Modular Design of Nonlinear Observers for State and Disturbance Estimation

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***Abstract*—This work studies the problem of design of nonlinear observers in the presence of exogenous disturbances. In particular, the present work proposes a systematic design method for nonlinear state and disturbance observers in modular form, consisting of an observer for the disturbance-free part of the system, along with a disturbance observer and a state-estimate corrector. The modular observer is first defined and characterized in a general setting and then, a systematic design method is developed on the basis of exact linearization with eigenvalue assignment. Necessary and sufficient conditions for feasibility of the proposed nonlinear modular observer design are derived. The convergence properties of the proposed nonlinear observer are tested through simulation studies in an illustrative example involving a biological reactor.**

I. Introduction

TECHNICAL limitations and/or high cost of sensors result in the non-availability of all state variables for direct on-line measurement, and this creates the need for on-line state estimation. Furthermore, the operation of a process or plant is subject to time-varying disturbances, associated with changes in key process parameters or improper operation of sensing instruments. For this reason, in addition to monitoring the state variables, there is a definite practical need for detection and estimation of disturbances.

The problem of combined state and disturbance estimation can be conceptually formulated as a state estimation problem for an extended system. In the case of linear systems, the well-known Luenberger observer offers a comprehensive solution. More specifically, in industrial applications of combined state and disturbance estimation, the Luenberger observer is designed and implemented in a modular configuration, consisting of an observer for the disturbance-free part of the system and, on top of it, a disturbance estimator and a state-estimate corrector ([3]).

The purpose of the present work is to develop a systematic nonlinear observer design method for state and disturbance estimation, so that the resulting observer possesses the modular configuration that is sought for in practice. The nonlinear observer design problem will be formulated within the general framework of exact observer linearization ([1],[2],[4],[5],[7]–[12]), and in particular, following the invariant-manifold formulation, originally proposed in [5] and further developed in [7],[10],[11],[12].

After a review of recent results on state and disturbance observers in non-modular form ([6]), the modular observer will be defined and characterized in terms of appropriate invariance conditions. Necessary and sufficient conditions for exact linearization with eigenvalue assignment will be derived, leading to a step-by-step design procedure for the modular observer. It will be shown that the proposed modular observer design is feasible whenever the simultaneous design of [6] is feasible. Under these conditions, it can be proven that both state and disturbance estimation errors converge to zero with assignable rates. Finally, the performance of the proposed observer is evaluated in an illustrative bioreactor application through simulation studies.

II. Problem Formulation

Consider a dynamic system

$$\begin{aligned} \dot{x} &= f(x,w) \\ y &= h(x,w) \end{aligned} \tag{1}$$

that represents the dynamics of a process, where x is the process state vector, y is the vector of measurements and w is the vector of unmeasurable process or sensor disturbances. The dynamics of the disturbances is generated by the exosystem

$$\dot{w} = s(w) \tag{2}$$

The problem of *state and disturbance estimation* is the one of estimating both the process states x and the disturbances w given on-line measurements of y. If one considers the extended system consisting of (1) and (2), i.e.

$$\begin{aligned} \dot{x} &= f(x, w) \\ \dot{w} &= s(w) \\ y &= h(x, w) \end{aligned} \tag{3}$$

the problem of state and disturbance estimation becomes a standard state estimation problem for system (3), where the state $\begin{bmatrix} x \\ w \end{bmatrix}$ of (3) must be estimated from on-line measurements of y.

The effect of disturbances is often neglected by engineers, whenever their magnitude is viewed to be small relative to the other variables affecting the operation of the process. In this case, the process dynamics is approximately represented as

$$\begin{aligned} \dot{x}^{df} &= f\left(x^{df}, 0\right) \\ y &= h\left(x^{df}, 0\right) \end{aligned} \tag{4}$$

where x^{df} represents the "disturbance-free" process state, i.e. the process state when $w = 0$.

When the effect of disturbances can be neglected, (4) can be used as the basis for designing an observer. This is the most convenient approach from an engineering point of view, since it simplifies the observer equations and implementation. When disturbances are large enough so that they must be accounted for, engineers still want to monitor process state estimates obtained from (4) ("disturbance-free part of the system"), together with disturbance estimates and estimates of the process states that account for disturbances ("corrected state estimates"). In practice, a typical industrial implementation of a linear Luenberger observer for state and disturbance estimation has the structure of Figure 1 ([3]).

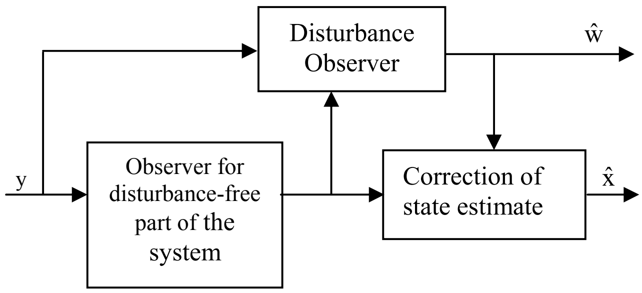


Fig. 1. Modular observer for state and disturbance estimation

The observer of Figure 1 has a modular structure: there is a basic observer for the disturbance-free part of the system and, on top of it, there is a disturbance observer. The disturbance estimate is used to correct the state estimate that was computed from the disturbance-free part of the system.

The modular configuration generates two sets of state estimates, one neglecting disturbances and another accounting for disturbances, and their comparison can be useful from a process monitoring point of view. Moreover, it gives the process engineer the flexibility of turning on or off the disturbance observer, or using alternative disturbance observers based on different assumptions on the nature of disturbances.

The problem of simultaneous state and disturbance estimation in non-modular structure was recently studied in [6] in the context of the observer linearization approach ([1],[2],[4]–[12]). In the present paper, a modular design method will be developed: the observer for the disturbance-free part of the system will be designed first and then the disturbance observer plus state-estimate corrector will be designed. The observer error linearization approach will be used for both steps of the design and the results will be compared with the simultaneous design of [6].

III. BACKGROUND

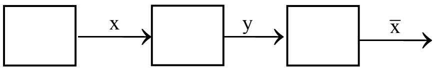
Definition 1: *Given a dynamic system*

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{5}$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$, $h: \mathbb{R}^n \to \mathbb{R}^p$ *are continuously differentiable functions, the system*

$$\begin{aligned} \dot{\bar{x}} &= \Gamma(\bar{x}, y) \\ \hat{x} &= \Xi(\bar{x}) \end{aligned} \tag{6}$$

where $\Gamma : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$, $\Xi : \mathbb{R}^n \to \mathbb{R}^n$ *are continuously differentiable functions, is called an* <u>*observer*</u> *for (5) if in the series connection*



the overall dynamics

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{\bar{x}} &= \Gamma(\bar{x}, h(x)) \end{aligned} \tag{7}$$

has the property that $x = \Xi(\bar{x})$ *is an invariant manifold. In particular, when* Ξ *is the identity map, (6) is called an* <u>*identity observer*</u>.

