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Dynamic Output Feedback $\mathcal{H}_\infty$ Control of Discrete-time Markov Jump Linear Systems Through Linear Matrix Inequalities

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Abstract— **This paper addresses the $\mathcal{H}_\infty$ dynamic output feedback control design problem of discrete-time Markov jump linear systems. Under the mode-dependent assumption, which means that the Markov parameters are available for feedback, the main contribution is on the complete characterization of all full order proper controllers such that the $\mathcal{H}_\infty$ norm of the closed loop system remains bounded by a given pre-specified level, yielding the global solution to the corresponding mode-dependent optimal control design problem, expressed in terms of pure Linear Matrix Inequalities – LMIs. A practical application, consisting the networked control of a vehicle platoon using measurement signals transmitted in a Markov channel, as initially proposed in [15], is considered.**

I. Introduction

In the last years, parameter dependent dynamic systems have received a great amount of attention due to its flexibility to represent with precision real world situations with practical appealing. In this framework, LPV and gain scheduling design problems appeared in the deterministic and stochastic contexts. The later class is composed by control systems where the open loop model presents sudden changes on their structures or parameters, which being modelled as Markovian processes becomes decisive for the increasing interest on the so called Markovian jump linear system - MJLS in both continuous and discrete-time domains. An important assumption to consider for MJLS design is if the Markov chain state, often called mode, is available or not to the controller at every instant of time. Based on that information the design is said to be either mode-dependent or mode-independent, respectively. In this paper only the first case is considered for the following main reasons: First, in many practical situations, the system parameters are measurable, see [6], [15], [18] and [19]. Second, as a limitation of the proposed design method based on LMIs, only full order mode-dependent controllers may be handled without introducing any kind of conservatism. The mode-independent version of the output feedback control problem needs further research effort towards its complete solution.

One of the first works in the literature dealing with this class of models was presented in [1]. After that, a large amount of theory and design procedures have been developed in order to extend the concepts of the deterministic systems to this special class, namely stability concepts and testable conditions [5], [12], [13]; optimal state feedback control [11]; state feedback $\mathcal{H}_\infty$ optimization and robustness via LMIs [7], [9], [17] and state feedback $\mathcal{H}_\infty$ via Riccati equations [2].

A problem of theoretical and practical importance in this area is the dynamic output feedback design. For the continuous-time case, many results are available in the literature being related to the complete solution of the associated $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems, see [8] and more recently [14]. However, the same is not true for discrete-time systems for which only few results are available, see [6], [15] and [19]. Indeed, the output feedback control problem with $\mathcal{H}_\infty$ criterion has been treated in [15] but restricting the attention to strictly proper controllers and, in addition, treating exclusively a very particular Markov chain characterized by the transition probability matrix with identical rows. Finally, [19] proposes to handle the problem by a relaxation technique applied to bilinear matrix inequalities.

In this paper, the $\mathcal{H}_\infty$ control problem is considered and, contrary to what has been done in [15], we do not make any assumption on the probability transitions of the Markov chain. We believe that the present paper innovates on the following directions:

- The set of all full order, proper and mode-dependent controllers imposing a pre-specified $\mathcal{H}_\infty$ norm level to the controlled output of the closed loop system is provided. As a consequence, from the solution of convex problems expressed in terms of pure LMIs, the global optimal $\mathcal{H}_\infty$ controllers are determined in only one shot avoiding an iterative process and convergence difficulties to get the global solution, [3].
- The controllers are parameterized by LMIs whose dimensions depend upon the dimension of the state variable of the open loop system and not on the number of modes of the Markov chain. This contributes decisively to decrease the computational burden involved.

The paper is organized as follows. In the next section, classical results as stability and $\mathcal{H}_\infty$ norms calculations using LMIs are presented. In the same section it is also introduced a one-to-one change of variables used throughout the text for the linearization of the nonlinear dependence of the previously mentioned norm with respect to the matrices of the controller state space realization. In Section III the $\mathcal{H}_\infty$ norm control design problem is solved where all results are necessary and sufficient for the class of controllers (full order, proper and mode-dependent) considered. Section IV is

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devoted to present a practical application of the theoretical results obtained so far. It consists on networked control of a vehicle platoon using measurement signals transmitted in a Markov channel modelled in [15] but, in our opinion, with a more realistic, from the practical viewpoint, transition probability matrix. Section V presents the main conclusions of the paper and brief considerations on further works.

