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Necessary and sufficient numerical conditions for asymptotic stability of linear time-varying systems

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Abstract— **In this paper, necessary and sufficient numerical conditions for stability and for asymptotic stability of linear continuous time-varying systems are derived. For a given set of initial conditions, a tube containing all the trajectories of the system is constructed in the state space. At each instant of time, there exists an initial condition inside the set such that the resulting trajectory attains the border of the tube. Based on the above formulation, necessary and sufficient conditions for stability and for asymptotic stability are expressed through the solution of a linear differential Lyapunov equation. The conditions can deal with the stability of periodic systems as well. One of the main characteristics of the proposed necessary and sufficient conditions is that the only assumption on the dynamical matrix of the linear time-varying system is continuity. Examples from the literature illustrate the superiority of the proposed conditions when compared to other methods.**

I. Introduction

The stability of linear continuous time-varying systems has been investigated in numerous papers [1–7]. Although from a theoretical point of view there exist necessary and sufficient conditions in the literature [8–10], a lot of effort has been dedicated to the search for numerically tractable necessary and sufficient conditions (see [11] and references therein). In many cases, only sufficient conditions are obtained, as for instance in the methods based on the analysis of the eigenvalues of a time-invariant system [12, 13].

As it is well known, even when the eigenvalues of the system have strictly negative real parts for all instants of time the linear time-varying system can be unstable (see for instance the second example in this paper). On the other hand, an asymptotically stable linear time-varying system can exhibit a system matrix with eigenvalues that have strictly positive real parts [12]. This gives an idea of the difficulty of assessing the stability of a linear time-varying system.

Other techniques use the Lyapunov theory, for instance, by associating to the original linear time-varying system time-invariant piecewise approximations from which sufficient conditions for stability are derived [1, 5, 7]. This is the case of the recently published paper [11], where classical Lyapunov equations are solved for a sequence of discrete points inside the time interval of interest. Associating a quadratic Lyapunov function to each point of the grid, a tube is constructed in the state space. For the selected set of initial conditions, all the trajectories of the original system lie strictly inside the tube. The dynamic matrix of the system is supposed to be continuous, norm-bounded and Hurwitz for all instants of time. The sufficient conditions are derived from the geometric properties of the tube.

In this paper, necessary and sufficient conditions for stability and for asymptotic stability of linear continuous time-varying system are given. Using a strategy similar to the one in [11], a tube is constructed in terms of the norm of the trajectories in the state space for a given set of initial conditions. Inside this set, there exists an initial condition such that the associated trajectory reaches the boundary of the tube. In this sense, the tube is not approximated as in [11], but it is the ellipsoidal tube of minimum "size" that contains all the trajectories of the system for a given ellipsoidal set of initial conditions. Another important difference is that in [11] the dynamic matrix is assumed to be continuous *and* Hurwitz for all t, while only the continuity of the dynamic matrix is assumed here. Using the properties of the tube, necessary and sufficient conditions for stability and for asymptotic stability are obtained in terms of the solution of a differential Lyapunov equation. The computational burden required is similar to the one in the approach from [11], where a set of standard Lyapunov equations (one at each point of the time grid) has to be solved and, between two points, a function need to be integrated. The conditions proposed here can easily be adapted to cope with the stability analysis of linear continuous time-varying periodic system.

The paper is organized as follows. In Section II, some definitions of stability are recalled. Section III presents the main results of the paper, i.e. a method to construct an ellipsoidal tube in the state space containing the trajectories of the system for a given ellipsoidal set of initial conditions, based on the solution of a differential Lyapunov equation, and the necessary and sufficient conditions for stability and for asymptotic stability. Examples from the literature illustrate the potentiality of the results when compared to other methods in Section IV. Conclusions and a discussion on future topics of research regarding this subject end the paper.

Notation

Notation is standard. The Euclidian vector norm is denoted by $\|\cdot\|$. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. Identity and null matrices are denoted by I and 0, respectively. The transpose of a matrix

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$A \in \mathbb{R}^{m\times n}$ is denoted by A'. $\lambda_{min}(A)$ ($\lambda_{max}(A)$) denotes the minimal (maximal) eigenvalue of matrix A.