In the above definition, the requirement that $x = \Xi(\bar{x})$ is an invariant manifold of (7), i.e. that

$x(0) = \Xi(\bar{x}(0)) \;\Rightarrow\; x(t) = \Xi(\bar{x}(t)) \quad \forall t > 0$,

translates to the following condition:

$$\left[\frac{\partial \Xi}{\partial \bar{x}}(\bar{x})\right] \Gamma(\bar{x}, h(\Xi(\bar{x}))) = f(\Xi(\bar{x})) \tag{8}$$

In the special case of identity observer, the above condition collapses to

$$\Gamma(\bar{x}, h(\bar{x})) = f(\bar{x}) \tag{9}$$

In the observer linearization design method, the observer is chosen so that it can be transformed to a linear system

$$\dot{z} = Az + By \tag{10}$$

with A, B being $n \times n$ and $n \times p$ matrices respectively and A having pre-assigned eigenvalues, under an appropriate coordinate transformation.

<u>Proposition 1</u> [5]: *Given* $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^p$ *continuously differentiable functions, it is possible to find a continuously differentiable function* $\Gamma : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$
that satisfies condition (9) and a coordinate transformation that maps $\dot{\bar{x}} = \Gamma(\bar{x}, y)$ *to (10) if and only if there exists an invertible continuously differentiable function* $\theta : \mathbb{R}^n \to \mathbb{R}^n$ *that satisfies the partial differential equation*

$$\frac{\partial \theta}{\partial x}(x) f(x) = A\theta(x) + Bh(x) \tag{11}$$

Assuming that such a function θ *exists, the choice*

$$\Gamma(x, y) = f(x) + L(x)\,(y - h(x)) \tag{12}$$

where

$$L(x) = \left[\frac{\partial \theta}{\partial x}(x)\right]^{-1} B \tag{13}$$

satisfies condition (9) and makes $\dot{\bar{x}} = \Gamma(\bar{x}, y)$ *transformable to (10) via the coordinate transformation* $z = \theta(\bar{x})$.

The result of Proposition 1 reduces the problem of observer linearization to the one of solving the PDE (11). Alternative local solvability conditions for (11) are available in the literature ([5],[7],[10],[11],[12]), depending on the smoothness assumptions on $f(x)$ and $h(x)$ and also, depending on the nature of the spectrum of $\frac{\partial f}{\partial x}(x)$ evaluated at the equilibrium point. The following Proposition provides local solvability conditions for the real-analytic Poincaré-domain case.

<u>Proposition 2</u> [5]: *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^p$ *be real analytic functions with* $f(0) = 0$, $h(0) = 0$ *and denote* $F = \frac{\partial f}{\partial x}(0)$, $H = \frac{\partial h}{\partial x}(0)$. *Suppose:*

1. *There exists an invertible matrix* T *such that* $TF = AT + BH$.
2. *All the eigenvalues of* A *are non-resonant with* $\sigma(F)$, *i.e. no eigenvalue* λ_j *of* A *is of the form* $\lambda_j = \sum_{i=1}^{n} m_i \kappa_i$

with $\kappa_i \in \sigma(F)$ *and* m_i *nonnegative integers, not all zero.*

3. 0 does not lie in the convex hull of $\sigma(F)$.

Then there exists a unique analytic solution of the PDE (11) locally around $x = 0$. *The solution has the property that* $\frac{\partial \theta}{\partial x}(0) = T$ *and so,* θ *is a local diffeomorphism.*

Note that assumptions 1 and 2 of the above Proposition imply that (H, F) is an observable pair. On the other hand, if (H, F) is an observable pair, it is always possible to find matrices A, B, T which satisfy the matrix equation of assumption 1, with T invertible and A having prescribed eigenvalues.

The results of Propositions 1 and 2 can now be applied to the problem of state and disturbance estimation, considering the extended system (3), where $x \in \mathbb{R}^m$ is the vector process states, $w \in \mathbb{R}^\ell$ is the vector of disturbances and $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{m+\ell}$ is the state of the entire system (3) that must be estimated. The corresponding observer PDE (11) for system (3) is

$$\frac{\partial \theta}{\partial x}(x,w) f(x,w) + \frac{\partial \theta}{\partial w}(x,w) s(w) = A\theta(x,w) + Bh(x,w) \tag{14}$$

and, assuming that it is solvable, the observer has the form

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} f(\hat{x},\hat{w}) \\ s(\hat{w}) \end{bmatrix} + L(\hat{x},\hat{w})\big(y - h(\hat{x},w)\big) \tag{15}$$

where $$L(x,w) = \begin{bmatrix} \frac{\partial \theta}{\partial x}(x,w) & \frac{\partial \theta}{\partial w}(x,w) \end{bmatrix}^{-1} B \tag{16}$$

Note that the computational effort in the observer design can be somewhat reduced if A is taken to be block-diagonal, with diagonal blocks of sizes $m \times m$ and $\ell \times \ell$. In particular, setting $A = \begin{bmatrix} A_0 & 0 \\ 0 & A' \end{bmatrix}$, $B = \begin{bmatrix} B_0 \\ B' \end{bmatrix}$, where A_0, A', B_0, B' are $m \times m$, $\ell \times \ell$, $m \times p$, $\ell \times p$ matrices respectively and partitioning $\theta(x,w)$ accordingly

$$\theta(x,w) = \begin{bmatrix} \pi(x,w) \\ \eta(x,w) \end{bmatrix}$$

with $\pi : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^m$, $\eta : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^\ell$, the observer PDE (14) breaks up into two uncoupled PDE's :

$$\frac{\partial \pi}{\partial x}(x,w) f(x,w) + \frac{\partial \pi}{\partial w}(x,w) s(w) = A_0\pi(x,w) + B_0 h(x,w) \tag{17}$$

$$\frac{\partial \eta}{\partial x}(x,w) f(x,w) + \frac{\partial \eta}{\partial w}(x,w) s(w) = A'\eta(x,w) + B' h(x,w) \tag{18}$$

As an immediate consequence of Proposition 2, we have

Proposition 3 [6]: *Let* $f : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^m$, $s : \mathbb{R}^\ell \to \mathbb{R}^\ell$ *and* $h : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^p$ *be real analytic functions with* $f(0,0) = 0$, $s(0) = 0$, $h(0,0) = 0$ *and denote* $F = \frac{\partial f}{\partial x}(0,0)$, $P = \frac{\partial f}{\partial w}(0,0)$, $S = \frac{\partial s}{\partial w}(0)$, $H = \frac{\partial h}{\partial x}(0,0)$, $Q = \frac{\partial h}{\partial w}(0,0)$.