The notation used throughout is standard. Capital letters denote matrices and small letters denote vectors. For scalars, small Greek letters are used. For real matrices or vectors ($'$) indicates transpose. For square matrices $\mathrm{Tr}(X)$ denotes the trace function of X being equal to the sum of its eigenvalues and, for the sake of easing the notation of partitioned symmetric matrices, the symbol ($\bullet$) denotes generically each of its symmetric blocks. The set of natural numbers is denoted by $\mathbb{N}$ while $\mathbb{K} = \{1, \cdots, N\}$. Given N^2 nonnegative real numbers p_{ij} satisfying $p_{i1} + \cdots + p_{iN} = 1$ for all $i \in \mathbb{K}$ and N positive definite matrices $X_j \in \mathbb{R}^{n\times n}$ for all $j \in \mathbb{K}$, the convex combination of these matrices with weights p_{ij} is denoted by $X_{pi} = \sum_{j=1}^{N} p_{ij} X_j$ for all $i \in \mathbb{K}$. Similarly, for positive definite matrices, the inverse of the convex combination of inverses is denoted as

$$X_{qi} = \left(\sum_{j=1}^{N} p_{ij} X_j^{-1} \right)^{-1} \tag{1}$$

Clearly, X_{pi} depends linearly on matrices $X_1, \cdots, X_N$ while the dependence of X_{qi} with respect to the same matrices is highly nonlinear. The same mathematical manipulations are adopted for positive definite matrices depending on two indexes $i, j \in \mathbb{K}\times\mathbb{K}$. The symbol $\mathcal{E}\{\cdot\}$ denotes mathematical expectation of $\{\cdot\}$. For any stochastic signal $\xi(k)$, defined in the discrete-time domain $k \in \mathbb{N}$, the quantity $\|\xi\|_2^2 = \sum_{k=0}^{\infty} \mathcal{E}\{\xi(k)'\xi(k)\}$ is its squared norm. The class of all signals $\xi(k) \in \mathbb{R}^r$, $k \in \mathbb{N}$ with $\|\xi\|_2^2$ finite is denoted $\mathcal{L}_2^r$.

II. Problem Formulation and Basic Results

A discrete-time Markovian jump linear system, denoted $\mathbb{G}$, is described by the following stochastic equations

$$\begin{aligned} x(k+1) &= A(\theta_k)x(k) + B(\theta_k)u(k) + J(\theta_k)w(k) \\ z(k) &= C_z(\theta_k)x(k) + D_z(\theta_k)u(k) + E_z(\theta_k)w(k) \\ y(k) &= C_y(\theta_k)x(k) + E_y(\theta_k)w(k) \end{aligned}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control, $w(k) \in \mathbb{R}^p$ is the external perturbation, $z(k) \in \mathbb{R}^r$ is the controlled output and $y(k) \in \mathbb{R}^q$ is the measured output. The state space matrices of system $\mathbb{G}$ depend upon a Markov chain taking values in the finite set $\mathbb{K}$ with the associated transition probability matrix given by $p_{ij} = Prob(\theta_{k+1} = j|\theta_k = i)$ which clearly satisfies the normalized constraints $p_{ij} \geq 0$ and $\sum_{j=1}^{N} p_{ij} = 1$ for each $i \in \mathbb{K}$. To ease the presentation, the following notations $A(\theta_k) := A_i$, $B(\theta_k) := B_i$, $J(\theta_k) := J_i$, $C_z(\theta_k) := C_{zi}$, $D_z(\theta_k) := D_{zi}$, $E_z(\theta_k) := E_{zi}$, $C_y(\theta_k) := C_{yi}$ and $E_y(\theta_k) := E_{yi}$ whenever $\theta_k = i \in \mathbb{K}$ are adopted. The first important concept related to the system $\mathbb{G}$ is stability. In the context of MJLS, there are several equivalent forms to define stability. See for instance, [12] where it has been shown that some definitions of stability are actually equivalent for a MJLS being referred to as second-moment stability - SMS. The next proposition [5], [11] presents a method to check stability from the existence of a positive definite solution of a set of coupled Lyapunov-like inequalities.

Lemma 1: System $\mathbb{G}$ is stable if and only if there exist, for all $i \in \mathbb{K}$, matrices $P_i = P_i' > 0$ such that

$$A_i' P_{pi} A_i - P_i < 0 \tag{2}$$

Although the inequalities (2) are already in the form of LMIs, as it will be clear in the sequel, they present some difficulties to be circumvented. Basically, the main technical difficulty is the way the summations that take the jump probabilities into account appear in the inequalities, involving the system dynamics and preventing one from using the standard transformations available in the literature to linearize the controller formulas. For the sake of comparison, recall that in continuous-time MJLS these summations appear as additive terms to the standard inequalities and never involve products with the dynamic matrices [8]. The following lemma provides an alternative characterization of stability for discrete-time MJLS. As it will be shown later, these inequalities are more appropriate for dynamic output feedback control design.