II. Problem statement and preliminaries

Consider linear continuous time-varying systems given by

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0 \tag{1}$$

with $x \in \mathbb{R}^n$, $A(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$, $t \to A(t)$ and the initial time t_0. It is assumed that the application $t \to A(t)$ is continuous in t. The following classical notions of stability, known as stability in the sense of Lyapunov (see for instance [8]), are recalled.

Definition 1: Let t_0 be any real number. If for all $\delta > 0$, there exists $r(t_0,\delta) > 0$ such that $\|x(t_0)\| < r(t_0,\delta)$ implies $\|x(t)\| < \delta$ for all $t \geq t_0$, then the system (1) is stable in the sense of Lyapunov.

As pointed out in [8], for ordinary differential equations and in particular for (1), if the system is stable for the initial time t_0, it is stable for subsequent initial times $t_1 > t_0$. The important consequence is that t_0 can be selected arbitrarily. However, in some particular situations, the dependence with respect to the initial time t_0 can be dropped, resulting in uniform stability.

Definition 2: If in Definition 1, $r(\cdot)$ does not depend on t_0, then the system (1) is said to be uniformly stable in the sense of Lyapunov.

Finally, if the equilibrium at the origin is attractive (see [8]), uniform asymptotic stability is obtained.

Definition 3: If the system (1) is uniformly stable and there exists an $\varepsilon > 0$, independent of t_0, such that $\|x(t_0)\| < \varepsilon$ implies $\lim_{t\to\infty} x(t) = 0$, then the system is said to be uniformly asymptotically stable.

The aim of this paper is to derive numerically tractable necessary and sufficient conditions to assess the stability and the asymptotic stability of system (1). To this end, some well known facts concerning the solutions of system (1) are recalled.

For any t_0 and $x(t_0) = x_0$, the solution of system (1) is given by

$$x(t,t_0,x_0) = \Phi(t,t_0)x(t_0), \ \forall t \geq t_0 \tag{2}$$

where $\Phi(t,t_0)$, the state transition matrix, is the unique solution of the matrix differential equation [9, 10]

$$\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0), \ \forall t \geq t_0, \quad \Phi(t_0,t_0) = I \tag{3}$$

It is also well known that $\Phi(t,t_0)$ is non singular for all $t \geq t_0$ [10].

III. Main results

This section presents the main results of the paper, namely, numerical tractable necessary and sufficient conditions for stability and for asymptotic stability.

Consider the following sets

$$\Omega_0 = \left\{x \in \mathbb{R}^n;\ x'x \leq \rho_0^2,\ \rho_0 > 0\right\} \tag{4}$$

and

$$E_t = \left\{x \in \mathbb{R}^n;\ x'x \leq \rho(t)^2, \quad \rho(t) > 0,\ \forall t \geq t_0\right\} \tag{5}$$

with $\Omega_0 \subseteq E_{t_0}$ (i.e. $\rho(t_0) \geq \rho_0$). Then, the following central result can be presented.

Theorem 1: Consider system (1) and suppose that $x(t_0) = x_0 \in \Omega_0$. Then $x(t) \in E_t$ for all $t \geq t_0$ if and only if $\rho(t)$ is such that

$$\rho(t) \geq \overline{\rho}(t) = \rho_0 \lambda_{\max}^{1/2}(X(t,t_0)) \tag{6}$$

where matrix $X(t,t_0)$ is the solution of the following Lyapunov differential matrix equation

$$\frac{\partial}{\partial t}X(t,t_0) = A(t)X(t,t_0) + X(t,t_0)A(t)', \quad X(t_0,t_0) = I \tag{7}$$

Proof: Consider the following optimization problem for a given instant of time t

$$\begin{cases} \max\limits_{x(t)} \ \rho(t)^2 = x(t)'x(t) \\ x_0'x_0 \leq \rho_0^2 \end{cases} \tag{8}$$

