**Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008**

TuC04.4

Computational Complexity Analysis of Set Membership Identification of a class of Nonlinear Systems.

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Abstract— This paper analyzes the computational complexity of set membership identification of a class of nonlinear systems consisting of the interconnection of a Linear Time Invariant plant and a static nonlinearity. Its main result shows that, even in cases where a portion of the plant is known, the problem is generically NP–hard both in the number of experimental data points and in the number of inputs or outputs of the nonlinearity. These results provide new insight into the reasons underlying the high computational complexity of several recently proposed algorithms and point out to the need for developing computationally tractable relaxations.

I. INTRODUCTION

Set membership identification of nonlinear systems consisting of the interconnection of a memoryless, static nonlinearity and a Linear Time Invariant (LTI) plant has received considerable attention in the past decade (see for instance [1], [2], [3], [5], [6], [7], [8], [9], [10], [11], [12], [13], [16], [17], [21] and references therein). These methods are attractive since they furnish hard bounds on the values of the unknown parameters of the plant, in a form that can be directly used for instance by robust control synthesis techniques. However, the resulting algorithms typically have high computational complexity (entailing for instance solving an optimization over rank one matrices [7], non-convex optimization problems [6], or a combinatorial number of Linear Programs [2]), or require assuming both that several intermediate signals are available for measurement and that portions of the plant are known [9], [10], [11], [13].

The goal of this paper is to shed some insight into the reasons underlying this high computational complexity. As shown here, this is an intrinsic difficulty of the general framework, rather than a feature of specific approaches. Our main result shows that, contrary to the case of *linear identification*, the problem of set–membership identification of Lur'e type systems consisting of the interconnection of a memoryless nonlinearity and a Linear Time Invariant Plant is generically NP–hard in both the dimension of the nonlinearity, and in the number of experiments, *even* when a portion of the system is known. It follows that nonlinear identification is an intrinsically difficult problem, even in cases involving relatively simple structures, such as Wiener or Hammerstein systems, and partially known elements.

Hence, the need to develop polynomial time relaxations, such as the ones in [7], [2], [6], [14], [19] or to assume the availability of intermediate signals [9], [10], [11]. In addition, these results also highlight a connection between high computational complexity and non-invertibility of the static nonlinearity.

II. PRELIMINARIES

In this section we introduce the notation used in the paper and precisely state the problem under consideration:

A. Notation

Z, R, C	set of integer, real, complex numbers, respectively
$\overline{\sigma}(\mathbf{A})$	maximum singular value of the matrix $\mathbf{A}$.
ℓ^p	extended Banach space of vector valued real sequences equipped with the norm:

$$\|x\|_p \doteq \left(\sum_{i=0}^{\infty} \|\mathbf{x}_i\|^p \right)^{\frac{1}{p}},$$

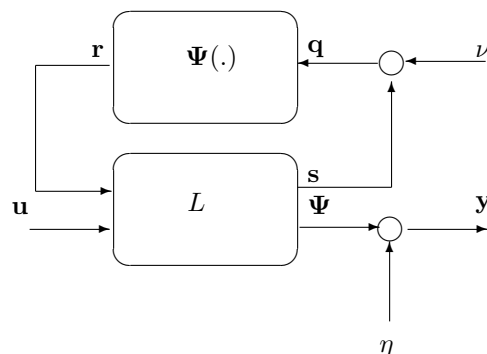|	
	$p \in [1, \infty)$ and $\|x\|_\infty \doteq \sup_i \|\mathbf{x}_i\|_\infty$.
$\mathcal{H}_\infty$	space of transfer functions analytic in $|z| \leq 1$, equipped with the norm $\|G\|_\infty \doteq ess \sup_{|z| < \rho} \overline{\sigma}(G(z))$.
$\overline{\mathcal{B}}\mathcal{H}_\infty(\gamma)$	closed γ–ball in $\mathcal{H}_\infty$: $\{H \in \mathcal{H}_\infty \colon \|H\|_\infty \leq \gamma\}$.

