Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Model Predictive Control for Unstable and Non-Minimum Constrained Processes

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Abstract

In this paper we propose a newer kind of robust model predictive controller (MPC), namely, Nash game approach based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ model predictive control. Two quadratic objective functions are considered, viz. one for $\mathcal{H}_2$ problem and another for $\mathcal{H}_\infty$ problem. The resulting open-loop control problem reduces to solving a pair of coupled non-symmetric algebraic Riccati equations. The pair of coupled Riccati equations are solved, by finding the invariant subspace spanning eigenvectors of a specially constructed matrix. The novel control algorithm is demonstrated using numerical example for non-minimum unstable system.

1 Introduction

Robust model predictive control (MPC) is one of the very demanding area of research in recent years (Cuzzola et al., 2002; Löfberg. J, 2003; Ramírez et al., 2006; Zhao et al., 2000). Even if there are good models available both for the plant and the disturbances affecting the plant, any unmodelled phenomenon and/or some unknown disturbances causing discrepancies in the system's performance could be taken care by the available feedback in MPC design. However the open-loop controller is not actually meant for that. Eventually, the controller should be robustly designed enough to meet such discrepancies which affects both stability and performance of the process. $\mathcal{H}_\infty$ robust MPC is a well known such kind of controller and its analogy to game theoretic strategy is also well documented in the literature (Başar and Olsder, 1995; Rao, 2000). Moreover, to benefit from the advantage of combining $\mathcal{H}_2$ and $\mathcal{H}_\infty$, a newer hybrid robust controller called mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller was developed in early 90's (Kaminer et al., 1993; Limebeer et al., 1994). However, this type of robust controller has not be much investigated in the MPC regime, except recently by Orukpe et al. (2007). In this present work, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ model predictive control problem is approached from a different perspective, i.e., using Nash game approach is proposed. Here, the existence of an optimal saddle point solution is claimed as the solution of the optimal control problem. The coupled non-symmetric algebraic Riccati equation resulting from such problem is solved in a systematic manner, with the construction of a special matrix, as shown in Freiling et al (1999). The efficacy of the present controller is compared against that of Orukpe et al. (2007) for non-minimum phase unstable processes, which are of much interest in process industries.

The paper has been organized in the following manner. Section 2 gives the systematic formulation of the problem. The existence of the optimal saddle point solution is shown in section 3. The procedure of solving the resulting coupled algebraic Riccati equations is shown in section 4. In section 5 the efficiency of the proposed method is showed using numerical example. Finally, in section 6 the conclusions are given.

2 Problem Formulation

Consider the discrete-time linear system affected with process disturbance:

$$\begin{aligned} x_{k+1} &= Ax_k + B_u u_k + B_\omega \omega_k, \quad x(0) = x_0 \\ z_k &= \begin{bmatrix} Cx_k \\ Du_k \end{bmatrix} \end{aligned} \tag{1}$$

where, $x_k = [x_k^1, x_k^2, \cdots, x_k^n]^T \in \mathbb{R}^n$ is the system state, $u_k = [u_k^1, u_k^2, \cdots, u_k^m]^T \in \mathbb{R}^m$ is the control input; $\omega_k \in \mathcal{L}_{2,[0,\infty]}$ is the unknown but bounded disturbance. Also, A, B_u and B_ω are matrices of appropriate dimensions. Also consider $Q = C^T C$ and $R := D^T D$.

The $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control performance are given by equations (2) and (3), respectively, as

$$J_\infty(u, \omega) = \frac{1}{2}\sum_{k=0}^{\infty}\left(x_k^T Q x_k - \gamma^2 \omega_k^T \omega_k\right) \tag{2}$$

$$J_2(u, \omega) = \frac{1}{2}\sum_{k=0}^{\infty} z_k^T z_k = \frac{1}{2}\sum_{k=0}^{\infty}\left(x_k^T Q x_k + u_k^T R u_k\right) \tag{3}$$

where γ^2 is the upper bound of the worst case performance (or the attenuation factor for the disturbance).