At each instant of time t, the optimal solution provides the maximal (in terms of $\rho(t)^2$) set E_t given by (5) containing $\|x(t)\|$ for all initial conditions inside Ω_0 defined in (4). The problem can be equivalently formulated only in terms of the initial condition x_0 as follows.

$$\begin{cases} \max\limits_{x_0} \ \rho(t)^2 = x_0'\Phi(t,t_0)'\Phi(t,t_0)x_0 \\ x_0'x_0 \leq \rho_0^2 \end{cases} \tag{9}$$

The Lagrangian of this optimization problem reads

$$\mathscr{L}(x_0,\beta) = x_0'\Phi(t,t_0)'\Phi(t,t_0)x_0 + \beta\left(x_0'x_0 - \rho_0^2\right) \tag{10}$$

with $\beta \leq 0$. The optimality conditions yield

$$\frac{\partial \mathscr{L}}{\partial x_0} = 0 \quad \Rightarrow \quad 2\Phi(t,t_0)'\Phi(t,t_0)x_0 + 2\beta x_0 = 0 \tag{11}$$

which can be rewritten as

$$2\beta\Phi(t,t_0)'\left(\Phi(t_0,t)'\Phi(t_0,t) + \beta^{-1}I\right)\Phi(t,t_0)x_0 = 0 \tag{12}$$

and

$$\frac{\partial \mathscr{L}}{\partial \beta} = 0 \quad \Rightarrow \quad x_0'x_0 - \rho_0^2 = 0 \tag{13}$$

To obtain a solution $x_0 \neq 0$ satisfying both conditions, one must have

$$\det\left(\Phi(t_0,t)'\Phi(t_0,t) + \beta^{-1}I\right) = 0 \tag{14}$$

Multiplying condition (11) on the left by x_0', it follows that

$$x_0'\Phi(t,t_0)'\Phi(t,t_0)x_0 + \beta x_0'x_0 = 0 \tag{15}$$

and therefore one can deduce that

$$\rho(t)^2 = -\rho_0^2\beta, \quad \forall t \geq t_0 \tag{16}$$

Moreover, from (14), it follows that $-\beta^{-1}$ is an eigenvalue of matrix $\Phi(t_0,t)'\Phi(t_0,t)$. Then, from (16) one has

$$\begin{aligned} \overline{\rho}^2(t) &= \max\ \rho^2(t) \\ &= \rho_0^2\lambda_{\min}^{-1}\left(\Phi(t_0,t)'\Phi(t_0,t)\right) = \ \rho_0^2\lambda_{\min}^{-1}\left(X(t,t_0)^{-1}\right) \\ &= \rho_0^2\lambda_{\max}(X(t,t_0)) \end{aligned} \tag{17}$$

If $\rho(t) \geq \overline{\rho}(t)^2$ for all $t \geq t_0$, the state of the system is confined in the family of sets E_t and the sufficiency part is proven. To prove the necessity, suppose $x(t) \in E_t$ and that for $t > t_0$, $\rho(t) < \overline{\rho}(t)$. This implies that there exist both $x_0 \in \Omega_0$ and $t > t_0$ such that $x(t)$ does not belong to the ellipsoid E_t, which leads to a contradiction. This concludes the proof. ■

Remark 1: It is important to note that Theorem 1 defines in the state space a tube containing all the trajectories of the system for each $t \geq t_0$. Note also that, for each t, the optimal solution of (8) is such that $x_0'x_0 = \rho_0^2$. Therefore, for some $x_0 \in \Omega_0$, $x_0'x_0 = \rho_0^2$, there always exists a trajectory $x(t)$ such that $x(t)'x(t) = \overline{\rho}^2(t)$ (i.e. a trajectory that reaches the boundary of the tube).

A. Necessary and sufficient conditions for stability

The result of Theorem 1 suggests the following test for stability.

