Sustainability of water ecosystem under uncertainty

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Abstract

The concept of sustainability, an abstract one by its nature, has been given a mathematical representation through the use of Fisher information as a measure. The measure is used to propose sustainability hypotheses for a dynamical system which has paved the way to achieve sustainable development through externally enforced control schemes. This work is an attempt to make further investigation into such possibilities through the application of optimal control theory on a predator-prey model. Different hypotheses of sustainability have been attempted. Application of control is shown to have a positive effect on the behavior of the system indicating the usefulness of the new measure and the hypotheses.

1 Introduction

Sustainability and sustainable development, defined as the ". . . development that meets the needs of the present without compromising the ability of future generations to meet their own needs"[1], is a relatively new concept in the field of ecology. This new concept has become increasingly important and popular in recent times, attracting active research.

Although the concept has been universally recognized to be of paramount importance, one of the major concerns is the abstract nature of it. Embodied in a multidisciplinary environment, a suitable mathematical representation, or a measure, of sustainability is essential for successful communication amongst various fields. To that effect, Cabezas and Fath [2] have proposed Fisher information as a measure of the sustainability of a dynamic system and have formulated the sustainability hypothesis with particular focus on natural ecosystems. Achievement of a sustainable system in this new setting needs long term planning which will account for the complex formulation. Advanced control schemes, to date sporadically implemented in environmental systems, offer an exciting option.

This work is a step in that direction involving the application of optimal control theory to the aquatic ecosystems with the objective of sustainable development. The work considers a simple two species predator-prey model to simulate the behavior of large water bodies such as lakes. To approximate natural systems more closely, uncertainty has also been introduced into the model. Different hypotheses of the sustainability theory have been implemented and evaluated on this model to arrive at some preliminary conclusions in this field. The next section introduces Fisher information as a measure of sustainability followed by the section introducing the concept of control in ecosystem as used in this work. Section 4 explains the model and the control related aspects of it and section 5 gives the results and discussions. The last section makes concluding remarks with pointers towards the future work.

2 Sustainability and Fisher Information

Cabezas and Fath [2] have proposed to use information theory in ecology to derive a measure for the sustainability of a system, the hypothesis being based on the argument that information is a fundamental quantity of any system, irrespective of the discipline [3]. They also hypothesize that information theory can serve as the basis for the construction of basic theory of sustainability. Amongst the various representations of system information, they use Fisher information as the quantity for this hypothesis [4].

Fisher information (F.I.), introduced by Ronald Fisher [5], is a statistical measure of indeterminacy. One of the interpretation of the Fisher information relevant for this work is that it is a measure of the state of order or organization of a system or phenomenon [3]. The Fisher information, I, for one variable is given as

$$I = \int \frac{1}{p(x)} \left(\frac{dp(x)}{dx} \right)^2 dx \tag{1}$$

where p is the probability density function (pdf) of variable x. This can be extended to a system of n variables. Fisher information, being a local property, dependent on the derivative of the density function, is sensitive to the perturbations that affect the density function and therefore can be used as an indicator of the system disorder. The sustainability hypothesis therefore states that: the time-averaged Fisher information of a system in a persistent regime does not change with time. Any change in the regime will manifest itself through a corresponding change in Fisher information value.

The probability density function p gives the likelihood of finding a system in a given state at a particular time. For dynamical systems, p is a function of time t and Fisher information can then be computed using chain rule as:

$$I = \int \frac{1}{p} \left(\frac{dp}{dt} \right)^2 dt = \int \frac{1}{p} \left[\sum_i^n \left(\frac{\partial p}{\partial x_i} \frac{dx_i}{dt} \right) \right]^2 dt \tag{2}$$

where all the symbols have the previously assigned meanings. Based on this interpretation, the two additional corollaries to the sustainability hypothesis are: (1) if the Fisher information of a system is increasing with time, then the system is maintaining a state of self-organization and (2) if the Fisher information of a system is decreasing with time, then the system is losing its state of self-organization. These corollaries give an idea of the quality of the change, if the system is changing its state. This theory forms the basis of this work. The next section describes the concept of external control of an ecosystem and its application towards sustainability.

3 Ecosystem and Control

3.1 Review and future possibilities

Development of control theory has primarily been motivated with the aim of improving the performance of engineering systems such as mechanical, electrical or chemical, amongst others. But environmental and biological systems offer an equally exciting avenue for the development and implementation of some of the advanced control strategies. With better *mathematical* representations of these natural system being developed, this appears to be a manageable task.

