01D05 Fundamental Research in Transport Processes III

Transient Effects on Secondary Flow Behavior in Double Bifurcation Model

Fong Yew Leong[a], Kenneth A. Smith[a,b], Chi-Hwa Wang[a,c]
[a] *MEBCS Program, Singapore-MIT Alliance*
[b] *Department of Chemical Engineering, Massachusetts Institute of Technology,*
[c] *Department of Chemical and Biomolecular Engineering, National University of Singapore*

Prepared for presentation at the 2006 AIChE Annual Meeting

San Francisco, California, November 12-17, 2005.

1. Introduction

Previous work has suggested that localized extension of the vortex lines in the upstream daughter tube is responsible for the generation of vorticities in the grand-daughter tubes [1]. This indicated that it may be possible to have a simple physical explanation for the secondary flow structures in the grand-daughter tubes.

The purpose of this work is to make use of simple conceptual ideas in order to rationalize the vorticity generation and transport that is occurring in the daughter tubes of a double bifurcation model. This provides the groundwork for the eventual understanding of how transient effects can influence the nature of such flows.

2. Scaling Laws

We first begin with the vorticity transport equation.

$$\frac{\partial \overline{\omega}}{\partial t} + \overline{u} \cdot \nabla \overline{\omega} = \overline{\omega} \cdot \nabla \overline{u} + \nu \nabla^2 \overline{\omega} \qquad (1)$$

For steady state analysis, we can neglect the temporal term, and that leaves us with the remaining three other terms to work on. Even though there are three scalar component equations of the vorticity transport vector equation (1), we can think of the flow as consisting of the main flow along of the axis of the tube (axial component) and the secondary flow perpendicular to the axis (secondary component), which we denote as x and y respectively.

In this way, the convective term in the axial and secondary directions can be written as

$$u\frac{\partial \omega_x}{\partial x} + v\frac{\partial \omega_x}{\partial y} \qquad\qquad u\frac{\partial \omega_y}{\partial x} + v\frac{\partial \omega_y}{\partial y} \qquad (2)$$

where u and v are the components of the velocity vector in the axial (x) and secondary (y) directions; and ω_x and ω_y are the respective components of the vorticity vector.

Similarly, the vortex stretching term can be written as

$$\omega_x \frac{\partial u}{\partial x} + \omega_y \frac{\partial u}{\partial y} \qquad\qquad \omega_x \frac{\partial v}{\partial x} + \omega_y \frac{\partial v}{\partial y} \qquad (3)$$

And finally, the viscous term can be written as

$$\nu \frac{\partial^2 \omega_x}{\partial x^2} \qquad\qquad \nu \frac{\partial^2 \omega_y}{\partial y^2} \qquad (4)$$

where ν is the kinematic viscosity of the fluid.

Notations used:

Cartesian coordinates for symmetric-planar bifurcation model (see Fig. 1)
x refers to the axial component of the daughter tube (x = x');
y refers to the component orthogonal to both the axial component and bifurcation axis;
z refers to the component parallel to the bifurcation axis;
U is the velocity scale for the axial velocity u;
V is the velocity scale for the secondary velocity v;
L is the length of the DT;
a is the radius of the DT;
R is the radius of curvature;
θ is the half-bifurcation angle;