Theorem 2: The system (1) is stable if and only if

$$\rho_M = \max_{t \geq t_0} \ \rho_0 \lambda_{\max}^{1/2}(X(t,t_0)) < \infty \tag{18}$$

where matrix $X(t,t_0)$ is the solution to Lyapunov differential matrix equation (7).

Proof: Theorem 1 states that, for initial conditions such that $x_0'x_0 \leq \rho_0^2$, one has

$$x'(t)x(t) \leq \overline{\rho}(t)^2 \tag{19}$$

Now, choose an arbitrary scalar $\delta > 0$, define a function $\gamma(t)$ satisfying

$$\delta = \gamma(t)\ \lambda_{\max}^{1/2}(X(t,t_0)), \quad \forall t \geq t_0 \tag{20}$$

and consider the sets

$$\left\{x_0 \in \mathbb{R}^n : x_0'x_0 \leq \gamma(t)^2,\ \gamma(t) > 0\right\} \tag{21}$$

Following the proof of Theorem 1, each set represents the set of initial conditions such that $x(t)'x(t) \leq \delta^2$ at time t. For all $t \geq t_0$, if the initial condition belongs to the set

$$\bigcap_{t \geq t_0} \left\{x_0 \in \mathbb{R}^n : x_0'x_0 \leq \gamma(t)^2\right\} = \left\{x_0 \in \mathbb{R}^n : x_0'x_0 \leq r_0^2\right\} \tag{22}$$

with

$$r_0 = \min_{t \geq t_0} \delta\ \lambda_{\max}^{-1/2}(X(t,t_0)) \leq \frac{\delta\rho_0}{\rho_M} \tag{23}$$

Then, for all $\delta > 0$ there exists

$$r = \frac{\delta\rho_0}{\rho_M} \text{ such that } \|x_0\| < r \ \Rightarrow\ \|x(t)\| < \delta\ , \forall t \geq t_0 \tag{24}$$

provided that $\rho_M < \infty$. Therefore, by Definition 1, the system is stable in the sense of Lyapunov. This proves the sufficiency. The necessity follows from the fact that Theorem 1 defines in the state space a tube containing all the trajectories of the system $\forall t \geq t_0$. Thus, there exists $t = \bar{t}$ corresponding to the $\overline{\rho}(t)$ maximal such that $x(\bar{t})'x(\bar{t}) = \rho_M^2$ and, if the system is stable, ρ_M must be finite. Uniformity comes from the fact that r is independent of the initial time $\bar{t}_0 \geq t_0$. ■

B. Necessary and sufficient conditions for asymptotic stability

The results from Theorem 2 can be extended to provide necessary and sufficient conditions for asymptotic stability.

Theorem 3: The system (1) is asymptotically stable if and only if

$$\rho_M < \infty \quad \text{and} \quad \lim_{t \to \infty} \overline{\rho}(t) = 0 \tag{25}$$

Proof: The stability is assured by $\rho_M < \infty$. Then, it is clear that if $\lim_{t\to\infty} \overline{\rho}(t) = 0$ one has $\lim_{t\to\infty} x(t) = 0$. By choosing $\varepsilon > 0$ independently of t_0, it is always possible to find ρ_0 at time t_0 such that

$$\left\{x \in \mathbb{R}^n;\ x_0'x_0 \leq \varepsilon^2\right\} \subset \left\{x \in \mathbb{R}^n;\ x_0'x_0 \leq \rho_0^2\right\} \tag{26}$$

Therefore, whenever $\|x_0\| < \varepsilon$ one has $\lim_{t\to\infty} x(t) = 0$ implying that the system is asymptotically stable. The necessity part follows similarly as in the proof of Theorem 1. ■

Theorems 2 and 3 provide necessary and sufficient conditions for testing the stability and the asymptotic stability of system (1). Since system (1) is linear, the results do not depend on the particular choice for ρ_0. The method requires the numerical integration of the Lyapunov differential linear equation (7) in the interval $[t_0, \infty)$. This can be easily done by some standard techniques and, in practice, a finite time interval can be used. As pointed out in [11], in many practical situations the systems operate in a finite time horizon. An important point concerning time-varying periodic systems is that conclusive responses about the stability of the system can be obtained by integrating (7) over one single period. This extension is presented in the following subsection.