Suppose:

1. There exists an invertible matrix T *such that* $T\begin{bmatrix} F & P \\ S & 0 \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A' \end{bmatrix} T + \begin{bmatrix} B \\ B' \end{bmatrix} \begin{bmatrix} H & Q \end{bmatrix}$.

2. All the eigenvalues of A_0 *and* A' *are non-resonant with* $\sigma(F) \cup \sigma(S)$.

3. 0 does not lie in the convex hull of $\sigma(F) \cup \sigma(S)$.

Then the PDE's (17) and (18) admit unique solutions locally around $(x,w) = (0,0)$. *The solutions have the property that*

$$\begin{bmatrix} \frac{\partial \pi}{\partial x}(0,0) & \frac{\partial \pi}{\partial w}(0,0) \\ \frac{\partial \eta}{\partial x}(0,0) & \frac{\partial \eta}{\partial w}(0,0) \end{bmatrix} = T$$

IV. Modular Observer

In the present section, a modular design of an observer for system (3) will be developed, as an alternative to the "simultaneous" design outlined in the previous section.

The proposed modular observer will consist of two distinct parts. An identity observer for the disturbance-free part of the system (system (4)), followed by a disturbance observer together with a corrector for the state estimate. The identity observer for the disturbance-free part of the system will have the form

$$\dot{\tilde{x}} = \Gamma_0(\tilde{x}, y) \tag{19}$$

whereas the disturbance observer and corrector will have the form

$$\dot{\hat{w}} = \Gamma_w(\tilde{x}, \hat{w}, y) \tag{20}$$

$$\hat{x} = \Upsilon(\tilde{x}, \hat{w}) \tag{21}$$

Figure 2 depicts the structure of the proposed modular observer

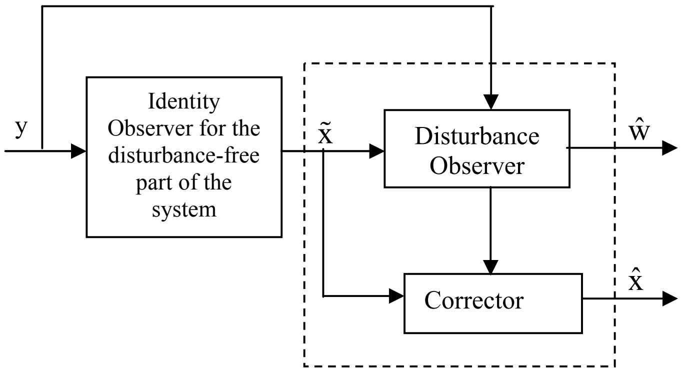


Figure 2: Structure of the modular observer (19) – (21).

The following Proposition is an immediate consequence of Definition 1, applied to the modular structure considered here.

Proposition 4: a) *System (19) will be an identity observer for (4) if*

$$\Gamma_0(\tilde{x}, h(\tilde{x}, 0)) = f(\tilde{x}, 0) \tag{22}$$

b) *The overall system (19) – (21) will be an observer for (3) if the following conditions are met*

$$\frac{\partial \Upsilon}{\partial \tilde{x}}(\tilde{x}, \hat{w})\Gamma_0(\tilde{x}, h(\Upsilon(x, \hat{w}), \hat{w})) + \frac{\partial \Upsilon}{\partial \hat{w}}(\tilde{x}, \hat{w}) s(\hat{w}) = f(\Upsilon(x, \hat{w}), \hat{w}) \tag{23}$$

and

$$\Gamma_w(\tilde{x}, \hat{w}, h(\Upsilon(x, \hat{w}), \hat{w})) = s(\hat{w}) \tag{24}$$

We can now proceed with the design of the modular observer via linearization and eigenvalue assignment. The rationale of the previous section will be followed, while adhering to the modular structure of (19)–(21).

Proposition 5: *It is possible to find* $\Gamma_0 : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ *that satisfies (22) and a coordinate transformation that maps (19) to*

$$\dot{z}_0 = A_0 z_0 + B_0 y \tag{25}$$

where A_0 *and* B_0 *are* $m \times m$ *and* $m \times p$ *matrices respectively, if and only if there exists an invertible function* $\theta_0 : \mathbb{R}^m \to \mathbb{R}^m$ *that satisfies the PDE*

$$\frac{\partial \theta_0}{\partial x}(x) f(x, 0) = A_0 \theta_0(x) + B_0 h(x, 0) \tag{26}$$

Assuming that such a function θ_0 *exists, the choice*

$$\Gamma_0(x, y) = f(x, 0) + \left[\frac{\partial \theta_0}{\partial x}(x)\right]^{-1} B_0(y - h(x, 0)) \tag{27}$$

satisfies condition (22) and makes (19) transformable to (25) via the coordinate transformation $z_0 = \theta_0(\tilde{x})$.

Proposition 6: *Assume that* Γ_0 *is given by (27), with* θ_0 *being an invertible function that satisfies (26). It is possible to find* $\Gamma_w : \mathbb{R}^m \times \mathbb{R}^\ell \times \mathbb{R}^p \to \mathbb{R}^m$ *that satisfies (24) and a coordinate transformation that maps the dynamics of (19)–(20) to*

$$\begin{bmatrix}\dot{z}_0\\ \dot{z}'\end{bmatrix}=\begin{bmatrix}A_0 & 0\\ 0 & A'\end{bmatrix}\begin{bmatrix}z_0\\ z'\end{bmatrix}+\begin{bmatrix}B_0\\ B'\end{bmatrix}y \tag{28}$$

if and only if there exists a function $\Phi:\mathbb{R}^m\times\mathbb{R}^\ell\to\mathbb{R}^\ell$ *such that the equation* $z'=\Phi(x,w)$ *is solvable with respect to* w *and*

$$\frac{\partial\Phi}{\partial x}(x,w)\,\Gamma_0(x,h(\Upsilon(x,w),w))+\frac{\partial\Phi}{\partial w}(x,w)\,s(w)=\ A'\Phi(x,w)+B'h(\Upsilon(x,w)) \tag{29}$$

Assuming that such a function Φ exists, the choice

$$\Gamma_w(x,w,y)=s(w)+\left[\frac{\partial\Phi}{\partial w}(x,w)\right]^{-1}\cdot\left(B'-\left[\frac{\partial\Phi}{\partial x}(x,w)\right]\left[\frac{\partial\theta_0}{\partial x}(x)\right]^{-1}B_0\right)\left(y-h\left(\Upsilon(x,w),w\right)\right) \tag{30}$$

satisfies condition (24) and makes (19)–(20) transformable to (28) via coordinate transformation $\begin{bmatrix}z_0\\ z'\end{bmatrix}=\begin{bmatrix}\theta_0(\tilde{x})\\ \Phi(\tilde{x},\hat{w})\end{bmatrix}$.