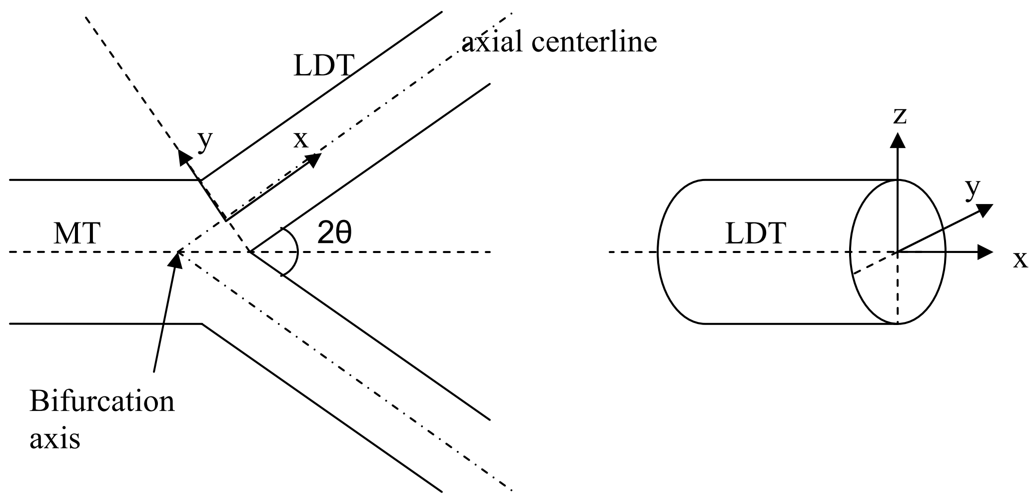

Fig. 1 Schematic Diagram of Planar Symmetric Bifurcation Model in Top-down view (Left) and Cartesian Coordinates used (Right)

If the axial component of the vorticity is the sole consideration, the inertial term can be estimated as

$$u\frac{\partial \omega_x}{\partial x} \sim \frac{U\Omega_x}{L}$$

And the viscous term can be scaled as

$$\nu\frac{\partial^2 \omega_x}{\partial y^2} \sim \frac{\nu\Omega_x}{\delta^2} \qquad (5)$$

Where δ is an unspecified viscous length-scale (or boundary layer thickness)

Consider the secondary flow problem. Based on what was previously observed in the numerical simulation, the vortex line (or ring) is distorted and stretched by the secondary flow as shown in the schematic diagram below in **Fig. 2**. This in turn led to an increase in the vorticities along the vortex line with respect to the grand-daughter tubes.

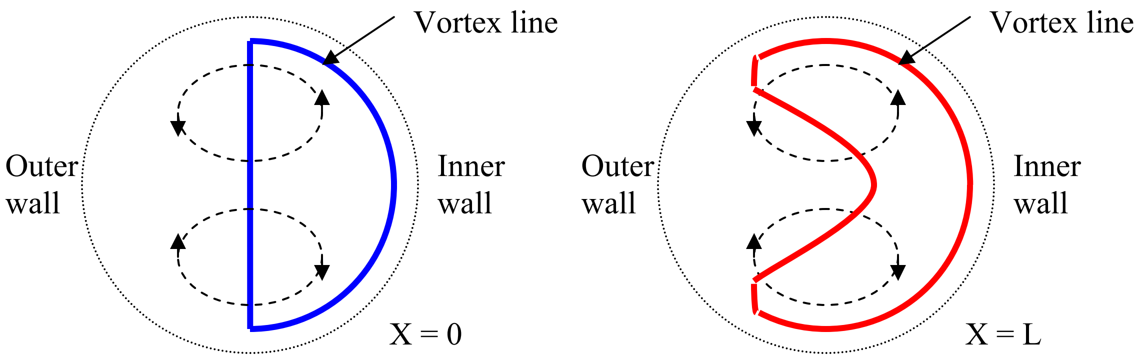

Fig. 2 (Left) Schematic Diagram of Vortex Line in Cross-section of DT. (Right) Same Vortex Line at an arbitrary distance L downstream. Dashed lines are secondary flow streamlines. Bold lines are vortex lines.

Hence, in the absence of secondary flow, there is no stretching of the vortex line in the cross-section of DT. In the case of the bifurcation geometry, the secondary flow is generated by the centrifugal acceleration, which is caused by the fluid having to negotiate a curvature to reach the daughter tube. Hence, secondary flow is insignificant without centrifugal acceleration (it is assumed that displacement effect caused by new intervening wall surface at the carinal ridge is negligible).