Lemma 2: System $\mathbb{G}$ is stable if and only if there exist, for all $i \in \mathbb{K}$, matrices $P_i = P_i' > 0$ such that

$$\begin{bmatrix} P_i & A_i' \\ A_i & P_{pi}^{-1} \end{bmatrix} > 0 \tag{3}$$

The nonlinear inequalities in Lemma 2 have several formal advantages over the linear ones appearing in Lemma 1. The notation used in Lemma 2 puts in evidence that the inequalities required for testing stability involve only matrices of the same index $i \in \mathbb{K}$ and the coupling between the indices can be dealt with by the linear equality constraint $P_{pi} = \sum_{j=1}^{N} p_{ij} P_j$, which does not involve the dynamic system matrices A_i, for all $i \in \mathbb{K}$. Such feature will be of extreme importance in deriving simple and effective formulas for an adequate parametrization of the controller state space matrices. The next definition is the generalization of the $\mathcal{H}_\infty$ norm from linear time invariant - LTI systems to the stochastic Markovian jump case under consideration. The formal definition of this important concept is as follows.

Definition 1: The $\mathcal{H}_\infty$-norm of a stable system $\mathbb{G}$ from the input w to the output z is given by

$$\|\mathbb{G}\|_\infty^2 = \sup_{0\neq w\in\mathcal{L}_2^p, \theta_0\in\mathbb{K}} \frac{\|z\|_2^2}{\|w\|_2^2} \tag{4}$$

It is interesting to observe that in the deterministic case characterized by $N = 1$ the previous definition reduces to the usual $\mathcal{H}_\infty$-norm of the LTI discrete-time system $\mathbb{G}$. The next lemma shows how the $\mathcal{H}_\infty$-norm of the system $\mathbb{G}$ can be calculated [4], [16].

Lemma 3: The system $\mathbb{G}$ is stable and satisfies the norm constraint $\|\mathbb{G}\|_\infty^2 < \gamma$ if and only if there exist matrices

$P_i = P_i' > 0$ such that

$$\begin{bmatrix} A_i & J_i \\ C_{zi} & E_{zi} \end{bmatrix}' \begin{bmatrix} P_{pi} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_i & J_i \\ C_{zi} & E_{zi} \end{bmatrix} - \begin{bmatrix} P_i & 0 \\ 0 & \gamma I \end{bmatrix} < 0 \quad (5)$$

holds for all $i \in \mathbb{K}$.

Lemma 3 is a bounded real lemma for the MJLS. It can be obtained from the conditions derived in [4], see also [15], [16] and the references therein. Notice that, as we have already mentioned, (5) reduces to the deterministic $\mathcal{H}_\infty$ norm condition for $N = 1$ and to be feasible it requires the existence of positive definite matrices P_i such that $A_i' P_{pi} A_i - P_i + C_{zi}' C_{zi} < 0$ for all $i \in \mathbb{K}$. This is possible if and only if $\mathbb{G}$ is stable, see [12].

From this result, there is no difficulty to calculate the norm $\|\mathbb{G}\|_\infty^2$ from the optimal solution of a convex programming problem expressed by LMIs. Indeed, applying the Schur Complement to (5) it is seen that

$$\|\mathbb{G}\|_\infty^2 = \inf_{(\gamma, P_i) \in \Psi} \gamma \quad (6)$$

where Ψ is the set of all positive definite matrices P_i and $\gamma \in \mathbb{R}$ such that the following LMI

$$\begin{bmatrix} P_i & 0 & A_i' P_{pi} & C_{zi}' \\ \bullet & \gamma I & J_i' P_{pi} & E_{zi}' \\ \bullet & \bullet & P_{pi} & 0 \\ \bullet & \bullet & \bullet & I \end{bmatrix} > 0 \quad (7)$$

is satisfied for each $i \in \mathbb{K}$. The determination of $\|\mathbb{G}\|_\infty^2$ using LMIs appears to be adequate and efficient. Indeed, we have to handle N linear matrix inequalities with N matrix variables and the coupling terms $P_{pi} = \sum_{j=1}^{N} p_{ij} P_j$ for all $i \in \mathbb{K}$. In addition, the calculation of the global optimal solution of the convex programming problem (6) does not need iterations and consequently no convergence condition has to be verified.