The proofs of Propositions 5 and 6 are omitted for brevity.

In summary, the design of the modular observer involves the following steps:

1. *Design of the identity observer for the disturbance-free part of the system:*

   $\Gamma_0(x,y)\ =\ f(x,0)+L_0(x)\left(y-h(x,0)\right)$

   where $L_0(x)\ =\left[\frac{\partial\theta_0}{\partial x}(x)\right]^{-1}B_0$ *and where* $\theta_0(x)$ *satisfies the PDE*

$$\frac{\partial\theta_0}{\partial x}(x)f(x,0)\ =\ A_0\theta_0(x)+B_0h(x,0) \tag{26}$$

2. *Design of the corrector:*

   $\Upsilon(x,w)$ *satisfies the PDE*

$$\frac{\partial\Upsilon}{\partial x}(x,w)\Gamma_0\left(x,h\left(\Upsilon(x,w),w\right)\right)+\frac{\partial\Upsilon}{\partial w}(x,w)s(w)=f\left(\Upsilon(x,w),w\right) \tag{23}$$

3. *Design of the disturbance observer:*

   $\Gamma_w(x,w,y)=s(w)+L'(x,w)\left(y-h\left(\Upsilon(x,w),w\right)\right)$

   where $L'(x,w)\ =\left[\frac{\partial\Phi}{\partial w}(x,w)\right]^{-1}\left(B'-\frac{\partial\Phi}{\partial x}(x,w)L_0(x)\right)$ *and where* $\Phi(x,w)$ *satisfies the PDE*

$$\frac{\partial\Phi}{\partial x}(x,w)\Gamma_0\left(x,h\left(\Upsilon(x,w),w\right)\right)+\frac{\partial\Phi}{\partial w}(x,w)s(w)=A'\Phi(x,w)+B'h\left(\Upsilon(x,w),w\right) \tag{29}$$

The conclusion is that the modular design of the observer reduces to solvability of three PDE's, (26) for θ_0, (23) for Υ and (29) for Φ.

Local solvability conditions for PDE (26) immediately arise from Proposition 2. However, PDE's (23) and (29) need further attention.

In the next section, it will be seen that if the PDE's (17) and (18) for simultaneous design are locally solvable, then PDE's (23) and (29) for modular design are also solvable.

V. Modular Versus Simultaneous Design

The following proposition provides the solution of the modular observer design problem if the solution of the simultaneous design problem is available.

Proposition 7: *Suppose that (17) and (18) are locally solvable in a neighborhood of* $(x=0, w=0)$ *and that*

$\left[\frac{\partial \pi}{\partial x}(x,0)\right]$ *and* $\begin{bmatrix} \frac{\partial \pi}{\partial x}(x,w) & \frac{\partial \pi}{\partial w}(x,w) \\ \frac{\partial \eta}{\partial x}(x,w) & \frac{\partial \eta}{\partial w}(x,w) \end{bmatrix}$ *are invertible matrices in a neighborhood of* $(x=0, w=0)$. *Then*

i. *The function* $\theta_0(x) = \pi(x,0)$ *is an invertible function in a neighborhood of* $x=0$ *and satisfies the PDE (26).*

ii. *The algebraic equation* $\pi(\hat{x}, w) = \pi(x,0)$ *is locally solvable with respect to* $\hat{x}$ *and its solution* $\hat{x} = \Upsilon(x,w)$ *satisfies the PDE (23).*

iii. *The function* $\Phi(x,w) = \eta(\Upsilon(x,w), w)$ *satisfies the PDE (29). Also, the algebraic equation* $z' = \Phi(x,w)$ *is locally solvable with respect to* w.

The proof of Proposition 7 is omitted for brevity.

VI. Modular Design For Sensing Error Estimation

A special case of state and disturbance estimation is the *state and sensing error estimation problem*, when the disturbances only affect the sensing devices, and in an additive way. In this case, the extended system (3) has the following special form:

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{w} &= s(w) \\ y &= h(x) + q(w) \end{aligned} \tag{30}$$

It has been shown in [6] that the solutions of the PDE's (17) and (18) are of the form

$$\begin{aligned} \pi(x,w) &= \psi(x) + \omega(w) \\ \eta(x,w) &= \psi'(x) + \omega'(w) \end{aligned} \tag{31}$$

where $\psi, \omega, \psi', \omega'$ satisfy the PDE's

$$\frac{\partial \psi}{\partial x}(x) f(x) = A_0 \psi(x) + B_0 h(x), \tag{32}$$

$$\frac{\partial \omega}{\partial w}(w) s(w) = A_0 \omega(w) + B_0 q(w), \tag{33}$$

$$\frac{\partial \psi'}{\partial x}(x) f(x) = A' \psi'(x) + B' h(x), \tag{34}$$

$$\frac{\partial \omega'}{\partial w}(w) s(w) = A' \omega'(w) + B' q(w) \tag{35}$$

Applying the results of Proposition 7, it follows that the solution to the modular observer design problem can be obtained as follows:

$$\begin{aligned} \theta_0(x) &= \psi(x) \\ \Upsilon(x,w) &= \psi^{-1}(\psi(x) - \omega(w)) \\ \Phi(x,w) &= \psi'(\Upsilon(x,w)) + \omega'(w) \end{aligned} \tag{36}$$

It should be noted that, in most engineering applications, disturbances with linear dynamics and acting linearly on the measurements are considered:

$s(w) = Sw$

$q(w) = Qw$

where S and Q are matrices of appropriate dimensions.