C. Periodic systems

Suppose now that the linear time-varying system (1) is periodic, i.e.

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0,\ A(t+T) = A(t) \tag{27}$$

Note that Theorem 2 can be applied to periodic system as well, but in this special case the test can be formulated taking into account the period T, as presented in the following corollary.

Corollary 4: The periodic system (27) is stable if and only if

$$\rho_T = \max_{t \in [t_1, t_1+T]} \ \rho_0 \lambda_{\max}^{1/2}(X(t,t_0)) < \infty \tag{28}$$

for all $t_1 \geq t_0$.

Proof: The proof follows from Theorem 2. ■

Concerning asymptotic stability, Theorem 3 can be extended as well to cope with periodic systems and the Lyapunov differential equation (7) can be integrated over one single period, as shown in next corollary.

Corollary 5: The periodic system (27) is asymptotically stable if and only if

$$\rho_T < \infty \quad \text{and} \quad \overline{\rho}(t_1+T) < \overline{\rho}(t_1), \ \ \forall t_1 \geq t_0 \tag{29}$$

with

$$\rho_T = \max_{t \in [t_1, t_1+T]} \ \rho_0 \lambda_{\max}^{1/2}(X(t,t_0)),\ t_1 \geq t_0 \tag{30}$$

$$\overline{\rho}(t) = \rho_0 \lambda_{\max}^{1/2}(X(t,t_0)) \quad (31)$$

and matrix $X(t,t_0)$ is the solution of the Lyapunov differential matrix equation (7).

Proof: Since the system is periodic, (29) implies that $x(t)$ is bounded for all $t \geq t_0$ and also that

$$\lim_{t\to\infty} \overline{\rho}(t) = 0 \Rightarrow \lim_{t\to\infty} x(t) = 0 \quad (32)$$

The necessity follows with the arguments developed in the proof of Theorem 1. ■

IV. NUMERICAL EXAMPLES

To illustrate the proposed conditions, some examples from the literature are considered. The numerical simulations were performed with `ode45` using default options in Matlab. The continuity of $A(t)$ assures the good behavior of the method, since the results are related to the numerical solution of a standard differential equation. In all the examples, $\rho_0 = 1$ has been chosen.

A. Example 1

This example concerns a mass-spring system where both damping and elastic constant coefficients are time-varying, as presented for instance in [11]. The dynamic matrix $A(t)$ is given by

$$A(t) = \begin{bmatrix} 0 & 1 \\ -(2-\alpha\sin t) & -(2-\alpha\cos t) \end{bmatrix} \quad (33)$$

where α is a positive constant parameter. It is simple to verify that this is a periodic system with period $T = 2\pi$ and also that, for $\alpha < 2$, the dynamic matrix $A(t)$ is Hurwitz for all t. Note that the Hurwitz stability assumption is central for the method proposed in [11], where this example was investigated in details. Indeed, some conditions in the literature require the time variation of the system matrix to be small [4–6], that is

$$\|\dot{A}(t)\| \leq \gamma \ \forall t \quad \text{or} \quad \frac{1}{T_1}\int_{t_0}^{t_0+T_1} \|\dot{A}(t)\| dt \leq \gamma \quad (34)$$

for γ sufficiently small and a suitable T_1. For this particular example, one has

$$\|\dot{A}(t)\| = \lambda_{max}\left(\dot{A}(t)'\dot{A}(t)\right) = \alpha \ \forall t \quad (35)$$

and therefore, as discussed in [11], it is not possible to improve the results obtained by Rosenbrock [1] with the sufficient conditions proposed in the aforementioned references.

To compare the potentialities of the sufficient conditions proposed in [1, 11] with the necessary and sufficient conditions proposed in this paper, the maximal value of α for which stability is assured by each method has been computed, as shown in Table I.