We are now in position to state the dynamic output feedback control design problems to be dealt with in the rest of this paper. Associated to $\mathbb{G}$, consider the full order mode-dependent linear controller

$$\mathbb{C} : \begin{cases} x_c(k+1) &= A_c(\theta_k) x_c(k) + B_c(\theta_k) y(k) \\ u(k) &= C_c(\theta_k) x_c(k) + D_c(\theta_k) y(k) \end{cases} \quad (8)$$

where $x_c(k) \in \mathbb{R}^n$, $x_c(0) = 0$ and the matrices A_{ci}, B_{ci}, C_{ci} and D_{ci} for all $i \in \mathbb{K}$ are of compatible dimensions. The goal is to determine these matrices in such a way that the $\mathcal{H}_\infty$ norm of the closed loop system is minimized. Connecting the controller $\mathbb{C}$ to the system $\mathbb{G}$, the controlled output is given by

$$\mathbb{F} : \begin{cases} \tilde{x}(k+1) &= \tilde{A}(\theta_k) \tilde{x}(k) + \tilde{J}(\theta_k) w(k) \\ z(k) &= \tilde{C}(\theta_k) \tilde{x}(k) + \tilde{E}(\theta_k) w(k) \end{cases} \quad (9)$$

where the indicated matrices are

$$\tilde{A}_i = \begin{bmatrix} A_i + B_i D_{ci} C_{yi} & B_i C_{ci} \\ B_{ci} C_{yi} & A_{ci} \end{bmatrix}, \tilde{J}_i = \begin{bmatrix} J_i + B_i D_{ci} E_{yi} \\ B_{ci} E_{yi} \end{bmatrix}$$

$$\tilde{C}_i = \begin{bmatrix} C_{zi} + D_{zi} D_{ci} C_{yi} & D_{zi} C_{ci} \end{bmatrix}, \tilde{E}_i = E_{zi} + D_{zi} D_{ci} E_{yi}$$

hence, the problem to be solved is written in the final form

$$\min_{A_{ci}, B_{ci}, C_{ci}, D_{ci}} \|\mathbb{F}\|_\infty^2 \quad (10)$$

It is important to make clear that the above formulation of the dynamic output feedback control design problem is highly non convex and difficult to solve, that is, in this form, it is not possible to calculate its global optimal solution. The reason is that the calculation of the objective function $\|\mathbb{F}\|_\infty^2$ depends upon a set of auxiliary variables (see problem (6)) which multiplies the controller variables producing, consequently, a non-convex problem. The way to circumvent this difficulty is to introduce a one-to-one change of variables able to linearize the nonlinear constraints to be handled.

From the previous determination of the $\mathcal{H}_\infty$ norm, it can be seen that the full order controller $\mathbb{C}$ imposes a closed loop system $\mathbb{F}$ of order twice the order of the plant $\mathbb{G}$. Hence the $\mathcal{H}_\infty$ norm calculations need auxiliary symmetric matrices $\tilde{P}_i \in \mathbb{R}^{2n \times 2n}$, for all $i \in \mathbb{K}$. Accordingly, let $\tilde{P}_i$ be $2n \times 2n$ real matrices partitioned as follows:

$$\tilde{P}_i = \begin{bmatrix} X_i & U_i \\ U_i' & \hat{X}_i \end{bmatrix}, \tilde{P}_i^{-1} = \begin{bmatrix} Y_i & V_i \\ V_i' & \hat{Y}_i \end{bmatrix}, \tilde{T}_i = \begin{bmatrix} Y_i & I \\ V_i' & 0 \end{bmatrix} \quad (11)$$

where all blocks are $n \times n$ real matrices. It is immediately verified that

$$\tilde{T}_i' \tilde{P}_i \tilde{T}_i = \begin{bmatrix} Y_i & I \\ I & X_i \end{bmatrix} \quad (12)$$

for all $i \in \mathbb{K}$. It is a well known fact, that if the matrix in (12) is constrained to be definite positive then it is always possible to determine the matrix blocks in (11) in order to get $\tilde{P}_i > 0$. Moreover, this can be accomplished even if matrix U_i or V_i for each $i \in \mathbb{K}$ is arbitrarily (nonsingular) fixed. Now, we proceed by considering $\tilde{P}_i > 0$ and adopting a similar reasoning to the convex combination of these matrices. From (11), the same partition yields

$$\tilde{P}_{pi} = \sum_{j=1}^{N} p_{ij} \tilde{P}_j = \begin{bmatrix} X_{pi} & U_{pi} \\ U_{pi}' & \hat{X}_{pi} \end{bmatrix} \quad (13)$$

and denoting

$$\tilde{P}_{pi}^{-1} = \begin{bmatrix} R_{1i} & R_{2i} \\ R_{2i}' & R_{3i} \end{bmatrix}, \tilde{Q}_i = \begin{bmatrix} I & X_{pi} \\ 0 & U_{pi}' \end{bmatrix} \quad (14)$$

it is verified that

$$\tilde{Q}_i' \tilde{P}_{pi}^{-1} \tilde{Q}_i = \begin{bmatrix} R_{1i} & I \\ I & X_{pi} \end{bmatrix} \quad (15)$$