In this case,

$\omega(w) = \Omega w$, where Ω is the solution of $\Omega S - A_0 \Omega = B_0 Q$

$\omega'(w) = \Omega' w$, where Ω' is the solution of $\Omega' S' - A' \Omega' = B' Q$

and thus

$$\Upsilon(x,w)=\psi^{-1}\left(\psi(x)-\Omega w\right)$$
$$\Phi(x,w)=\psi'\left(\Upsilon(x,w)\right)+\Omega' w \tag{37}$$

VII. BIOREACTOR APPLICATION

Consider a typical bioreactor, where biochemical reactions take place, resulting in biomass production and substrate consumption following Monod kinetics:

$$\begin{aligned}\frac{dX}{dt}&=-DX+\frac{\mu_{\max}S}{K+S}X\\ \frac{dS}{dt}&=D\left(S_f-S\right)-\frac{1}{Y}\frac{\mu_{\max}S}{K+S}X\end{aligned} \tag{38}$$

where X is the biomass concentration, S the substrate concentration, S_f the inlet substrate concentration, D the dilution rate, K a reaction constant, Y the yield coefficient and μ_{max} the maximal specific growth rate. It is assumed that the process parameters satisfy the inequality $\frac{\mu_{max}}{D}>1+\frac{K}{S_f}$, which guarantees the existence of a unique positive equilibrium for the bioreactor dynamics. Moreover, it guarantees that the equilibrium is hyperbolically stable.

The biomass X is measurable on line, but the measurement could be subjected to a systematic error w. This is assumed to remain constant over a certain period of time, but potentially undergoing step changes. The objective is to estimate both the bioreactor's state and the systematic error (step disturbance) w. Therefore the dynamic system under consideration is:

$$\begin{aligned}\frac{dX}{dt}&=-DX+\frac{\mu_{max}S}{K+S}X\\ \frac{dS}{dt}&=D\left(S_f-S\right)-\frac{1}{Y}\frac{\mu_{max}S}{K+S}X\\ \frac{dw}{dt}&=0\\ y&=X+w\end{aligned} \tag{39}$$

and the objective is to design a modular-form observer for this system, to estimate the unmeasured substrate concentration S and the error w, following the method developed in the previous section.

For the design of the nonlinear observer, a MAPLE code has been written, which solves the PDEs (32), (34) and performs the inversion and function compositions in (37) up to a finite truncation order N, calculates the state-dependent observer gains and performs numerical simulations, in order to test the observer.

In the present study, the following parameter values were used in all simulations:

$S_f=50.0$, $D=0.4$, $K=2.0$, $Y=0.5$, $\mu_{max}=0.9$

for which the equilibrium is:

$X_s=24.2$, $S_s=1.6$, $w_s=0.0$

and corresponds to zero sensing error.

The Jacobian of the dynamics of (39), evaluated at equilibrium forms an observable pair with $\begin{bmatrix}1 & 0 & 1\end{bmatrix}$, i.e. system (39) is linearly observable.

The process (39) and the observer were simulated with the following initial conditions:

$X(0)=20.0,\quad \hat{X}(0)=22.0$

$S(0)=7.0,\quad \hat{S}(0)=3.0$

$w(0)=1.0,\quad \hat{w}(0)=0.0$

This accounts for a unit step change in the sensing error, in the presence of non-equilibrium initial conditions.

Representative Observer Responses

Figures 3-6 depict representative observer responses, obtained for the following design parameters:

$$A = \begin{bmatrix} -1 & 0 \\ -6 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} -3.9 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 \end{bmatrix}$$

i.e for observer eigenvalues at -1 and -1.5 for the process states and -3.9 for the disturbance state. All calculations (solution of PDEs, inversion and function compositions) were performed up to truncation order N=5 which, as will be seen in the sequel, provides adequate accuracy.

In Figures 3 and 4, the process state response is compared with two alternative state estimates – one neglecting the disturbance and one accounting for it. Neglecting the disturbance gives rise to a permanent error (offset). The corrector part of the modular observer provides the necessary correction to completely eliminate the offset.

Figure 5 compares the applied step disturbance with the disturbance estimate. The disturbance observer part of the modular observer quickly detects the presence of the step disturbance and accurately estimates it without permanent error.

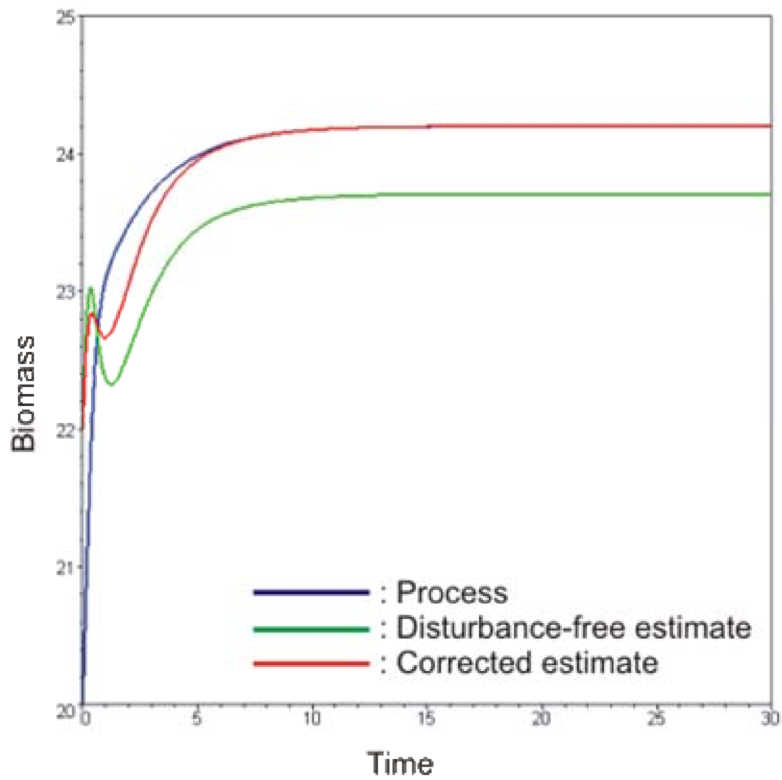


Fig. 3: Disturbance-free and corrected biomass concentration estimates, for truncation order N=5.