To further illustrate the evolution of function $\overline{\rho}(t)$, Figure 1 shows the time simulation in the interval $t \in [0, 14]$ for $\alpha = 1.53$ (the limit of stability assured by the algorithm in [11]) and for $\alpha = 3.1$, two situations where the asymptotic stability can be confirmed since $\overline{\rho}(4+2\pi) < \overline{\rho}(4)$. Figure 2 shows $\overline{\rho}(t)$ for $\alpha = 3.1$ and also $\|x(t)\|$ for 100 initial conditions

TABLE I
MAXIMUM VALUES OF α FOR EXAMPLE 1.

Method	α
[1]	< 1.3
[11]	< 1.6
Corollary 5	$< \mathbf{3.162}$

such that $x_0'x_0 = \rho_0^2$ randomly generated. It can be noted that there is no gap between the computed tube and the set of all the trajectories of the system, as predicted by the theory.

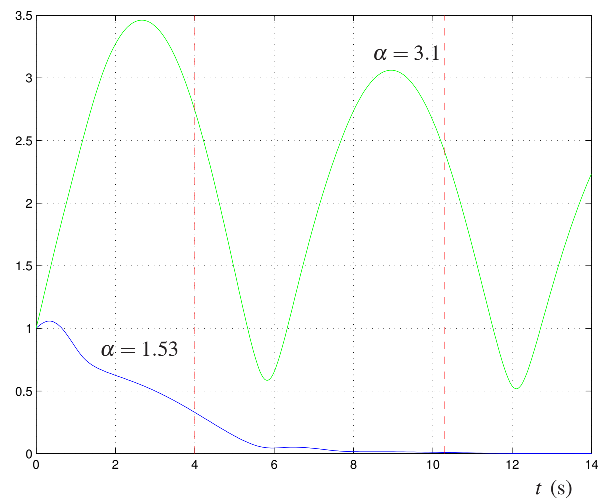

Fig. 1. Function $\overline{\rho}(t)$ for $\alpha = 1.53$ and for $\alpha = 3.1$ (the vertical bars indicate 4 and $4+2\pi$) in Example 1.

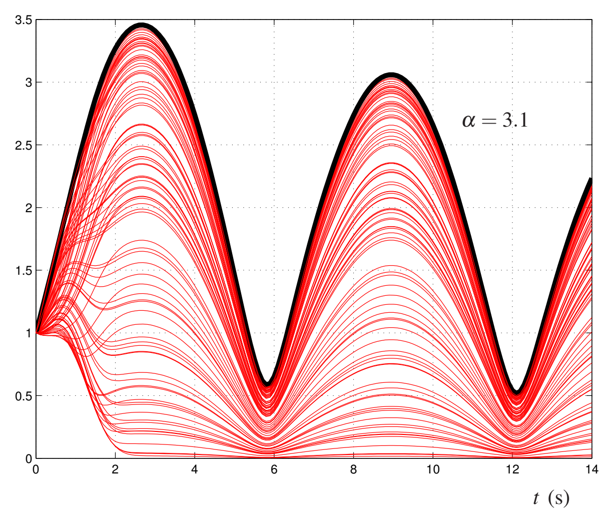

Fig. 2. Function $\overline{\rho}(t)$ and $\|x(t)\|$ for 100 initial conditions such that $x_0'x_0 = \rho_0^2$ randomly generated for $\alpha = 3.1$ in Example 1.

The value of α leading to the limit of asymptotic stability has been found to be approximately equal to 3.162 through a line search using the conditions of Corollary 5. This indicates that the system from Example 1 is asymptotically stable if and only if $\alpha < 3.162$. The limit situation is illustrated by the time behavior of $\overline{\rho}(t)$ in Figure 3 and also in Figure 4, where the phase planes for different initial conditions x_0, $x_0'x_0 = 1$, leading to distinct periodic solutions, are depicted.