B. Problem Statement

Fig. 1. Lur'e Systems Identification Setup

In this paper we are interested in analyzing the computational complexity analysis of the problem of set membership

identification of Lur'e systems using time–domain data. Specifically, we will consider systems of the form shown in Figure 1 consisting of the interconnection of a LTI system L and a memoryless nonlinearity $\mathbf{\Psi}(.)$. The corresponding equations are given by:

$$\begin{bmatrix} \mathbf{q}_k \\ \mathbf{y}_k \end{bmatrix} = (\mathbf{L} * \begin{bmatrix} r \\ u \end{bmatrix})_{\mathbf{k}} + \begin{bmatrix} \nu_k \\ \eta_k \end{bmatrix}$$
$$\mathbf{r}_k = \mathbf{\Psi}(\mathbf{q_k})$$

(1)

where $*$ denotes convolution and the signals $\mathbf{u} \in R^{n_u}$ and $\mathbf{y} \in R^{n_y}$ represent the experimental data: a known finite input sequence and its corresponding output sequence, corrupted by unknown but norm–bounded measurement noise η. Note that in general the intermediate signals $\mathbf{s} \in R^{n_p}$ (the output of the LTI system), ν (noise at the input to the nonlinearity) and $\mathbf{r}$, the output of the nonlinearity, are, in general, not measurable. In this context, the set membership nonlinear identification problem of interest here can be stated as:

Problem 1: Given: (i) *a priori* information consisting of a set membership description of the admissible plants, non-linearities and noise, $\mathcal{S}, \mathcal{NL}, \mathcal{N} \doteq \mathcal{N}_\eta \times \mathcal{N}_\nu$, respectively, and (ii) *a posteriori* experimental data $\{\mathbf{y}_k, \mathbf{u}_k\}_{k=0}^{n_m-1}$, determine:

1) if the *a priori* and *a posteriori* information are consistent, i.e., the consistency set

$$\mathcal{T}(\mathbf{y}, n_m, \mathcal{N}) \doteq \{ L \in \mathcal{S} :$$
$$y_k = \left[\mathcal{F}_u(L, \mathbf{\Psi})(\begin{bmatrix} \nu \\ \mathbf{u} \end{bmatrix}) \right]_k + \eta_k,$$
$$\text{for some } \mathbf{\Psi} \in \mathcal{NL}$$
$$\text{and some sequences } (\nu_k, \eta_k) \in \mathcal{N},$$
$$k = 0, 1, \ldots, n_m - 1. \}$$

(2)

is nonempty.
2) If $\mathcal{T}(\mathbf{y}, n_m, \mathcal{N}) \neq \emptyset$, find a nominal model $\{L, \mathbf{\Psi}(.)\}$ that interpolates the experimental data[1]

In particular, in its simplest form the set description of the admissible set of linear plants and noise are (see for instance [14]) $\mathcal{S} \doteq \bar{\mathcal{B}}\mathcal{H}_\infty(K)$, $\mathcal{N}_\eta \doteq \{\eta : \|\eta_k\|_p \le \epsilon_\eta\}$, and $\mathcal{N}_\nu \doteq \{\nu : \|\nu_k\|_p \le \epsilon_\nu\}$, for some known constants K, ϵ_η, ϵ_ν.

III. BACKGROUND RESULTS ON COMPUTATIONAL COMPLEXITY

In order to establish that Problem 1 is NP–hard, we need the following preliminary results concerning the computational complexity of two decision problems:

Lemma 1 ([15]): Given a vector $\mathbf{a} \in Z^n$, the problem of determining if there exists a vector $\mathbf{x}$, $|x_i| = 1$ such that $\mathbf{a}^T\mathbf{x} = 0$ (the knapsack problem) is NP-complete.

Lemma 2 ([4], page 307): For a given vector $\mathbf{a} \in Z^n$, there exists a polynomial time computable $(2n+1) \times (2n+1)$ symmetric matrix $\mathbf{A_a}$ and a polynomial time computable

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number $\epsilon_\mathbf{a} \in (0, 1)$ such that $\max_{\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{D}^{2n+1}} |\mathbf{v_1^T A_a v_2}| = 1$ if there exists a solution $\mathbf{x} \in \{-1, 1\}^n$ to $\mathbf{a}^T\mathbf{x} = 0$, and is less than or equal to $1 - \epsilon_\mathbf{a}$ otherwise.

Corollary 1: The problem of checking whether $\max_{\mathbf{v}_1, \mathbf{v}_2 \in D^{2n+1}} |\mathbf{v_1^T A_a v_2}| > 1 - \frac{\epsilon_\mathbf{a}}{2}$ is NP-hard, since the knapsack problem can be reduced to it in polynomial time.
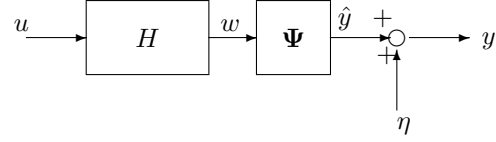

Fig. 2.   Wiener System Structure

The results above can be used to establish the following results concerning the computational complexity of two identification problems:

Lemma 3: [20] Consider the Wiener system structure shown in Figure 2, with a quadratic nonlinearity of the form $\hat{y} = \mathbf{w}^T \mathbf{A} \mathbf{w}$, where $\mathbf{A}$ is a known, given matrix. Assume that $\mathcal{N} \doteq \{\eta : \|\eta_k\|_\infty \le \epsilon\}$. Then

(i) The problem of identifying $H(.)$ is NP hard in $n \doteq \max\{\text{number of inputs}, \text{number of outputs}\}$ to H, the linear portion of the plant.
(ii) The problem of identifying $H(.)$ is NP hard in the number of experimental data pairs $(\mathbf{y}, \mathbf{u})$.