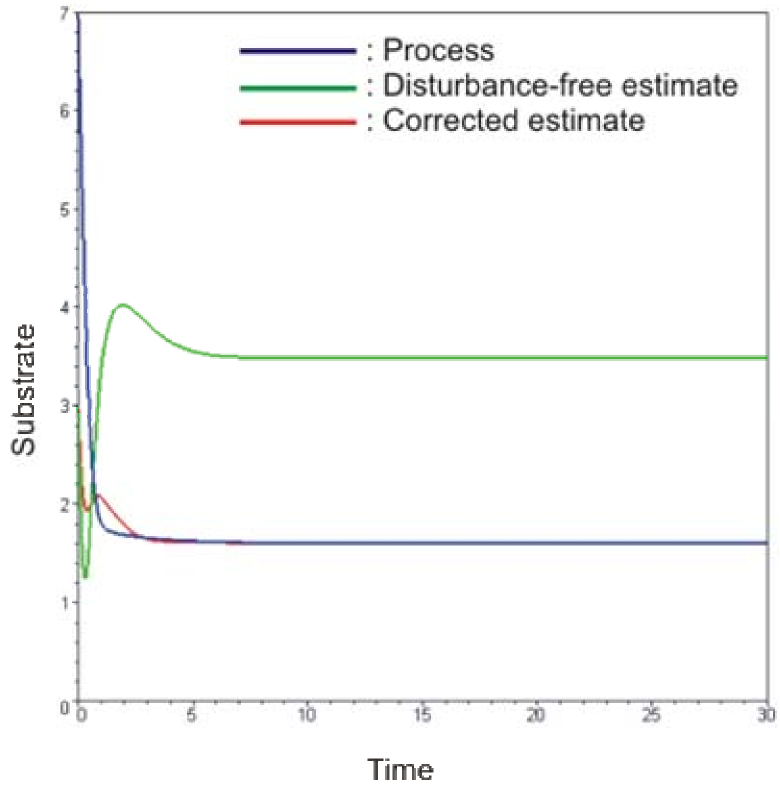


Fig. 4: Disturbance-free and corrected substrate concentration estimates, for truncation order N=5.

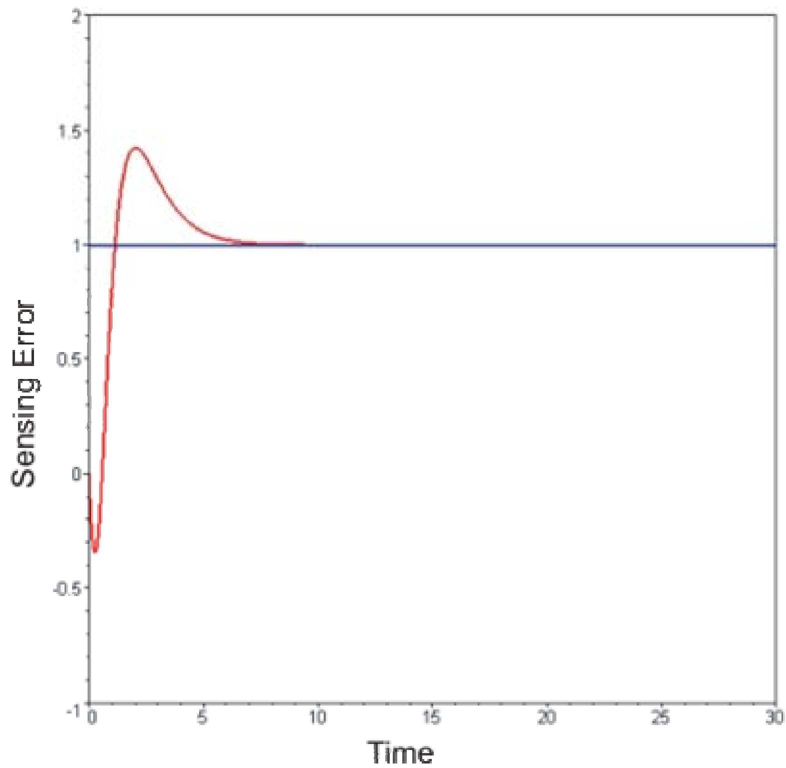


Fig. 5: Sensing error estimation, for truncation order N=5.

Effect of measurement noise

In addition to systematic error in the sensing device, noise is expected to be present in the measurement signal. In this case, the observer is driven by $y = x + w + v$, where w, as before, represents the systematic error in the measurement and v is the measurement noise. In order to test the ability of the proposed observer to perform well in the presence of noise, the foregoing numerical calculations were all repeated by adding white noise to the measurement signal. White noise was simulated using normally distributed random numbers of zero mean and given standard deviation. Figure 4 depicts simulated white noise of standard deviation 0.2.

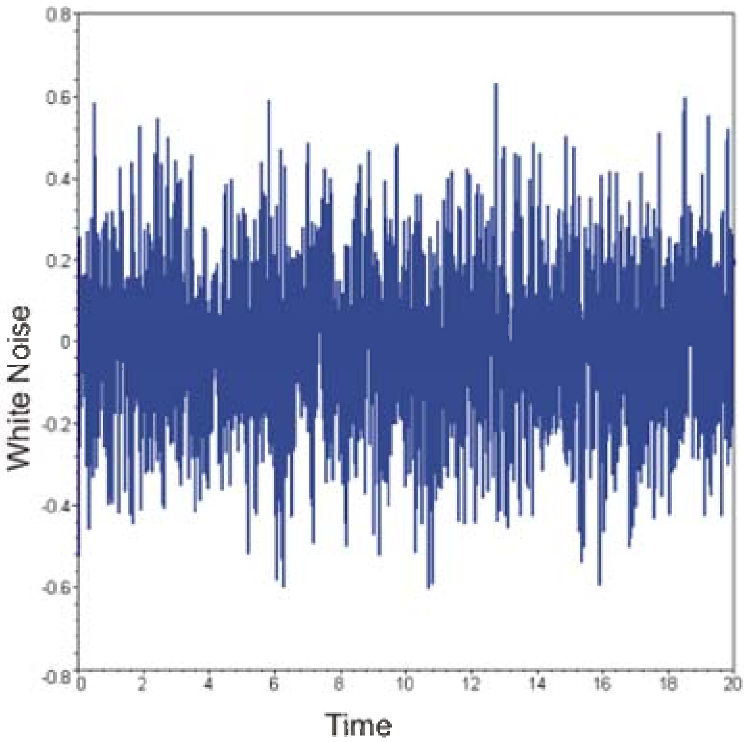


Fig. 6: Simulated white noise of standard deviation 0.2.

Figures 7-9 depict the response of the modular observer in the presence of the above white noise. It is seen that the observer performs well despite the presence of this level of noise. In general, the tuning of the observer eigenvalues must be performed in accordance with the level of noise – low level of noise allows using faster eigenvalues for faster convergence of the observer, whereas higher level of noise necessitates using slower eigenvalues.