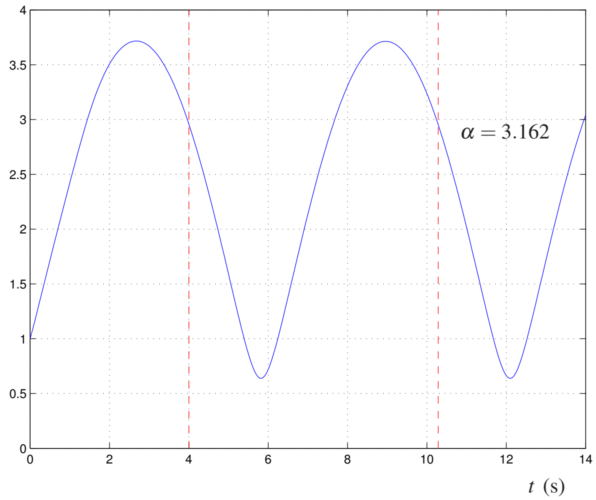

Fig. 3. Function $\overline{\rho}(t)$ for $\alpha = 3.162$ (the vertical bars indicate 4 and $4+2\pi$) in Example 1.

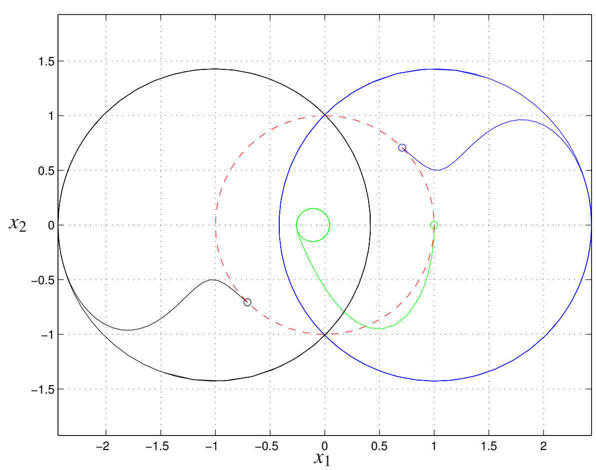

Fig. 4. Phase planes for $\alpha = 3.162$ in Example 1, for initial conditions $\left[\ 1/\sqrt{2}\ \ 1/\sqrt{2}\ \right]'$, $\left[\ -1/\sqrt{2}\ \ -1/\sqrt{2}\ \right]'$ and $\left[\ 1\ \ 0\ \right]'$. The dashed line indicates the circle $x'x = 1$.

To end with this example, the time simulation of $x(t)$ is shown for the initial condition $x_0 = \left[\ 1/\sqrt{2}\ \ 1/\sqrt{2}\ \right]'$ in two different situations: stable ($\alpha = 3.1$, Figure 5) and unstable ($\alpha = 3.2$, Figure 6) behaviors, corroborating the results obtained with Corollary 5.

B. Example 2

The second example is borrowed from [14] and corresponds to an unstable system, periodic with $T = \pi$, given by

$$A(t) = \begin{bmatrix} -1+1.5\cos^2 t & 1-1.5\sin t\cos t \\ -1-1.5\sin t\cos t & -1+1.5\sin^2 t \end{bmatrix} \qquad (36)$$

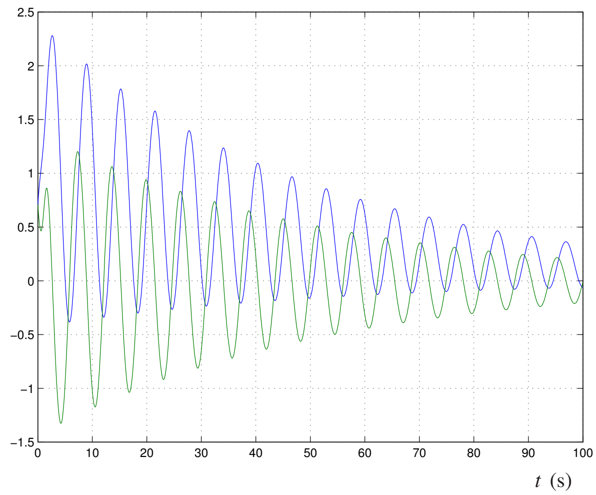

Fig. 5. Trajectories $x(t)$ for $x_0 = \left[\ 1/\sqrt{2}\ 1/\sqrt{2}\ \right]'$ and $\alpha = 3.1$ in Example 1.

