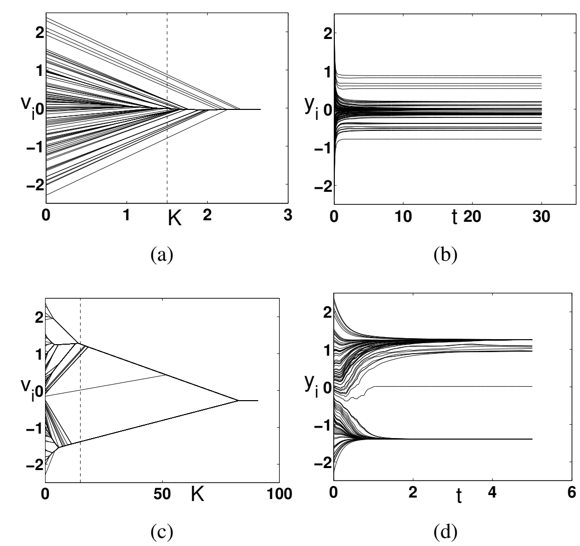

Fig. 4. Opinion formation: Fig. (a) and (c) show the opinions v_i as a function of the coupling strength K. Fig. (b) and (d) show the time evolution for a constant K (1.5 and 15 respectively). For Fig. (a) and (b) the parameters A_i, γ_i and F_{ij} are all equal to one (except $F_{ii} = 0$, $\forall i \in I_N$), for Fig. (c) and (d) the parameter values are given by equations (8) to (10).

In Fig. 4(a) we show the evolution of the opinions v_i eventually reached as a function of K. The v_i-values were calculated by means of an algorithm based on the inequalities (5). We considered 100 agents with b_i chosen from a Gaussian distribution with zero mean and standard deviation one. The parameters A_i, γ_i, and F_{ij} $(i \neq j)$ were all taken equal to one (except that $F_{ii} = 0$, $\forall i \in I_N$). Notice a steady convergence to complete agreement as a function of K. In Fig. 4(b) the time evolution of the opinions y_i for $K = 1.5$, as obtained by numerical integration of the mathematical model, is shown.

In a second simulation (Fig. 4(c), again obtained by an algorithm based on the inequalities (5)) we kept the same parameters b_i, but the values for A_i and γ_i were changed to account for the fact that people with extreme opinions are reluctant to change their point of view (smaller A_i) while making more efforts to persuade other people (larger γ_i). Also F_{ij} decreases with increasing values of $|b_j - b_i|$, reflecting the idea that people tend to pay more attention to people with a similar opinion:

$$\gamma_i = 1 + 2b_i^2, \tag{8}$$

$$A_i = \frac{1}{1 + b_i^2}, \tag{9}$$

$$F_{ij} = \exp(-2|b_j - b_i|). \tag{10}$$

In Fig. 4(d) we show the time evolution of the y_i for $K = 15$, again obtained by numerical integration. (For the numerical integration in Fig. 4(b) and 4(d) the Euler method was used with a time step of $0.03/K$.)

While in the first case it seems possible to take a decision by a unanimous consent, in the second case — which is more realistic — it is far more favorable to let a majority vote decide, as one notices a deadlock of extreme opinions for K around 15. Total consensus can only be reached under much higher pressure compared to the pressure needed for reaching a decision by a majority vote and might require unreasonable concessions from all parties involved.

As an important distinction with other existing models (for an overview, see [12]) we want to emphasize that the model (7) allows the coexistence of several groups, each characterized by its own group opinion — as opposed to models focusing on total consensus or the coexistence of only two opinions (such as in [23] or [24]) — while still allowing analytical exploration — as opposed to models for which the results rely on simulations (as in [9]).

VII. ACKNOWLEDGMENTS

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