The Hamiltonian function (H_2) of $\mathcal{H}_2$ problem is given as,

$$H_2 := J_2(u_k, \omega_k) + \bar{p}_{k+1}^T\left[(Ax_k + B_u u_k + B_\omega \omega_k) - x_{k+1}\right] \tag{4}$$

$$= \frac{1}{2}\left(x_k^T Q x_k + u_k^T R u_k\right) + \bar{p}_{k+1}^T\left[(Ax_k + B_u u_k + B_\omega \omega_k) - x_{k+1}\right] \tag{5}$$

where $\bar{p}_k$ is the Lagrange multiplier. To minimize the function H_2, we need to partially differentiate H_2 w.r.t. its components, say, x_k and u_k and setting them to zero.

$$\frac{\partial H_2}{\partial x_k} = 0: \quad Qx_k + A^T \bar{p}_{k+1} - \bar{p}_k = 0 \tag{6}$$

$$\frac{\partial H_2}{\partial u_k} = 0: \quad Ru_k + B_u^T \bar{p}_{k+1} = 0 \tag{7}$$

From equations (6) and (7) we have

$$\bar{p}_k = Qx_k + A^T \bar{p}_{k+1} \tag{8}$$

$$u_k = -R^{-1} B_u^T \bar{p}_{k+1} \tag{9}$$

Assume that $\bar{p}_k$ can be given in the form,

$$\bar{p}_k = P_k^I x_k \tag{10}$$

where P_k^I is an $n \times n$ real symmetric positive definite (Lyapunov) matrix. Using equation (10) into equation (9) gives

$$u_k = -R^{-1} B_u^T P_{k+1}^I x_{k+1} \tag{11}$$

Now, the $\mathcal{H}_\infty$ problem's Hamiltonian function (H_∞) is given as,

$$\begin{aligned} H_\infty &:= J_\infty(u_k, \omega_k) + \bar{\bar{p}}_{k+1}^T \left[(Ax_k + B_u u_k + B_\omega \omega_k) - x_{k+1}\right] && (12) \\ &= \frac{1}{2}\left(x_k^T Q x_k \; - \; \gamma^2 \omega_k^T \omega_k\right) + \bar{\bar{p}}_{k+1}^T \left[(Ax_k + B_u u_k + B_\omega \omega_k) - x_{k+1}\right] && (13) \end{aligned}$$

where $\bar{\bar{p}}_{k+1}$ is again the Lagrange multiplier. Partially differentiating the Hamiltonian function (H_∞) w.r.t. its components, say, x_k and ω_k and equating them to zero,

$$\begin{aligned} \frac{\partial H_\infty}{\partial x_k} &= 0: \quad Qx_k + A^T \bar{\bar{p}}_{k+1} - \bar{\bar{p}}_k = 0 && (14) \\ \frac{\partial H_\infty}{\partial \omega_k} &= 0: \quad -\gamma^2 \omega_k + B_\omega^T \bar{\bar{p}}_{k+1} = 0 && (15) \end{aligned}$$

From the equations (14) and (15) we have,

$$\begin{aligned} \bar{\bar{p}}_k &= Qx_k + A^T \bar{\bar{p}}_{k+1} && (16) \\ \omega_k &= \gamma^{-2} B_\omega^T \bar{\bar{p}}_{k+1} && (17) \end{aligned}$$

Again considering a similar assumption as in equation (10),

$$\bar{\bar{p}}_k = P_k^{II} x_k \tag{18}$$

Using equation (18) in (15) we get,

$$\omega_k = \gamma^{-2} B_\omega^T P_{k+1}^{II} x_{k+1} \tag{19}$$

Now, from equations (11) and (19) we could write the closed-loop system equation as

$$x_{k+1} = [I + B_u R^{-1} B_u^T P_{k+1}^I - B_\omega \gamma^{-2} B_\omega^T P_{k+1}^{II}]^{-1} A x_k := \Phi_k x_k \tag{20}$$

where Φ is the closed-loop state transition matrix. By substituting equations (18) and (20) in (16), the discrete algebraic Riccati equation ($\mathcal{H}_\infty$-DARE) obtained from $\mathcal{H}_\infty$ problem is,

$$P_k^{II} x_k \;=\; Qx_k + A^T P_{k+1}^{II}[I + B_u R^{-1} B_u^T P_{k+1}^I - B_\omega \gamma^{-2} B_\omega^T P_k^{II}]^{-1} A x_k \tag{21}$$