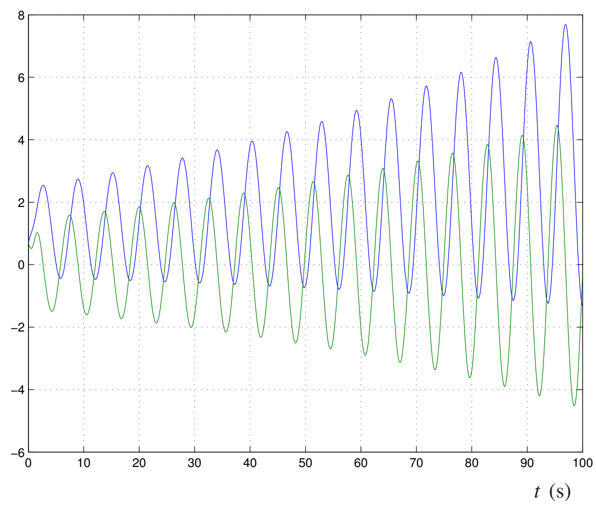

Fig. 6. Trajectories $x(t)$ for $x_0 = \left[\ 1/\sqrt{2}\ 1/\sqrt{2}\ \right]'$ and $\alpha = 3.2$ in Example 1.

which is Hurwitz for all t. Actually, the eigenvalues of $A(t)$ are given by $-0.25 \pm j0.25\sqrt{7}$ and do not depend on t. The state transition matrix is given by [14]

$$\Phi(t,0) = \begin{bmatrix} \exp(0.5t)\cos t & \exp(-t)\sin t \\ -\exp(0.5t)\sin t & \exp(-t)\cos t \end{bmatrix} \qquad (37)$$

indicating that the trajectories can diverge to infinity for initial conditions arbitrarily close to the origin. Note that methods based on sufficient conditions as in [11] cannot conclude about instability. On the other hand, the approach proposed here detects instability, as illustrated in Figure 7, where $\overline{\rho}(t)$ is shown together with $\|x(t)\|$ for 100 initial conditions randomly generated.

V. Conclusion

Necessary and sufficient conditions for stability and for asymptotic stability of linear continuous time-varying systems have been given. The conditions are based on the

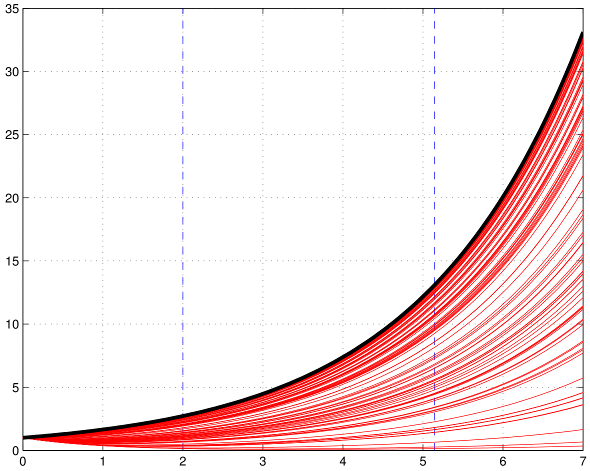

Fig. 7. Function $\overline{\rho}(t)$ and $\|x(t)\|$ for 100 initial conditions randomly generated (the vertical bars indicates 2 and $2+\pi$) in Example 2.

numerical solution of a linear differential Lyapunov equation, associated to a tube in the state space that confines all the trajectories of the system. For periodic systems, it suffices to integrate the differential Lyapunov equation over one single period. Extensions of the proposed conditions to cope with control design problems, particularly in the case of periodic systems, are being investigated by the authors.

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