Attempts to decipher the control philosophies of natural systems have been carried out for a long time for biological and more recently for environmental systems. For natural ecosystems, such studies are mainly directed towards identifying the dominant factors or species in the dynamic behavior of the system. Work by Hairston et al [6] is regarded as one of the earliest attempts towards systematically understanding these aspects. For aquatic ecosystems, work in this direction includes [7, 8, 9, 10]. Most of the work in this area is centered around Food Web Models as the representation of the natural ecosystem. These contributions have led to the formulation and extension of the "trophic cascade hypothesis" [7] for aquatic ecosystems.

Although often natural and hence uncontrollable, some of these factors are open to human intervention and therefore provide an opportunity to control ecosystems to our benefits. Most often the control aspect is associated with pollution prevention. Thus, there are restrictions imposed on the quantity and quality of waste that can be disposed in lakes and rivers by industries. These strategies are found to be good enough for the short term goal of preservation of ecosystems from pollutants. But the concept of sustainability demands the consideration of short term as well as long term objectives. A simplistic policy of a constant upper limit may be useless and a time dependent strategy may be needed. Moreover, there are multiple parameters which are partially or completely under human control and therefore can be manipulated for the betterment of the system. In such a case, rigorous mathematical analysis needs to replace heuristics and logic in decision making. Use of advanced control strategies, therefore, might not just be an option but rather a necessity.

Optimal control is at the forefront of the advanced control strategies. For the given objective function, optimal controller theoretically gives the best control strategy for the system. This work uses the theory of optimal control to derive control schemes for aquatic ecosystems with sustainability related objectives.

3.2 Sustainability and optimal control

Optimal control theory has been central to many a new technological developments. The theory presents three possible methodologies to derive the optimal control law, dynamic programming, calculus of variation (Euler-Lagrange equation) and Pontryagin's maximum principle. In this work, Pontryagin's maximum principle has been used for the same. As with every optimal control problem, it needs a mathematical representation of the objectives and constraints.

The sustainability considerations, as explained in the previous section, lead to two desirable situations of a system. (1) The Fisher information is constant and (2) If changing, the change in Fisher information is positive. Therefore the two options for the objective function in optimal control problem are:

- Variance minimization of the Fisher information
- Maximization of the time averaged value of Fisher information

Variance minimization of Fisher information has been previously tried using Taguchi's offline quality control method [11] on a twelve compartment food web model, where time independent decision variables are considered. This work considers both the mentioned objectives for optimal control with time dependent decision variables. The next section elaborates on this, starting with a brief description of the food chain model modeling the aquatic ecosystem and issues related to its control.

4 Model description

4.1 Predator-Prey model

The model used for the representation of the aquatic ecosystem is a two-species predator-prey model, with one prey specie and one predator specie. Derived from the more general class of Lotka-Volterra-type models, these type of models give a simple mathematical relationship to replicate the dynamics observed in natural systems. Most of the earlier literature considers either a two or three level food chain model. The model can be represented as:

$$\frac{dy_1}{dt} = g_1\Big(1 - \frac{y_1}{k}\Big)y_1 - ly_1y_2\Big(\frac{1}{1+\beta y_1}\Big) \qquad (3)$$

$$\frac{dy_2}{dt} = g_2y_1y_2\Big(\frac{1}{1+by_1}\Big) - m_2y_2 \qquad (4)$$

where y_1 and y_2 are the prey and predator populations respectively and g_1 (prey growth rate), g_2 (predator feeding rate), l (prey loss rate due to predator feeding) and m_2 (predator mortality rate) are parameters of the system. k is the prey density (prey carrying capacity) of the system and β is the predator satiation term. Please refer to [4] for the model details.

4.2 Model control aspects

Pontryagin's Maximum principle is used to derive the optimal control law for this model for the two different objectives mentioned in the last section. Prey density k has been selected as the control variable, not only because of its possible manipulation by humans amongst all the parameters, but also to account for future possible extensions of this work. The other parameters are set as $g_1 = m_2 = 1.0$, $l = g_2 = 0.01$ and $\beta = 0.005$.

Standard procedure is followed to derive the optimal control law equations [12], which are too complex to reproduce here. The system consists of a set of four ordinary differential equations (state and costate equations) and one algebraic equation (optimality condition). Starting conditions of the system (prey and population density at the start) are known and since the final states are free, costate variables at the final time are known [12]. The derivation of the optimal control law requires solution of the two point boundary value problem, which, for this highly complex differential-algebraic system of equations, is a cumbersome task. Numerical technique of the steepest ascent of Hamiltonian is therefore used to perform the boundary value problem solution [12, 13]. The final state of the system is not constrained and the objective does not contain final time function. The time horizon for the control problem is considered to be large enough so that the control law is only state dependent. The objective functions are given as:

- Maximization of Fisher information:

$$J = \text{Max } \frac{1}{T}\int_0^T \left(\frac{a(t)^2}{v(t)^4}\right) dt \qquad (5)$$

- Minimization of Fisher information variance

$$J = \text{Min } \int_0^T (FI_t - FI_{constant})^2 \, dt \qquad (6)$$

where

$$v(t) = \sqrt{\sum_{i=1}^{2}\left(\frac{dy_i}{dt}\right)^2}$$

$$a(t) = \frac{1}{v(t)}\left[\sum_{i=1}^{2}\frac{dy_i}{dt}\,\frac{d^2y_i}{dt^2}\right]$$

and FI_t is given by (1) and $FI_{constant}$ is a constant value around which the minimization of Fisher information variation is to be achieved.