It is important to stress that the four block matrices which define the inverse $\tilde{P}_{pi}^{-1}$ depend nonlinearly on the four block matrices of $\tilde{P}_{pi}$. However, since $R_{1i}^{-1} = X_{pi} - U_{pi} \hat{X}_{pi}^{-1} U_{pi}'$, setting U_i such that $U_i = -\hat{X}_i$ the partitioned matrix in (15) becomes

$$\tilde{Q}_i' \tilde{P}_{pi}^{-1} \tilde{Q}_i = \begin{bmatrix} (X_{pi} + U_{pi})^{-1} & I \\ I & X_{pi} \end{bmatrix} \quad (16)$$

From the above discussion, we mention again that the particular choice $U_i = -\hat{X}_i$ can be made with no loss of generality and constrain matrix U_i to be symmetric and negative definite. Furthermore, (11) provides $U_i = -\hat{X}_i = Y_i^{-1} - X_i$ which enables us to rewrite (16) in the final form

$$\tilde{Q}_i' \tilde{P}_{pi}^{-1} \tilde{Q}_i = \begin{bmatrix} Y_{qi} & I \\ I & X_{pi} \end{bmatrix} \tag{17}$$

Moreover, in the general case, that is without the particular choice $U_i = -\hat{X}_i$, matrices R_{1i} satisfy

$$\begin{aligned} R_{1i}^{-1} &= X_{pi} - U_{pi}\hat{X}_{pi}^{-1}U_{pi}' \\ &\geq \sum_{j=1}^{N} p_{ij}(X_j - U_j\hat{X}_j^{-1}U_j') \\ &\geq \sum_{j=1}^{N} p_{ij}Y_j^{-1} \\ &\geq Y_{qi}^{-1} \end{aligned} \tag{18}$$

for all $i \in \mathbb{K}$. The relations (17) and (18) are the key results to be used afterwards for dynamic output feedback control synthesis. Additionally, the results to be presented in the sequel are based on the linearization of the matrix inequalities involved in the norm calculations. Hence, let us introduce the following one-to-one change os variables

$$\begin{aligned} \begin{bmatrix} A_{ci} & B_{ci} \\ C_{ci} & D_{ci} \end{bmatrix} &= \begin{bmatrix} U_{pi} & X_{pi}B_i \\ 0 & I \end{bmatrix}^{-1} \times \\ &\times \begin{bmatrix} M_i - X_{pi}A_iY_i & F_i \\ L_i & K_i \end{bmatrix} \times \\ &\times \begin{bmatrix} V_i' & 0 \\ C_{yi}Y_i & I \end{bmatrix}^{-1} \end{aligned} \tag{19}$$

which from matrices (M_i, F_i, L_i, K_i), the dynamic output feedback controller matrices $(A_{ci}, B_{ci}, C_{ci}, D_{ci})$ are uniquely determined and vice-versa for each $i \in \mathbb{K}$. Indeed, notice that in (19), the inverses exist whenever matrices U_i and V_i are nonsingular for all $i \in \mathbb{K}$. The importance of this change of variables is that it makes possible to convert the $\mathcal{H}_\infty$ output feedback control design into a convex programming problem expressed in terms of LMIs.

III. $\mathcal{H}_\infty$ Mode-dependent Control Design

Based on the previous results our main purpose in this section is to calculate the global optimal solution of the $\mathcal{H}_\infty$ mode-dependent dynamic output feedback control design problem. Connecting the full order controller $\mathbb{C}$ defined in (8) to the open loop system $\mathbb{G}$, the problem to be dealt with can be expressed as $\inf \gamma$ where the infimum is taken with respect to the scalar γ and the matrix variables $\tilde{P}_i$, A_{ci}, B_{ci}, C_{ci} and D_{ci} for all $i \in \mathbb{K}$ satisfying the inequality

$$\begin{bmatrix} \tilde{P}_i & 0 & \tilde{A}_i' & \tilde{C}_i' \\ \bullet & \gamma I & \tilde{J}_i' & \tilde{E}_i' \\ \bullet & \bullet & \tilde{P}_{pi}^{-1} & 0 \\ \bullet & \bullet & \bullet & I \end{bmatrix} > 0 \tag{20}$$

where the closed loop system state space matrices defines $\mathbb{F}$ and $\tilde{P}_i$ is partitioned as indicated in (11) for each $i \in \mathbb{K}$. The solution of the $\mathcal{H}_\infty$ mode-dependent control design problem is summarized in the next theorem.

