Realizable Algebraic Reynolds Stress Model for Single Phase and for Multiphase Turbulent Flows

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EXTENDED ABSTRACT

A direct numerical simulation (DNS) of the instantaneous Navier-Stokes equation and the continuity equation provides a means to understand low Reynolds number turbulent flows of single-phase, Newtonian fluids in simple geometries. The ensemble average of these two equations yields the unclosed RANS-equation and the average continuity equation. Clearly, an appropriate closure model for the Reynolds stress is needed to support simulations of the RANS-equation for high Reynolds number flows in complex geometries.

Similarly, a DNS analysis of the instantaneous, phase-averaged, two-fluid model (or its corresponding mixture model) provides a means to understand low Reynolds number turbulent flows of two or more interpenetrating Newtonian fluids. Individual realizations produced by these closed models provide a means to understand the behavior of defined mixture properties and to test various multiphase constitutive assumptions. The Reynolds average of the mixture model also yields a set of unclosed equations (RAM- model).

Application of the RANS-equation (or the RAM-model) to high Reynolds number flows requires an appropriate closure for the average flux of fluctuating momentum caused by the fluctuating velocity field. For single phase flows, a closure for $<\underline{u}'\underline{u}'>$ is needed; for multiphase flows, $<\underline{u}'_{\rm mix}\,\underline{u}'_{\rm mix}>$ is required. Both of these operators are non-negative and, thereby, have non-negative eigenvalues. Closure models that do not guarantee this feature for all turbulent flows are called *unrealizable models*.

One of the most commonly used *realizable* closure models for the RAM-equation is based on an 'eddy viscosity" concept, which relates the Reynolds stress to the local mean strain rate, $<\underline{\underline{S}}_{\rm mix}>$, and the three non-trivial invariants of $<\underline{\underline{S}}_{\rm mix}>$ (see Shih et al., 1994, Koppula and Petty, 2005). The use of this idea for the mixture model gives the following algebraic closure for the kinematic turbulent momentum flux:

$$<\underline{u}'_{mix}\,\underline{u}'_{mix}> \; = \; tr\,(<\underline{u}'_{mix}\,\underline{u}'_{mix}>)\,\underline{\underline{\Re}}(<\underline{\underline{S}}_{mix}>)$$

$$= 2k_{mix}\left[\frac{1}{3}\underline{\underline{I}} - \tau_B(I_{<S>_{mix}}, II_{<S>_{mix}}, III_{<S>_{mix}})(<\underline{\underline{S}}_{mix}> - \frac{1}{3}tr(<\underline{\underline{S}}_{mix}>)\underline{\underline{I}})\right]. \qquad (1)$$

Unfortunately, this quisilinear model misrepresents the normal stress differences in fully developed pipe and channel flows as well as all other turbulent flow fields. More significant, however, is the observation that Eq.(1) is frame indifferent, which is inconsistent with the underlying physics of turbulent transport phenomena. The original idea of an “eddy viscosity”, albeit intuitively appealing, was used as early as 1877 by Boussinesq and afterwards by many others (see p.162, Bird et al., 2002; Pope, 1999) as a means to account for the enhanced mixing of momentum in turbulent flows due to turbulent fluctuations.

Parks et al. (1996) developed a preclosure for the Reynolds stress by using a smoothing approximation related to the space-time structure of turbulence. The preclosure equation links fluctuations in the instantaneous Reynolds stress to the Reynolds stress by using an operator that depends on the local mean field gradient and a time scale associated with the local space-time structure of turbulence. For multiphase flows governed by the mixture model, the Reynolds stress can be expressed as follows:

$$<\underline{u}'_{mix}\,\underline{u}'_{mix}> \; = \; tr<\underline{u}'_{mix}\,\underline{u}'_{mix}>\underline{\underline{R}} \qquad (2)$$

$$\underline{\underline{R}} = \frac{\underline{\underline{A}}^T\cdot<\underline{f}'\underline{f}'>\cdot\underline{\underline{A}}}{tr\,(\underline{\underline{A}}^T\cdot<\underline{f}'\underline{f}'>\cdot\underline{\underline{A}})}\,. \qquad (3)$$

In the above equation, the A-operator is defined as

$$\underline{\underline{A}} = [\,\underline{\underline{I}} + \tau_c\nabla<\underline{u}_{mix}>]^{-1} \qquad (4)$$

$$\tau_c = C_C^*\,k_{mix}\,/\,\varepsilon_{mix}\,. \qquad (5)$$

In a non-inertial frame, the A-operator depends on the angular velocity $\underline{\Omega}$ of the frame:

$$\underline{\underline{A}} = \left[\underline{\underline{I}} + \tau_C(\nabla < \underline{u}_{mix} > + \underline{\underline{\Omega}})\right]^{-1} . \tag{11}$$