Similarly, the discrete algebraic Riccati equation ($\mathcal{H}_2$-DARE) obtained from $\mathcal{H}_2$ problem, by substituting equations (10) and (20) in (8),

$$P_k^I x_k = Qx_k + A^T P_{k+1}^I [I + B_u R^{-1} B_u^T P_{k+1}^I - B_\omega \gamma^{-2} B_\omega^T P_k^{II}]^{-1} A x_k \tag{22}$$

3 Minimal values of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cost functions

In this section the minimal values of the cost functions are given. Let us rewrite the closed-loop system equation with the feedback gain matrices for the system given in (1) as;

$$x_{k+1} = (A + B_u K_u + B_\omega K_\omega)\, x_k \tag{23}$$

which could be otherwise given as,

$$x_{k+1} = \bar{A}_\omega x_k + B_u u_k \tag{24}$$

$$x_{k+1} = \bar{A}_u x_k + B_\omega \omega_k \tag{25}$$

where, $\bar{A}_\omega := A - B_\omega K_\omega$ and $\bar{A}_u := A - B_u K_u$. Assume that there exists an optimal feedback gain matrix K_ω from solving a $\mathcal{H}_\infty$ cost function and K_u from solving a $\mathcal{H}_2$ cost function.

Before finding the minimal value of the function, let us rewrite some of our earlier expressions in a form that suits with our new format of the closed loop equation as shown in equations (24) and (25). The $\mathcal{H}_2$ problem's Hamiltonian function (H_2) is rewritten as,

$$H_2 := J_2(u_k, \omega_k) + \bar{p}_{k+1}^T \left[(\bar{A}_\omega x_k + B_u u_k) - x_{k+1} \right] \tag{26}$$

which eventually yields,

$$\frac{\partial H_2}{\partial x_k} = 0 : \quad \bar{p}_k = Q x_k + \bar{A}_\omega^T \bar{p}_{k+1} \tag{27}$$

Therefore, the equation (24) from using equations (11) and (10),

$$\begin{aligned} x_{k+1} &= \bar{A}_\omega x_k - B_u B_u^T P_{k+1}^I x_{k+1} \\ \left[I + B_u B_u^T \bar{P}_{k+1}^I \right] x_{k+1} &= \bar{A}_\omega x_k \end{aligned} \tag{28}$$

Likewise, the $\mathcal{H}_\infty$ problem's Hamiltonian function (H_∞) is also rewritten as,

$$H_\infty := J_\infty(u_k, \omega_k) + \bar{\bar{p}}_{k+1}^T \left[(\bar{A}_u x_k + B_\omega \omega_k) - x_{k+1} \right] \tag{29}$$

which yields,

$$\frac{\partial H_\infty}{\partial x_k} = 0 : \quad \bar{\bar{p}}_k = Q x_k + \bar{A}_u^T \bar{\bar{p}}_{k+1} \tag{30}$$

The equation (25) from using equations (19) and (18),

$$\begin{aligned} x_{k+1} &= \bar{A}_u x_k + \gamma^{-2} B_\omega B_\omega^T \bar{P}_{k+1}^I x_{k+1} \\ \left[I - \gamma^{-2} B_\omega B_\omega^T \bar{P}_{k+1}^I \right] x_{k+1} &= \bar{A}_u x_k \end{aligned} \tag{31}$$

3.1 Minimal value of the $\mathcal{H}_2$ cost function

To find the minimum value of the $\mathcal{H}_2$ control performance, premultiply both sides of the equation (27) by x_k^T and using (10), we get,

$$x_k^T P_k^I x_k = x_k^T Q x_k + x_k^T \bar{A}_\omega^T P_{k+1}^I x_{k+1} \tag{32}$$

Substituting equation (28) into the previous equation, we get,

$$x_k^T Q x_k = x_k^T P_k^I x_k - x_{k+1}^T P_{k+1}^I x_{k+1} - x_{k+1}^T P_{k+1}^I B_u B_u^T P_{k+1}^I x_{k+1} \tag{33}$$