Theorem 1: There exist a mode-dependent output feedback controller of the form (8), a scalar $\gamma > 0$ and symmetric matrices $\tilde{P}_i > 0$ satisfying the inequalities (20) for all $i \in \mathbb{K}$ if and only if there exist a scalar $\gamma > 0$, symmetric matrices X_i, Y_i, Z_{ij} and matrices M_i, L_i, F_i, K_i, H_i of compatible dimensions satisfying the LMIs

$$\begin{bmatrix} \begin{bmatrix} Y_i & I & 0 \\ \bullet & X_i & 0 \\ \bullet & \bullet & \gamma I \end{bmatrix} & \Pi_i' \\ \Pi_i & \begin{bmatrix} H_i + H_i' - Z_{pi} & I & 0 \\ \bullet & X_{pi} & 0 \\ \bullet & \bullet & I \end{bmatrix} \end{bmatrix} > 0 \tag{21}$$

where

$$\Pi_i' = \begin{bmatrix} Y_iA_i' + L_i'B_i' & M_i' & Y_iC_{zi}' + L_i'D_{zi}' \\ A_i' + C_{yi}'K_i'B_i' & A_i'X_{pi} + C_{yi}'F_i' & C_{zi}' + C_{yi}'K_i'D_{zi}' \\ J_i' + E_{yi}'K_i'B_i' & J_i'X_{pi} + E_{yi}'F_i' & E_{zi}' + E_{yi}'K_i'D_{zi}' \end{bmatrix}$$

and

$$\begin{bmatrix} Z_{ij} & H_i' \\ \bullet & Y_j \end{bmatrix} > 0 \tag{22}$$

for all $i, j \in \mathbb{K} \times \mathbb{K}$. Furthermore, whenever (21)-(22) are satisfied, a suitable solution for (20) is provided by (19) with $U_i = Y_i^{-1} - X_i$ and $V_i = Y_i$ for all $i \in \mathbb{K}$.

Proof: For the necessity, assume that (20) holds. Partitioning $\tilde{P}_{pi}^{-1}$ as in (11) and multiplying (20) to the right by the matrix $\mathrm{diag}[\tilde{T}_i, I, \tilde{Q}_i, I]$ and to the left by its transpose we obtain the LMI (21) with R_{1i} at the place of $H_i + H_i' - Z_{pi}$ in the fourth row and fourth column block and where the matrices (M_i, F_i, L_i, K_i) are provided by the inverse transformation (19).

On the other hand, taking into account that (18) implies $Y_{qi} \geq R_{1i}$, for $H_i = Y_{qi}$ and $Z_{ij} = Y_{qi}Y_j^{-1}Y_{qi} + \varepsilon I$ with $\varepsilon > 0$ we see that (22) is verified and we obtain

$$\begin{aligned} H_i + H_i' - Z_{pi} &= Y_{qi} - \varepsilon I \\ &\geq R_{1i} - \varepsilon I \end{aligned} \tag{23}$$

hence, taking $\varepsilon > 0$ sufficiently small, inequality (21) holds and the claim follows.

For the sufficiency, assume that (21) and (22) hold. From (22) we have $Z_{ij} > H_i'Y_j^{-1}H_i$ and consequently multiplying these inequalities by p_{ij} and summing up for all $j \in \mathbb{K}$ we obtain

$$\begin{aligned} H_i + H_i' - Z_{pi} &= H_i + H_i' - \sum_{j=1}^{N} p_{ij}Z_{ij} \\ &\leq H_i + H_i' - H_i'Y_{qi}^{-1}H_i \\ &\leq Y_{qi} - (H_i - Y_{qi})'Y_{qi}^{-1}(H_i - Y_{qi}) \\ &\leq Y_{qi} \end{aligned} \tag{24}$$

which implies that (21) remains valid if the diagonal term on the forth column and row is replaced by Y_{qi} and consequently $X_{pi} > Y_{qi}^{-1} > 0$. Hence, imposing $U_i = Y_i^{-1} - X_i$ we get $V_i = Y_i$ and it is verified that the matrix $U_{pi} = Y_{qi}^{-1} - X_{pi}$ is nonsingular which enable us to determine the matrices $(A_{ci}, B_{ci}, C_{ci}, D_{ci})$ from the change of variables (19). Moreover, taking into account that this choice provides

$$\tilde{P}_{pi} = \begin{bmatrix} X_{pi} & Y_{qi}^{-1} - X_{pi} \\ \bullet & X_{pi} - Y_{qi}^{-1} \end{bmatrix} > 0 \tag{25}$$