Proof: The proof proceeds by showing that the decision problem of Corollary 1 can be reduced in polynomial time to a Wiener identification problem with a quadratic nonlinearity (see [20] for details). ∎
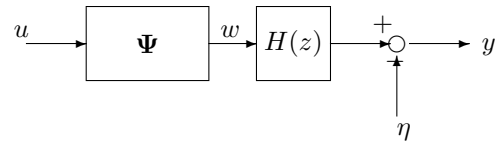

Fig. 3.   Hammerstein System Structure

Lemma 4: [20] Consider a Hammerstein system structure of the form shown in Figure 3, with $H(z)$ known and $\mathcal{N} \doteq \{\eta : \|\eta_k\|_\infty \le \epsilon\}$. Then:

(i) For a set description of the nonlinearity of the form

$$\mathcal{NL} = \{\mathbf{\Psi} : \mathbf{\Psi} = \textstyle\sum_{i=1}^{n} x_i \psi_i(u), |x_i| = 1$$
$$\text{where}: \quad \psi_i(u) = \begin{cases} 1 & i-1 \le u < i \\ 0 & \text{otherwise} \end{cases}$$

(3)

the problem of identifying $\mathbf{\Psi}(.)$ is NP hard in the number of experimental data pairs $(\mathbf{y}, \mathbf{u})$.

(ii) For a set description of the nonlinearity of the form

$$\mathcal{NL} = \{\mathbf{\Psi} : \mathbf{\Psi} = [\psi_1(.), \ldots, \psi_n(.)]^T,$$
$$\psi_i = x_i \text{sign}(u), |x_i| = 1$$

(4)

the problem of identifying $\mathbf{\Psi}(.)$ is NP hard in the number of inputs to $H(.)$, the linear portion of the plant.

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[1]If $\mathcal{T} = \emptyset$, then the experimental data $\{\mathbf{y}, \mathbf{u}\}$ invalidates the *a priori* assumptions about the class of models and noise, that is, the experimental data cannot be explained by models in these sets.

Proof: The proof proceeds by showing that the knap-sack problem of Lemma 1 can be reduced in polynomial time to a Hammerstein identification problem with nonlinearities of the form (3) or (4) (see [20] for details). ∎

IV. COMPUTATIONAL COMPLEXITY ANALYSIS OF SET MEMBERSHIP IDENTIFICATION OF LUR'E TYPE SYSTEMS

With the results above, we can now establish the main result of this paper:

Theorem 1: The problem of set membership identification of Lur'e type systems of the form (1) is generically NP hard both in the number of experimental data pairs $(\mathbf{y}, \mathbf{u})$ and the number of inputs/outputs of the static nonlinearity $\mathbf{\Psi}(.)$, even in cases where either $L(.)$, the linear portion of the plant, or $\mathbf{\Psi}(.)$, the static nonlinearity, are known.

Proof: For the known nonlinearity case, choose in (1) a quadratic nonlinearity, $\epsilon_\nu = 0$, and a linear plant L of the form:

$$L = \begin{pmatrix} \mathbf{0} & \mathbf{H} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \qquad (5)$$

and apply Lemma 3. For the known linear plant case, select an unknown nonlinearity of the form (3) or (4), $\epsilon_\nu = 0$, and L of the form:

$$L = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{H} & \mathbf{0} \end{pmatrix} \qquad (6)$$

and apply Lemma 4. ∎

V. CONCLUSIONS

This paper shows that the problem of set membership identification of Lur'e type systems is generically NP–hard, even in cases where a portion of the plant is known exactly. These results highlight the fact that, as opposed to the case of *linear identification*, these problems are intrinsically difficult, shedding some insight into the high computational cost of existing approaches and pointing out to the need to search for computationally tractable relaxations. An interesting feature borne out by the analysis presented here is the key role played by the non-invertibility of the nonlinearity in reducing the knapsack problem to a Lur'e type identification and thus establishing that the problems are NP–hard. Thus, the issue of whether these problem are NP–hard in the case of *invertible* nonlinearities is still open.

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