In Eq.(11), the velocity gradient is relative to the non-inertial frame and the rotational dyadic-valued operator is related to the angular velocity of the frame as follows

$$\underline{\underline{\Omega}} = \underline{\underline{\underline{\varepsilon}}} \cdot \underline{\Omega} . \tag{12}$$

C_C^* is a universal function of the local mixture Reynolds number. The turbulent kinetic energy $k_{mix} (\equiv < \underline{u}'_{mix} \cdot \underline{u}'_{mix} > / 2 \geq 0)$ and the turbulent dissipation $\varepsilon_{mix} (\equiv \nu < (\nabla \underline{u}'_{mix})^T : \nabla \underline{u}'_{mix} > \geq 0)$ satisfy validated scalar-valued transport equations. The “prestress” $< \underline{f}'\underline{f}' >$ is caused by fluctuations in the instantaneous Reynolds stress and pressure fluctuations.

If the “prestress” is isotropic, then $< \underline{f}'\underline{f}' > \propto \underline{\underline{I}}$. For this situation, the normalized momentum flux is

$$\underline{\underline{R}}^o = \frac{\underline{\underline{A}}^T \cdot \underline{\underline{A}}}{tr(\underline{\underline{A}}^T \cdot \underline{\underline{A}})} \equiv \underline{\underline{K}} \quad . \tag{6}$$

It is noteworthy that Eq.(6) reduces to the B-closure (i.e., Eq.(1) above) for $\| \tau_c \nabla < \underline{u}_{mix} > \| \ll 1$. However, Eq.(1) does not apply throughout the flow field.

The IPS-stress, defined by Eq.(6) above, is realizable for all turbulent flows and, most significantly, depends on the frame of rotation because the mean velocity gradient in not frame indifferent, as is the mean strain rate. This characteristic is consistent with the fundamental physical idea that the Reynolds stress, unlike the molecular stress, transports momentum by fluctuations influenced by Coriolis forces.

Both $\underline{\underline{K}}$ (see Eq.(6)) and $\underline{\underline{R}}$ are symmetric and non-negative operators. Therefore, the eigenvalues of these two operators are real, non-negative, and are restricted to the positive orthant of a hyperplane in a Euclidean three-dimensional space wherein tr($\underline{\underline{K}}$) = 1 and tr($\underline{\underline{R}}$) = 1. With the assumption that

$$\underline{\underline{R}} = \Re(\underline{\underline{K}}) , \tag{7}$$

the Cayley-Hamilton theorem of linear algebra implies that

$$\underline{\underline{R}} = \underline{\underline{K}} + C_2(\underline{\underline{K}} \cdot \underline{\underline{K}} - II_K \underline{\underline{K}}) \quad . \tag{8}$$

The condition that the eigenvalues on the boundaries of the K-plane map onto the boundaries of the R-plane imply that for all turbulent flows (see Koppula et al., 2006),

$$C_2 = -\alpha = -\alpha^*(II_K - \frac{1}{3}) \quad . \tag{9}$$

In the above equation, $II_K \equiv tr(\underline{\underline{K}} \cdot \underline{\underline{K}})$. An analysis of Eq. (8) shows that $\underline{\underline{R}}$ is a non-negative operator provided

$$0 \le \alpha^* \le 21 . \tag{10}$$

DNS results for fully-developed channel flows were used to determine the two closure coefficients introduced by Eq.(5) and Eq.(9). For $y^+ > 10$, $C_C^* \cong 0.65$ and $\alpha^* \cong 9.0$.

For simple shear flows in non-inertial frames with an angular velocity co-linear with the vorticity vector, the A-operator depends on two independent groups N_S and N_Ω that compare the turbulent time scale $k_{mix} / \varepsilon_{mix}$ with the time scales associated with the mixture shear rate and the angular velocity of the frame, respectively:

$$\tau_C \nabla < \underline{u}_{mix} > = N_S \, \underline{e}_y \underline{e}_z \quad , \quad N_S \equiv C_C^* \frac{k_{mix}}{\varepsilon_{mix}} \frac{d < u_z >_{mix}}{dy} \tag{13}$$

$$\tau_C \underline{\underline{\Omega}} = N_\Omega (\underline{e}_y \underline{e}_z - \underline{e}_z \underline{e}_y) \quad , \quad N_\Omega \equiv C_C^* \frac{k_{mix}}{\varepsilon_{mix}} \Omega_x . \tag{14}$$