it is immediately verified that (17) holds and that $R_{1i}^{-1} = Y_{qi}^{-1}$. The conclusion is that inequality (21) with the diagonal term on the forth column and row replaced by Y_{qi} can be rewritten as

$$\begin{bmatrix} \tilde{T}_i'\tilde{P}_i\tilde{T}_i & 0 & \tilde{T}_i'\tilde{A}_i'\tilde{Q}_i & \tilde{T}_i'\tilde{C}_i' \\ \bullet & \gamma I & \tilde{J}_i'\tilde{Q}_i & \tilde{E}_i' \\ \bullet & \bullet & \tilde{Q}_i'\tilde{P}_{pi}^{-1}\tilde{Q}_i & 0 \\ \bullet & \bullet & \bullet & I \end{bmatrix} > 0 \qquad (26)$$

which multiplied to the right by $\mathrm{diag}[\tilde{T}_i^{-1}, I, \tilde{Q}_i^{-1}, I]$ and to the left by its transpose provides the inequality (20) and the proof is concluded. ■

Defining the set of matrix variables $\mathcal{Y} = (X_i, Y_i, Z_{ij}, M_i, F_i, L_i, K_i, H_i)$ for all $i, j \in \mathbb{K} \times \mathbb{K}$ and the convex set Ξ of all feasible solutions of the LMIs (21) and (22), the optimal solution of the $\mathcal{H}_\infty$ control design problem is determined from

$$\inf_{\gamma, \mathcal{Y} \in \Xi} \gamma \qquad (27)$$

making clear that the mode-dependent output feedback design problem under consideration has been converted into a convex programming problem expressed in terms of LMIs.

The result reported in Theorem 1 outperforms the previous results available in the literature dealing with this class of control design problems, [15] and [19]. First, on the contrary of [15] where it is imposed the very restrictive constraint on the transition probability matrix, namely $p_{ij} = p_j$ for all $i, j \in \mathbb{K} \times \mathbb{K}$ and only strictly proper controllers are considered, we do not make any assumption on the structure of the transition probability matrix and general proper controllers are designed. Moreover, comparing to [19] where an iterative method to solve BMIs are needed, here the problem is solved in one shot by any LMI solver.

IV. Practical Application

Consider the vehicle following problem described in [15]. Let x_0 denote the position of the leading car and x_i denote the position of the ith follower. The reference trajectory for the lead vehicle is denoted r_0 and the tracking error for the lead vehicle is $e_0 = r_0 - x_0$. The other vehicle spacing errors are $e_i = x_{i-1} - x_i - \delta_i$ where δ_i is the desired vehicle spacing. The control objective is to enforce all tracking errors e_i to zero. Following [15], although the dynamic behavior for an individual vehicle is nonlinear, the use of a two-layered control scheme [10], allows us to consider a reasonable third order model for the vehicle dynamics according to

$$\frac{d}{dt}\begin{bmatrix} x_i(t) \\ v_i(t) \\ a_i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix}\begin{bmatrix} x_i(t) \\ v_i(t) \\ a_i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix} u_i(t) \qquad (28)$$

where $x_i(t)$, $v_i(t)$ and $a_i(t)$ are the position, velocity and acceleration of the ith vehicle, $\tau = 100ms$ is the time constant of the first-order lag. The system is discretized with $T_s = 20ms$ sample rate, considering a zero-order hold on the control input.

It is assumed that the measurement used to control the system is transmitted using a wireless network protocol such that errors can be detected but not corrected. At every sample time, a vehicle communicates its measurements to the network. If the received package is corrupted, the controller discards it and waits for the next package. We assume all communicated measurements from the vehicles are either received or corrupted and the Markov chain can be modelled with two modes $\{L, R\}$ for lost and received packages, respectively. It is further assumed that the statistics of the network are known so we have both the probability of the next broadcasted package be correct after a good one is received $p_{R,R}$ and the probability of a package being lost by error after a bad one is received $p_{L,L}$. The data communicated through the network is modelled as follows

$$\hat{y}_c(k) = \begin{cases} y_c(k), & \text{if } \theta(k) = R \\ \emptyset, & \text{if } \theta(k) = L \end{cases} \qquad (29)$$

where $\emptyset$ denotes a corrupted package of information and y_c is the vector of communicated measurements available for feedback. All cars in the platoon have on board sensors to capture the measurements from their particular motions. Those measurements are denoted as y_o, which leads to the following output available to the controller in every time instant