5 Results and discussion

5.1 Deterministic model results

The model, in the absence of any external control, has been simulated for a fixed value of $k = 900$ (control variable) for a time horizon of 60 units (in steps of 1 time unit). The Fisher information and the standard deviation of the Fisher information for this system are reported in table 1.

The results for the controlled model are illustrated in table 1, which report the F.I. and the standard deviation of the F.I. values for the uncontrolled and the controlled systems. It can be seen that the maximization of the F.I. and minimization of its variance are achieved in the respective cases. It should also be noted that the optimization of one objective results in degradation of the other objective. An equally important factor in this analysis is the dynamic behavior of the system. The time dependent plots of the two populations for the three cases are shown in figure 1. From the figure, it can be concluded that the behavior of the two controlled systems is distinctly better than that of the uncontrolled system, because of the smaller cycle time. A smaller cycle time means a more closed system which does not deviate appreciably from a few preferred states. Moreover, the performance of the controlled system with the aim of F.I. maximization is better that with the aim of F.I. variance minimization for the same reasons.

5.2 Model results with uncertainty

Ecosystem models are very rarely accurately known. Uncertainty consideration is therefore essential to get results of practical relevance. In the current predator-prey model, the mortality rate of the predators m_2 is considered to be uncertain. In an aquatic ecosystem, predators are the carnivorous fishes (piscivores), which in turn are removed (killed) and eaten by humans which makes

Table 1: Results of optimal control for deterministic predator-prey model

Type of analysis	Fisher Information	Standard deviation of Fisher Information
Uncontrolled model	2.4807×10^{-5}	5.1846×10^{-6}
Controlled model: F.I. Maximization	4.4048×10^{-5}	8.4960×10^{-6}
Controlled model: F.I. Variance Minimization	3.0077×10^{-5}	2.1674×10^{-6}

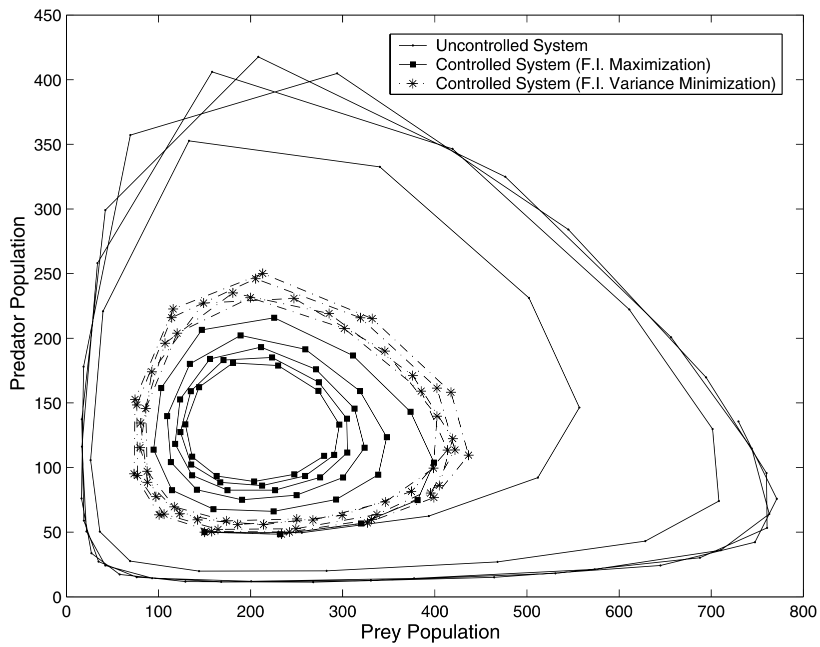

Figure 1: Predator and prey populations plots for deterministic model

their mortality rate uncertain. For this work, m_2 is assumed to have a normal distribution with mean of 1.0 and a standard deviation of 0.2. Random samples of the mortality rate are considered and the mortality rate is considered to change at every time step.