From equation (11), we have

$$\begin{aligned} u_k &= -B_u^T P_{k+1}^I x_{k+1} \\ \Rightarrow u_k^T u_k &= \left(-x_{k+1}^T P_{k+1}^I B_u\right)\left(-B_u^T P_{k+1}^I x_{k+1}\right) \\ &= x_{k+1}^T P_{k+1}^I B_u B_u^T P_{k+1}^I x_{k+1} \end{aligned} \tag{34}$$

By adding equations (33) and (34), we have

$$x_k^T Q x_k + u_k^T u_k = x_k^T P_k^I x_k - x_{k+1}^T P_{k+1}^I x_{k+1} \tag{35}$$

By substitution of this equation into the equation of $J_2(u, \omega)$ gives,

$$\begin{aligned} J_2(u^*, \omega^*) &= \frac{1}{2}\sum_{k=0}^{\infty} x_k^T P_k^I x_k - x_{k+1}^T P_{k+1}^I x_{k+1} \\ &= \frac{1}{2} x_0^T P_0^I x_0 \end{aligned} \tag{36}$$

Equation (36) is due to the cancelation of the terms when the summation is expanded over the time horizon $k = [0, \infty)$ and also using $lim_{k\to\infty} x_k = 0$. Note that the above assertion starts and is extended to the control law formulation with the assumption that there exists an optimal feedback gain matrix for the other (maximizing) player ω_k^*. So the $\mathcal{H}_2$ performance index given in equation (3) can be taken as $J_2(u, \omega^*)$. From the above argument and the result shown in equation (36), it could be directly stated that,

$$J_2(u^*, \omega^*) \le J_2(u, \omega^*) \tag{37}$$

3.2 Minimal value of the $\mathcal{H}_\infty$ cost function

The minimal value of the $\mathcal{H}_\infty$ cost function is found out in the following. Premultiplying equation (30) by x_k^T and using (18) we get

$$x_k^T P_k^{II} x_k = x_k^T Q x_k + x_k^T \bar{A}_u^T P_{k+1}^{II} x_{k+1} \tag{38}$$

Using equation (31) in the above equation, we get

$$\begin{aligned} x_k^T P_k^{II} x_k &= x_k^T Q x_k + x_{k+1}^T [I - \gamma^{-2} P_{k+1}^{II} B_\omega B_\omega^T] P_{k+1}^{II} x_{k+1} \\ x_k^T Q x_k &= x_k^T P_k^{II} x_k - x_{k+1}^T [I - \gamma^{-2} P_{k+1}^{II} B_\omega B_\omega^T] P_{k+1}^{II} x_{k+1} \end{aligned} \tag{39}$$

From equation (19), we can have

$$\omega_k^T\omega_k = \gamma^{-4}x_{k+1}^T P_{k+1}^{II} B_\omega B_\omega^T P_{k+1}^{II} x_{k+1} \tag{40}$$

Multiplying equation (40) by γ^2 and then subtracting it from equation (39) gives,

$$x_k^T Q x_k - \gamma^2 \omega_k^T \omega_k = x_k^T P_k^{II} x_k - x_{k+1}^T P_{k+1}^{II} x_{k+1} \tag{41}$$

Using the previous equation in the equation of $J_\infty(u, \omega)$ i.e., the $\mathcal{H}_\infty$ cost function gives,

$$\begin{aligned} J_\infty(u^*, \omega^*) &= \frac{1}{2}\sum_{k=0}^{\infty} x_k^T P_k^{II} x_k - x_{k+1}^T P_{k+1}^{II} x_{k+1} \\ &= \frac{1}{2} x_0^T P_0^{II} x_0 \end{aligned} \tag{42}$$

Using the similar argument as given in section 3.1, rather, the other player being the minimizing player, u_k^*, with the assumption that there exist an optimal feedback gain matrix (K_u) for the minimizing player, the $\mathcal{H}_\infty$ cost function can be taken as $J_\infty(u^*, \omega)$. Moreover, from the result of the equation (42), the following condition holds good;

$$J_\infty(u^*, \omega^*) \leq J_\infty(u^*, \omega) \tag{43}$$

So the saddle point of the Nash game is ensured once the coupled algebraic Riccati equations (AREs), given in equations (22) and (21), are solved for their solutions.