The secondary flow equation of motion and the order of magnitude estimates are then specified in the following manner:

Convective: $u\dfrac{\partial v}{\partial x}$ and $v\dfrac{\partial v}{\partial y}$ Estimate: $\dfrac{UV}{L}$ and $\dfrac{V^2}{a}$ (7)

Centrifugal: $\dfrac{u^2}{R}$ Estimate: $\dfrac{U^2}{R}$ (8)

Viscous: $\nu\dfrac{\partial^2 v}{\partial y^2}$ Estimate: $\dfrac{\nu V}{\delta^2}$ (9)

Since the scales for the convective and viscous terms (7, 9) are functionally dependent on V, a hypothetical scaling argument can be made about the centrifugal acceleration term, in an analogous way as [2]. It is in fact balanced by the dominant term from the remaining three other terms (7) and (9).

Two key parameters are identified, namely $\left(\frac{U\delta}{\nu}\right)\left(\frac{\delta}{L}\right)$ and $\left(\frac{L}{a}\right)\left(\frac{a}{R}\right)^{1/2}$, which is used to scale the secondary flow. As a result, three separate transport regimes (axial convective, radial convective and viscous) are demarcated as shown in **Fig. 3** below.

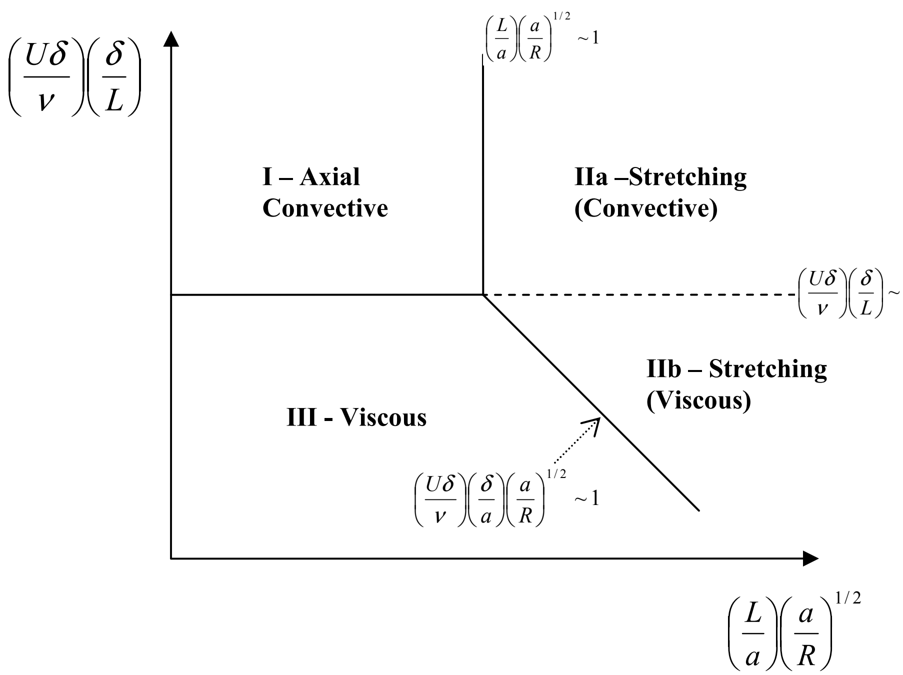

Fig. 3 Schematic Diagram of Flow Regimes Based on Scaling Arguments (Vorticity transport equation for secondary vorticity)

With the general scaling laws at hand, more specific cases can be examined. Two separate cases are presented here:

3. Case One: Pure Convective-Diffusive Transport

In this assumed case, the generation of secondary vorticity occurs instantaneously over a very short time-scale, so that the secondary vorticity generation profile resembles a Dirac delta. This allows us to neglect the vortex stretching term from the vorticity transport equation (since the bulk of the transport is convective-diffusive). The simplification is substantial because vorticity only enters the problem as a transportable scalar quantity to be specified as an initial condition.