$$y(k) := \begin{bmatrix} y_o(k) \\ \hat{y}_c(k) \end{bmatrix} \qquad (30)$$

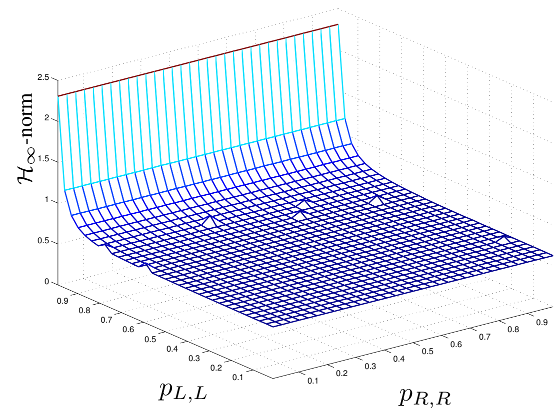

Fig. 1. $\mathcal{H}_\infty$-norm versus package loss rate

As in [15], for the controller design a model with two cars are considered. However, the important difference with the approach proposed in [15], though, is that we are able to calculate, from the results reported in this paper, the optimal controller associated to any value of the transition probability matrix and not only for those satisfying $p_{R,R} = 1 - p_{L,L}$. Figure 1 shows the grid with the $\mathcal{H}_\infty$ norm of the controlled system for all possible values of the package loss rates.

It is interesting to notice, although not intuitive, that the $\mathcal{H}_\infty$ norm does not change very much with respect to $p_{R,R}$ whenever $p_{L,L}$ is kept constant. However, if one takes two points on Figure 1 with different $p_{R,R}$ and the same

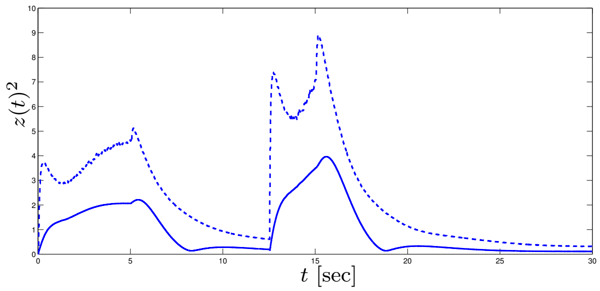

Fig. 2. Mean square errors

$p_{L,L}$, even though the $\mathcal{H}_\infty$ norm might be the same, the designed controllers are in general different. That cannot be accomplished by the control designed from [15] because it can only be calculated for systems such that $p_{L,L} = 1 - p_{R,R}$. Figure 1 shows spikes in some points due to lack of numerical precision.

Finally, we compare the Markov controller proposed by this design with the deterministic $\mathcal{H}_\infty$ optimal output feedback controller. For both systems a Monte Carlo simulation was run with 5×10^3 iterations. The following probability matrix for the package error rates

$$\begin{bmatrix} p_{L,L} & p_{L,R} \\ p_{R,L} & p_{R,R} \end{bmatrix} = \begin{bmatrix} 0.92 & 0.08 \\ 0.02 & 0.98 \end{bmatrix}$$

has been considered. The input has been calculated in order to impose that the lead vehicle should start with a constant acceleration of $3m/s^2$ for 5 seconds, constant speed for the next 7.5 seconds, should brake with an acceleration of $-6m/s^2$ for the remaining 2.5 seconds and after that acceleration should be zero. The parameters used for the control output z were the same used in [15]. The mean square errors can be seen in Figure 2, where it is clear the better performance of the Markovian controller (solid line) when compared to the deterministic one (dashed line).

V. Conclusion

In this paper a new approach to the dynamic output feedback control design of discrete-time MJLS is proposed. The set of all output feedback mode-dependent controllers imposing to the closed loop system a pre-specified $\mathcal{H}_\infty$ norm level is parametrized by convex constraints expressed by means of pure LMIs, likewise of what has been done in the continuous-time case, [8]. The controllers are obtained without any additional constraints as for instance to be strictly proper. The performance is enhanced due to the use of general proper controllers and also the possibility to deal with optimal control problems where the transition probability matrix is not restricted to have the same rows as considered in [15]. The controller is always obtained from the solution of a convex programming problem assuming that it has access to the system modes for all $k \in \mathbb{N}$. That assumption is of practical appealing if one considers the application for networked control, where transmission protocols can easily include package error detection. The inclusion of the system statistics and the correspondent models to obtain the closed loop controller has a computational cost. Nonetheless, we believe it is a worth cost to pay, given the better performance illustrated in Figure 2.

As a further research subject, we believe that the results reported in this paper can be applied to networked control problems where the Markov channel is described by more accurate models yielding higher order MJLS systems to be handled.

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