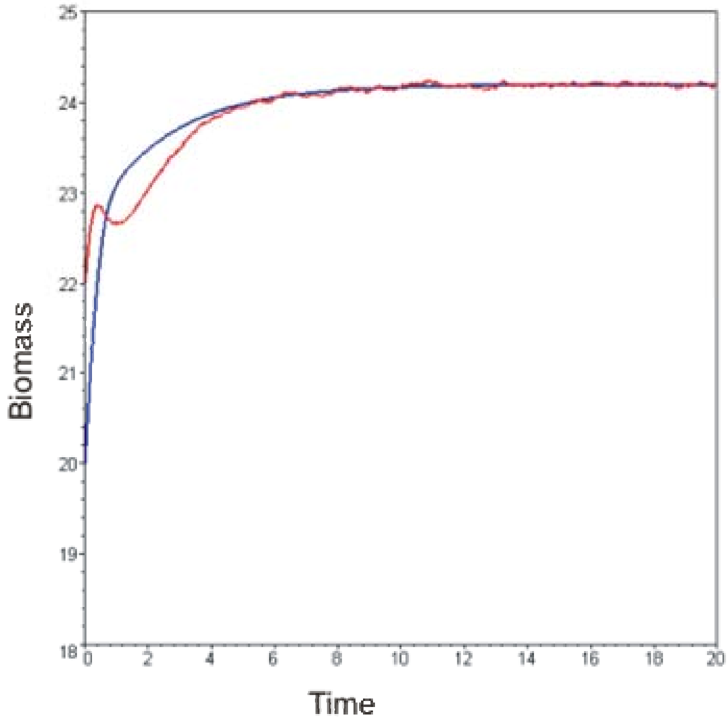


Fig. 7: True and estimated biomass concentration in the presence of white noise of standard deviation 0.2, for truncation order N=5.

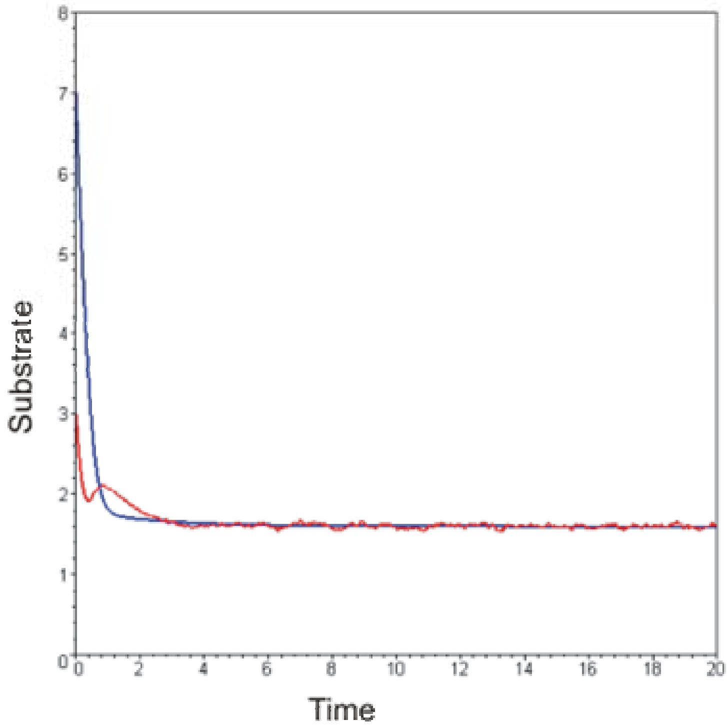


Fig. 8: True and estimated substrate concentration in the presence of white noise of standard deviation 0.2, for truncation order N=5.

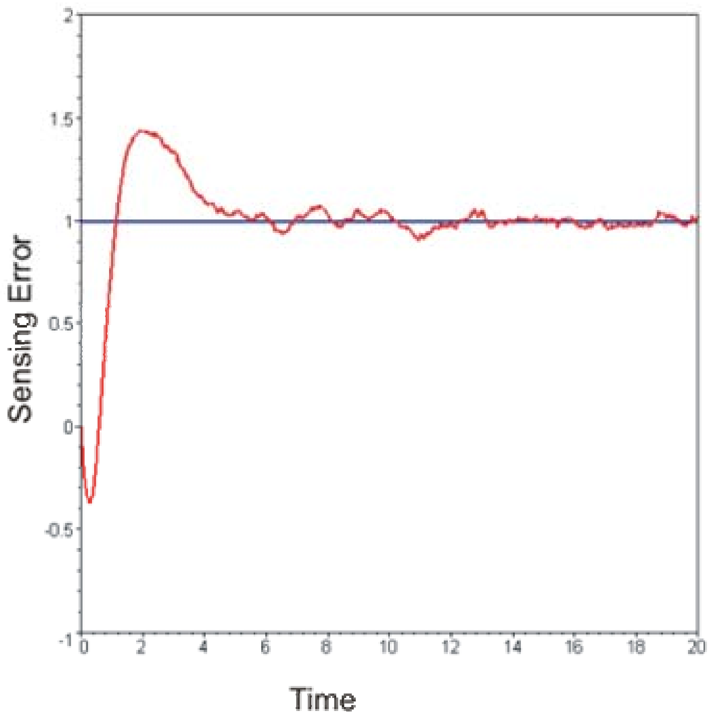


Fig. 9: True and estimated sensing error in the presence of white noise of standard deviation 0.2, for truncation order N=5.

Effect of truncation order

As mentioned earlier, all calculations (solution of PDEs, inversion and function compositions) were performed up to a certain truncation order N, and it is important to find out what magnitude of N is necessary for the accuracy of the calculations. Figures 10-12 depict the effect of truncation order on the observer response for the given set of eigenvalues (-1.0,-1.5,-3.9). It is seen that numerical convergence with respect to N is achieved for N=3 and above.

The case N=1 corresponds to a constant gain observer, with gains being equal to what a linear design would have given for the linearized system. We see from Figures 10-12 that the constant-gain observer cannot describe the system at all and gives rise to an unstable response. The quadratic approximation (N=2) gives rise to a stable response, but with a small offset present. The approximations for N=3 and higher give rise to stable observer response without offset.