Taking the curl of the momentum equation and simplifying,

$$U\frac{\partial \omega_y}{\partial x} = \nu \frac{\partial^2 \omega_y}{\partial z^2} \tag{10}$$

The boundary conditions for the secondary vorticity ω_y are

$$\omega_y(x,0) = 0 \tag{11}$$

$$\omega_y(x,a) = 0 \tag{12}$$

$$\omega_y(0,z) = \Omega_y(z) \tag{13}$$

The first boundary condition (11) is based on the condition of null shear stress at the plane of symmetry (z = 0). The second boundary condition (12) requires the assumption that the boundary layer thickness of the upper wall is thin ($\delta_v << a$), so that the stationary point of the velocity profile ($\partial u / \partial z = 0$) is positioned at $z \cong a$, where $0 \le z \le a$. The last boundary condition (13) is an entrance condition and also the upper limit of the vorticity without further vortex stretching along $0 \le x \le L$.

The solution obtained (via FFT) is

$$\overline{\omega}_y(x,z) = \sqrt{2}\sum_{n=1}^{\infty}\Omega_{yn}\exp\left(-\frac{(n\pi)^2\,\overline{x}}{\mathrm{Re}_a}\left(\frac{a}{L}\right)\right)\sin n\pi\overline{z} \tag{14}$$

$$\Omega_{yn} = \omega_{yn}(0) = \int_0^1\left(\sqrt{2}\sin n\pi\overline{z}\right)\Omega_y(\overline{z})d\overline{z} \tag{15}$$

where the variables on overbars indicate that they have been rendered dimensionless via

$\overline{\omega}_y = \omega_y\left(a/U\right)$, $\overline{x} = x/L$ and $\overline{z} = z/a$

4. Case Two: Generation of Secondary Vorticity and Convective Transport

In the attempt to resolve the convective-diffusive development of secondary through analytical means, the generation process of secondary vorticity has been neglected in Case One. An attempt is made here to rationalize the generation of secondary vorticity in the daughter tubes

How is secondary vorticity generated in the daughter tube? As previously mentioned, the pair of counter-rotating vortical secondary flow structure found in daughter tubes is mainly caused by the centrifugal acceleration of the fluid having to negotiate a segment of finite curvature. Although these vortices are predominantly axial in direction, the fluid shear distorts and extends the vortex lines along the cross-sectional plane, thus increases the secondary vorticity by stretching.

The upper symmetrical half of the cylinder cross-section is now considered as shown in **Fig. 4**. Several assumptions have to be made. First, a characteristic secondary flow streamline is defined with the center positioned at the center of the upper half of the cross-section, or a distance of $Z' = a/2$, where a is the radius of the tube. The characteristic minor semi-axis of the ellipse is Z' and the characteristic major semi-axis is Y', where $Y' = \sqrt{2}Z' = \sqrt{2}a/2$.

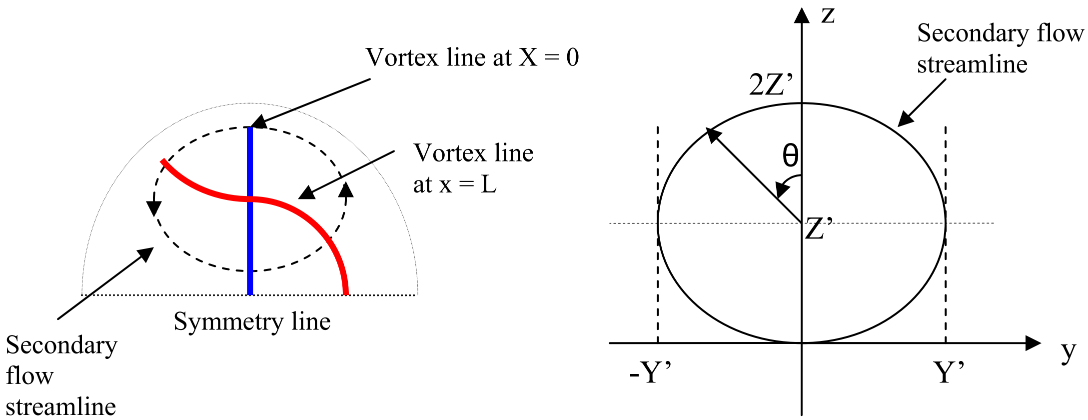

Fig. 4 Schematic Diagram of Vortex Lines in Cross-section of DT (Left) and the characteristic secondary flow streamline (Right)