Simulation of the uncontrolled model with uncertainty was first carried out and the values of the F.I. and standard deviation of the F.I. along the path are reported in table 2. Then the uncertain system has been subjected to optimal control using both the objective functions. Trajectories for these cases are generated and the F.I. and its standard deviation value for the path are also reported in table 2. Here again, the values indicate an improved performance in both the controlled cases. Contrary to the deterministic case, the optimization of one performance objective does not necessarily degrade the other objective. The population plots for the uncertain case are shown in figure 2. The plots indicate that the system performance has indeed improved due to the application of external control (for reasons previously mentioned). Although the performance with the aim of F.I. maximization is slightly better than that for F.I. variance minimization, the difference is not as pronounced as for the deterministic system.

These observations will be used to make some concluding remarks in the next section.

Table 2: Results of optimal control for predator-prey model with uncertainty

Type of analysis	Fisher Information	Standard deviation of Fisher Information
Uncontrolled model	3.4609×10^{-5}	1.8412×10^{-5}
Controlled model: F.I. Maximization	6.8837×10^{-5}	2.0934×10^{-5}
Controlled model: F.I. Variance Minimization	5.8257×10^{-5}	1.2764×10^{-5}

5.3 Computational considerations

Since the system of equations being solved is quite complex, computational problems need to be carefully avoided. Since the absolute value of the Fisher information is quite low, the objective function is linearly scaled using a constant to avoid numerical errors during the solution of the equations. Experience shows that this greatly reduces the convergence time. The termination constant and the step size for the steepest ascent method needs to be carefully chosen for converging results. Most often, it is a compromise between faster convergence and running the risk of making the solution divergent. It was also observed that a good initial guess is important to have convergence.

6 Conclusion and future work

The application of optimal control theory on an aquatic ecosystem has been considered here, with the aim of making the aquatic system sustainable. To that effect, Fisher information, a measure recently proposed to quantify sustainability, is used. The sustainability hypothesis states that the time averaged Fisher information of a sustainable system should remain constant and if changing, the value should increase. Considering both these objectives, control laws were derived for a simple two-species predator-prey model. To approximate the natural system, uncertainty was introduced in the model through changing mortality rate of the predator.

The results for both, deterministic and uncertain, cases show that the performance of the aquatic system has improved after the application of the control. Not only have the F.I. and its variance values improved, but these improvements have been translated into better time dependent variations of the populations of the two species. This indicates a good correlation between the mathematical theory of sustainability and the actual behavior of the natural system. It has also been observed for the considered system that the objective of F.I. maximization gives better system dynamics than that with the aim of F.I. variance minimization.

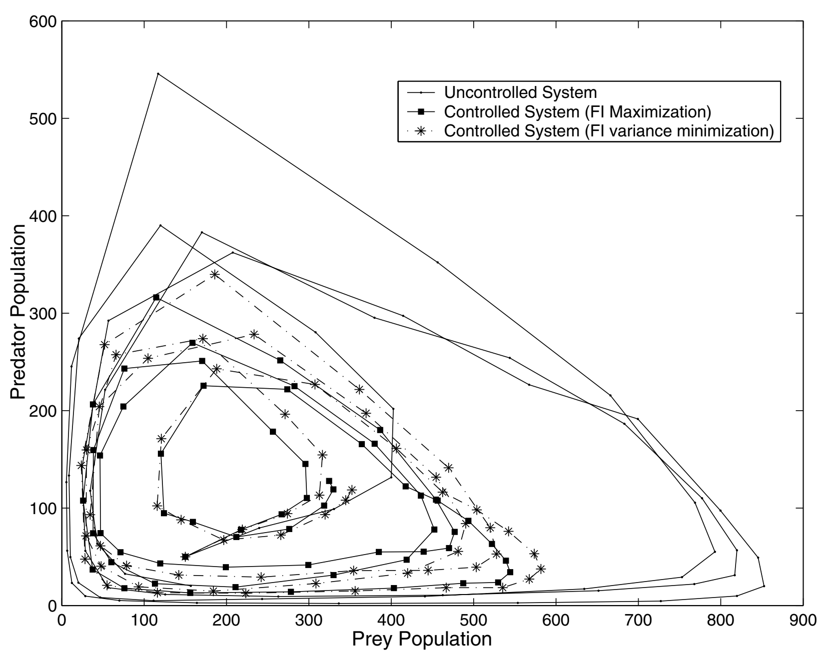

Figure 2: Predator and prey populations plots for model with uncertainty

This work forms the basis to tackle the problem of environmental pollution with an eye on sustainability. Pollutants discharged in water bodies are known to affect the ecosystem dynamics through changes in productivity, mortality, vegetation etc. The aim is to help decision making to preserve the aquatic ecosystems from the harmful effects of pollutants and contaminants. It is expected that sustainability considerations will lead to decision which are beneficial in the long run. But the task is expected to involve more complex system models (3-4 species) along with a combination of time independent and dependent uncertainties and a more specific understanding of control options. This work is expected to provide valuable groundwork to achieve the long term objectives.

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