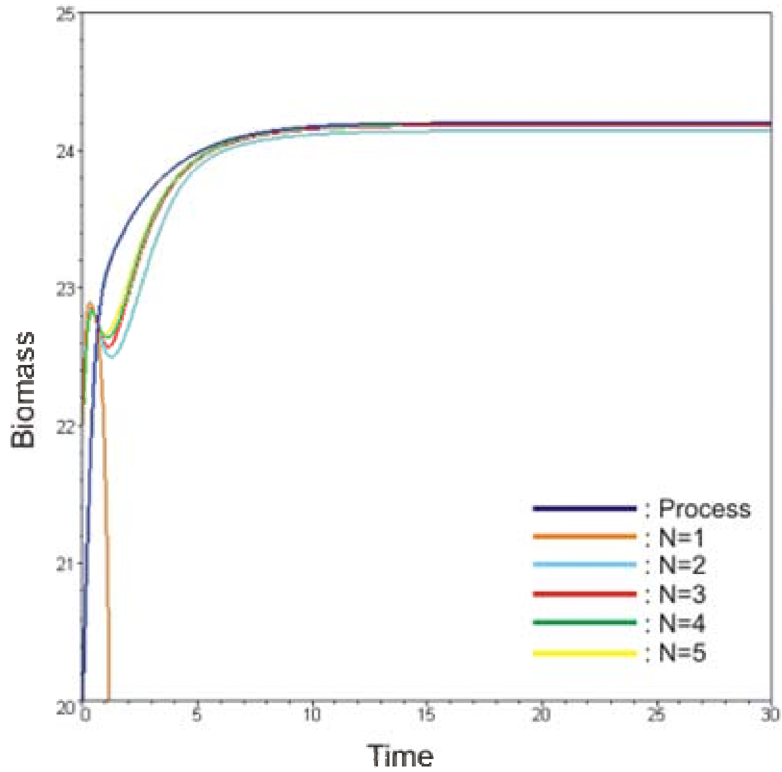


Fig. 10: True and estimated biomass concentration for different truncation orders.

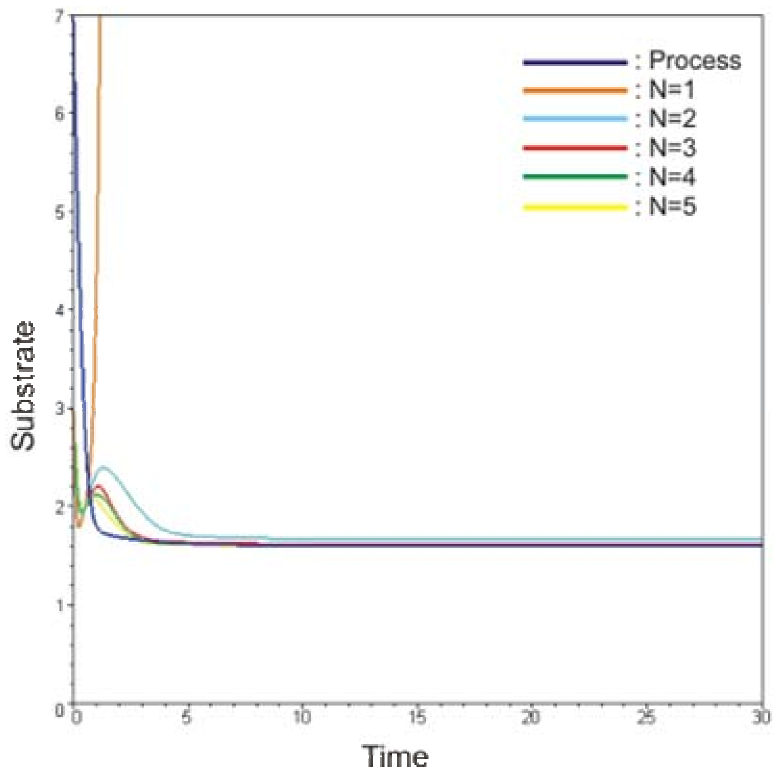


Fig. 11: True and estimated substrate concentration for different truncation orders.

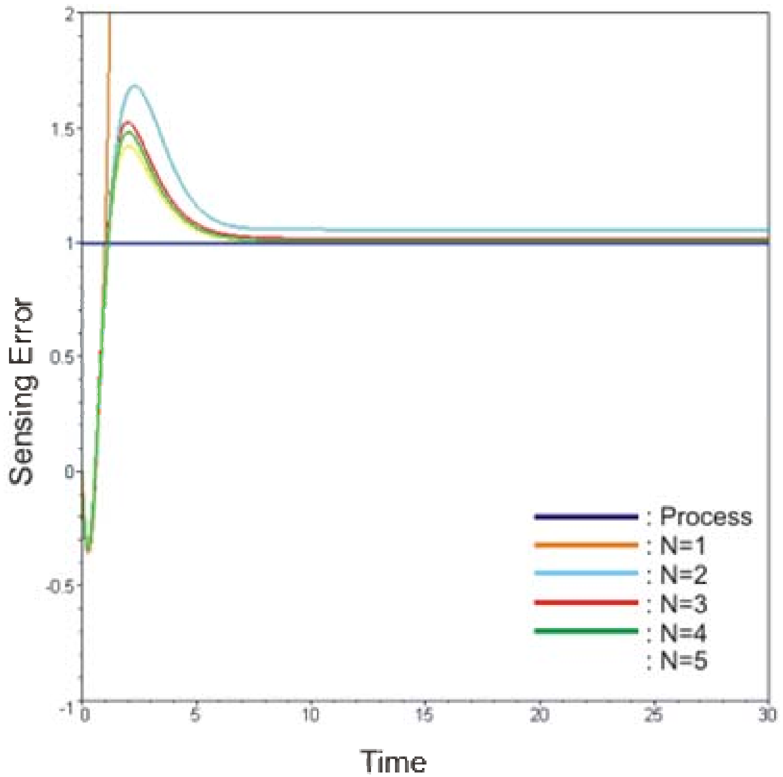


Fig. 12: True and estimated sensing error for different truncation orders.

VIII. CONCLUSIONS

This work developed a systematic method for the design of nonlinear observers for state and disturbance estimation, in the context of the observer linearization approach. The proposed design method leads to an observer with modular structure

$$\dot{\tilde{x}} = f(\tilde{x},0) + L_0(\tilde{x})\big(y - h(\tilde{x},0)\big)$$

$$\dot{\hat{w}} = s(\hat{w}) + L'(\tilde{x},\hat{w})\big(y - h\big(\Upsilon(\tilde{x},\hat{w}),\hat{w}\big)\big)$$

$$\hat{x} = \Upsilon(\tilde{x},\hat{w})$$

with the corrector $\Upsilon(x,w)$ satisfying the PDE

$$\frac{\partial \Upsilon}{\partial x}(x,w)\left[f(x,0) + L_0(x)\big(y - h(x,0)\big)\right] + \frac{\partial \Upsilon}{\partial w}(x,w)s(w)$$
$$= f\big(\Upsilon(x,w),w\big)$$

and the state-dependent gains $L_0(x)$ and $L'(x,w)$ being chosen for linearization with eigenvalue assignment.

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