The stretching of vortex line in the y-component is of interest. The ellipse as shown in **Fig. 4** (Right) can be expressed in polar coordinates as

$$y = -Y'\sin\theta$$
$$z = Z'(1+\cos\theta) \quad (16)$$

The analysis is restricted to a parametric angle of $0 \le \theta \le \pi/2$. The time scale parameter is defined as $t \sim x/U$ where x is the axial component. The derivative of y with respect to time t (in subscripts) is written as

$$y_t = -Y'\cos\theta \cdot \theta_t \quad (17)$$

Now the angular velocity can be expressed as a function of the secondary velocity

$$\theta_t = \frac{V}{r} \quad (18)$$

Therefore,

$$\left|\frac{y_t}{V}\right| = \left((\tan\theta)^2 + \left(\frac{Z'}{Y'}\right)^2\right)^{-1/2} \quad (19)$$

Where $\theta = \int_0^t \frac{V}{r} dt$

With the assumed secondary velocity scale of $V \sim \left(\frac{a}{R}\right)^{1/2} U$, an estimation of the vortex stretching term can be generalized as

$$\left|\frac{y_t}{a}\right| \sim \left[(aR)\left((\tan\theta)^2 + \left(\frac{Z'}{Y'}\right)^2\right)\right]^{-1/2} U \tag{20}$$

For this assumed case with negligible viscous effects, the vorticity transport equation is simplified in the same way as was shown in Case One.

$$\frac{\partial \omega_y}{\partial x} \approx \left[(aR)\left((\tan\theta)^2 + \left(\frac{Z'}{Y'}\right)^2\right)\right]^{-1/2} (\omega_y) \tag{21}$$

It is assumed that the radius of curvature depends weakly on x. For the case of $\theta \to 0$ which occurs near the bifurcation point $x \to 0$, eqn (16) simplifies to

$$\frac{\partial \ln \omega_y}{\partial x}\Big|_{x\to 0} \approx \frac{1}{\sqrt{aR}}\frac{Y'}{Z'} \tag{22}$$

which implies exponential dependence of the secondary vorticity on early axial distance.

On the other hand, for $\theta \to \pi/2$ which occurs at some unknown axial distance x = L', eqn (16) simplifies to

$$\frac{\partial \omega_y}{\partial x}\Big|_{x\to L'} \to 0 \tag{23}$$

which indicates that the rate of vortex stretching tends to zero at that point.

5. Conclusion

From the earlier arguments, it is recognized that in the absence of viscous effects, a plot of secondary vorticity as a function of axial distance in fact suggests a sigmoidal trend, i.e. an early exponential growth (22) followed by decreasing growth rates and eventually a stationary point (23).

Subsequently, at larger values of axial distance x, vorticity generation due to vortex stretching becomes insignificant and the viscous effects become dominant (14). In this case, the earlier presented result for Case 1 applies. Numerical simulation through the solution of the momentum equations has shown qualitative agreement with the predicted trends.

Acknowledgements

This work is supported by the Singapore-MIT Alliance (SMA) under grant number C382-427-003-091 and the Agency of Science, Technology and Research (A*STAR) under the grant number R279-000-208-305.

References

[1] Leong F.Y., C.H. Wang, K.A. Smith, *Secondary Flow Behavior and Particle Transport in Bifurcations*. AIChE Annual Meeting, Cincinnati, 2005.

[2] Jan D.L., A.H. Shapiro, R.D. Kamm. *Some features of oscillatory flow in a model bifurcation*. J. Appl. Physio. *67(1)*, 147-159, 1989.