4 Solution of the coupled AREs

Solving the pair of coupled AREs to get the values of the Lyapunov function matrices, P^I and P^{II} will fetch us the optimal (saddle point) solution of the Nash game approach to robust control of the linear system (1). Freiling et al. (1999), has given the necessary condition(s) that need to be satisfied, so as to get the solution of the AREs. From the AREs in equations (22) and (21) and the closed loop system equation (20), the following theorem can be stated.

Theorem 4.1 *If* $S(P^I, P^{II}) := \mathtt{span}\begin{pmatrix} I \\ P^I \\ P^{II} \end{pmatrix} \subset \mathbb{C}^{3n\times n}$ *is an invariant subspace of* M_{Na} *with* $\mathtt{det}(I + S_1 P^I + S_2 P^{II}) \neq 0$ *then* $\begin{pmatrix} P^I \\ P^{II} \end{pmatrix}$ *is the solution of the AREs, such that* $\begin{pmatrix} x \\ \bar{p} \\ \bar{\bar{p}} \end{pmatrix}(k+1) = M_{Na}\begin{pmatrix} x \\ \bar{p} \\ \bar{\bar{p}} \end{pmatrix}(k)$*, where,* $S1 := B_u R^{-1} B_u^T$ *,* $S2 := B_\omega \gamma^{-2} B_\omega^T$ *and*

$$M_{Na} := \begin{pmatrix} A^{-1} & A^{-1}S_1 & A^{-1}S_2 \\ QA^{-1} & A^T + QA^{-1}S_1 & QA^{-1}S_2 \\ QA^{-1} & QA^{-1}S_1 & A^T + QA^{-1}S_2 \end{pmatrix}.$$

Corollary 4.1 *If* $\mathtt{span}(v_1, \cdots, v_n)$ *is an invariant subspace such that* $\mathtt{det}X \neq 0$ *for* $\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} := (v_1, \cdots, v_n)$*, then* $\begin{pmatrix} P^I \\ P^{II} \end{pmatrix} := \begin{pmatrix} Y_1 X^{-1} \\ Y_2 X^{-1} \end{pmatrix}$ *is the solution of the AREs, if* $\mathtt{det}(I + S_1 P^I + S_2 P^{II}) \neq 0$.

Using the above theorem and the immediate corollary, the coupled algebraic Riccati equations resulting from the open-loop Nash games can be solved, enabling us to obtain the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC using Nash game approach.

Remark 4.1 *In the present study, we have* $Q_1 = Q_2 := Q$.

Remark 4.2 *The solution of the coupled AREs could exist if and only if the system's states are completely controllable and observable. The interested readers are refereed to Freiling et al. (1999) for further details.*

The invariant subspace spanning eigenvectors of M_{Na} are the required solution of the coupled Riccati equations (22) and (22) as per Corollary 4.1, such that the closed loop eigenvalues lies within the unit circle (i.e., $\mathtt{eig}(A + B_u K_u) \leq 1$, where $K_u := -R^{-1} B_u^T P_{k+1}^I \Phi_k x_k$ from equation (11) and (20)).

5 Numerical Example

The performance of both the control algorithms is checked for a non-minimum phase unstable system. The discrete time state-space model of such a system with sampling time, $T = 0.1$ is taken

$$A = \begin{bmatrix} 2.3150 & -1.3500 \\ 1.0000 & 0.0000 \end{bmatrix} \quad B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B_\omega = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = [0.1133 \quad -0.1191]$$

The system is subjected to a constant disturbance, $\omega = 0.1$. The poles and zero of the system are $1.1575 \pm 0.1010i$ and 1.0512, respectively. With $\gamma^2 = 100$ and $x_0 = [1 \;\; 0]^T$, the performance of Nash game approach is compared against Orukpe et al.'s algorithm and the results are furnished in Fig. 1. Although the present approach gives an oscillatory response (which could be improved by decreasing the input weighing matrix), it gives highly appreciable response than the other mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC algorithm.