The redistribution of energy caused by the coupling between the velocity gradient and the frame rotation operator with the turbulence is presently being assessed with DNS simulations. This comparison is a critical test for any algebraic closure. It is noteworthy that the classical B-closure (see Figure 1 and Eq.(1)) predicts that the frame rotation has no influence on the redistribution of turbulent energy, contrary to DNS simulations and experiments. The B-closure predicts that $R_{xx} = R_{yy} = R_{zz} = 1/3$ for all values of N_S and N_Ω (see Figures 1 and Figures 2). This unphysical result does not support the use of Eq.(1) as a closure model for single phase or multiphase flows. However, the algebraic prestress (APS-) model defined by Eq.(8) above does show a redistribution of energy across the flow field that is qualitatively in agreement with experimental data as well as DNS simulations.

As can be seen in Figure 2, the effect of frame rotation on homogeneous turbulence (i.e., $N_S = 0$) is to redistribute the turbulent energy to the component of the fluctuating velocity that is aligned in the same direction as the angular velocity. The energy is equally distributed between the other two components ($R_{yy} = R_{zz}$) for all values of N_Ω. For simple shear flows (i.e., $N_S > 0$), the angular velocity is co-linear with the mean field vorticity. For N_S = + 5, Figure 2 shows that the distribution process is more complicated. The APS-closure predicts that all the turbulent energy is transferred to the vorticity direction as $N_\Omega \to \pm\infty$ (see Figure 2).

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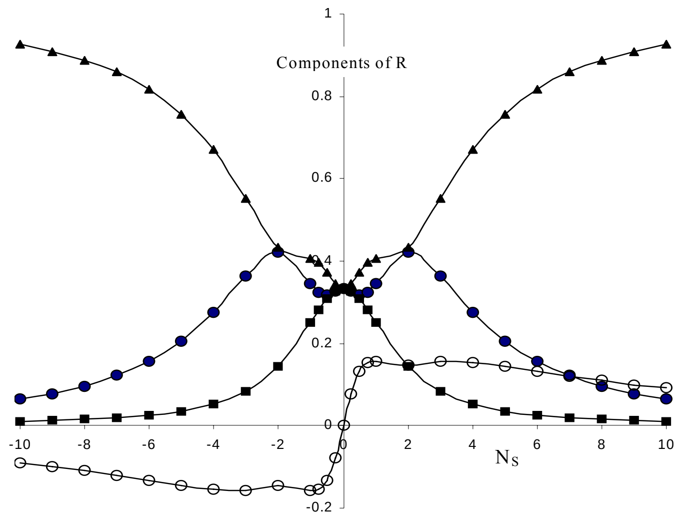

Figure 1. APS theory prediction of the variation of the components of $\underline{\underline{R}}$ with the mean strain rate for simple shear flows (● R_{xx}; ■ R_{yy}; ▲ R_{zz}; ○ $-R_{yz}$).

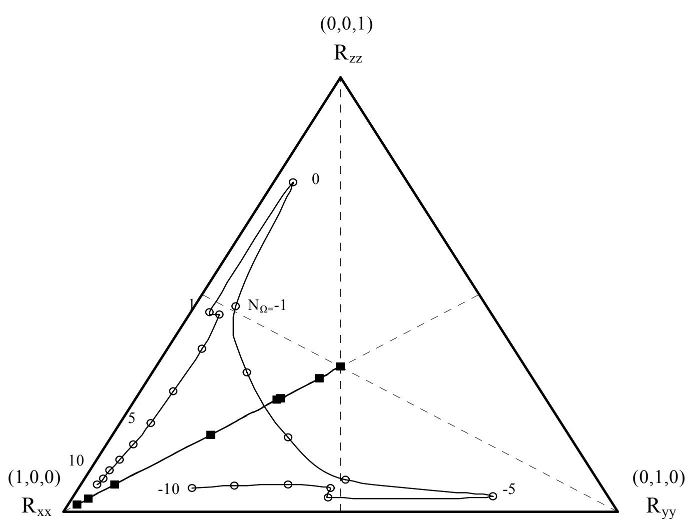

Figure 2. The effect of fame rotation (N_{Ω}) on the partition of turbulent energy for homogeneous turbulence (■ $N_S = 0$) and for homogeneous simple shear (O $N_S = 5$).