6 Conclusion

A Nash game theory based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust MPC is proposed. The coupled algebraic Riccati equations resulting from this approach is solved efficiently by constructing a special matrix, and from finding its generalized eigenvectors that span its invariant subspace (Freiling et al., 1999). The performance of the proposed robust predictive control algorithm is compared against the one proposed by Orukpe et al. (2007). The proposed

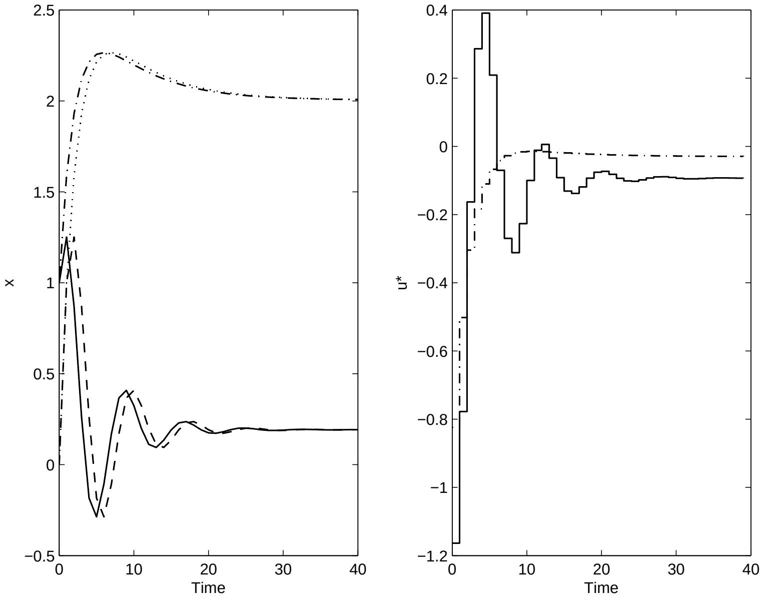

Fig 1 Comparison of Nash game approach mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust MPC (solid and dashed lines) against the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC proposed in Orukpe et al (dash-dot and dotted lines) for non-minimum unstable system for $\gamma^2 = 100$.

algorithm gives better results when used for non-minimum, unstable systems, which are common in many process industries. It should be noted that the present algorithm could only be used for systems, whose states are both completely controllable and observable.

In the present work, the algorithm has been developed only for unconstrained case. Enhancement of the present work to handle constraints, especially input constraints, is underway.

References

1. Başar, T. and Olsder, G. J. (1995) "Dynamic noncorporative game theory", *SIAM*, Philadelphia.

2. Cuzzola, F. A., Geromel, J. C. and Morari, M. (2002) "An improved approach for constrained robust model predictive control", 38, pp.1183-1189.

3. Freiling, G., Jank, G. and Abou-Kandil, H. (1999) "Discrete-time Riccati equations in open-loop Nash and Stackelberg games", *Euro. J. of Contr.*, 5, pp. 56-66.

4. Kaminer, I., Khargonekar, P. P. and Rotea, M. A. (1993) "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for discrete-time systems via convex optimization", *Automatica*, 29 (1), pp. 57-70.

5. Löfberg, J. (2003) "Minimax approaches to robust model predictive control", *Ph.D. Dissertation*, Linköping Univ., Sweden.

6. Limebeer, D.J.N., Anderson, B. D. O. and Hendel, H. (1994) "A Nash game approach to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control", *IEEE. Trans. Auto. Contr.*, 39 (1), pp. 69-82.

7. Orukpe, P. E., Jaimoukha, I. M. and El-Zobaidi, H. M. H. (2007) "Model predictive control based on mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control approach", *Amer. Contr. Conf.*, New York, USA.

8. Ramírez, D. R., Alamo, T., Camacho, E. F. and de la Peña, D. M. (2006) "Min-max MPC based on a computationally efficient upper bound of worst case cost", *J. Process Contr.*, 16, pp. 511-519.

9. Rao, C. V. (2000) "Moving horizon strategies for the constrained monitoring and control of nonlinear discrete-time systems", *Ph.D. Dissertation*, Univ. Wisconsin-Madison.

10. Zhao, H., Li, W. and Bentsman, J. (2000) "$\mathcal{H}_\infty$ prediction and unconstrained $\mathcal{H}_\infty$ predictive control:multi-input-multi-output case", *Int. J. Robust and Nonlin. Contr.*, 11(1), pp. 59-86.