ADCONIP 2014 Hiroshima

On the Decentralized H-infinity PID Controller Synthesis in Finite Frequency Domain

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Abstract—This work extends the current results of SISO (single-input-single-output) case H-infinity PID Controller synthesis in finite frequency domain to the subject of multivariable decentralized control. Sufficient conditions for the existence of such a decentralized PID controller are derived in terms of linear matrix inequalities (LMIs). A numerical example is given that establishes the efficacy of the proposed design.

Keywords—*H-infinity, decentralized control, GKYP, PID*

I. Introduction

In the past few decades we have witnessed the popularity of PID (proportional-integral-derivative) control [1] in the industries and the progress in its design methodologies, from SISO case to MIMO (multi-input-multi-output) case [2], and from centralized to decentralized [3,4] control structure. Despite its simplicity, the design of the PID gains is not an easy task, and it is well known that it is a non-convex problem. Recently, a method named open-loop shaping using the so called generalized Kalman-Yakubovich-Popov (GKYP) lemma [5] was proposed and applied to the design of SISO, PID controllers [6] where the objective is to minimize the H-infinity gain over a (semi)finite frequency range. In this paper we extend the results of [6] to the subject of multivabiable decentralized PID control based partially on our recent research results in [7] (where only the low frequency case has been addressed). The rest of the work is organized as follows: Section II gives the problem statement and preliminaries. Section III presents the main results. A numerical example is given in Section IV for illustration purpose. Section V is the Conclusion. A sketch of the proof of Theorem 2 is provided in the Appendix.

II. Problem Statement and Preliminaries

Notation: Let $\mathbb{R}$ be the set of real numbers, and $\mathbb{R}^{p\times m}$ denotes the set of all real $p\times m$ matrices. For a matrix G and G^T denote its transpose. The Hermitian part of a square matrix G is denoted by $He\{G\}:=G+G^*$. RH_∞ is the set of real-rational proper transfer functions with poles in the open left half complex plane. Let Ω be a closed interval in $\mathbb{R}$ and X be a complex-valued function of a single complex variable, $X^\sim(s):=X^T(-s)$; $\|X\|_\Omega := \sup_{\omega\in\Omega}\bar{\sigma}(X(j\omega))$, where $\bar{\sigma}(\bullet)$ denotes the largest singular value of the argument. A transfer function X is called inner if $X\in RH_\infty$ and $X^\sim X = I$; $X^\perp$ is called a complementary inner factor (CIF) of X if $[X \quad X^\perp]$ is square and is inner. A square function $X\in RH_\infty$ is called strictly positive real (SPR) if $He\{X(j\omega)\}>0$ for all $\omega\in\mathbb{R}\cup\{\infty\}$. Symbol $*$ in a matrix inequality is readily inferred by symmetry.

Consider an L-channel linear time-invariant system P described by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + \sum_{i=1}^{L} B_{2,i}u_i \\ z &= C_1 x + D_{11}w + \sum_{i=1}^{L} D_{12,i}u_i \\ y_i &= C_{2,i}x \quad (i=1,2,\ldots,L) \end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state, $w(t)\in\mathbb{R}^{m_1}$ is the exogenous input, $z(t)\in\mathbb{R}^{p_1}$ is the observed output, and $u_i(t)\in\mathbb{R}^{m_{2i}}$ and $y_i(t)\in\mathbb{R}^{p_{2i}}$ represent the control input and measurement output of channel i (i=1,2,…,L), respectively. The matrices A, B_1, $B_{2,i}$, C_1, C_2, D_{11} and $D_{12,i}$ are constant and of appropriate dimensions. To expedite calculations involving transfer functions, we shall use the following notation:

$$P(s)=\begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s)\end{bmatrix} := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right] \tag{2}$$

where

$$B_2=\left[B_{2,1}\cdots B_{2,L}\right], C_2=\left[C_{2,1}^T\cdots C_{2,L}^T\right]^T, D_{12}=\left[D_{12,1}\cdots D_{12,L}\right]. \tag{3}$$

Let $m_2=\sum_{i=1}^L m_{2i}$ and $p_2=\sum_{i=1}^L p_{2i}$. The objective of this work is formally stated as follows.

Problem 1: Let $\gamma>0$ and an interval $\Omega=[\varpi_1,\varpi_2]$ $(0\le\varpi_1<\varpi_2)$ be given. Consider plant (1), find a decentralized PID controller K described by:

$$u(t)=K_p y(t)+K_I\int_0^t y(\tau)d\tau + K_D(d\,y(t)/dt) \tag{4}$$

where $K_\chi = diag(K_\chi^1,\cdots,K_\chi^L), \chi=P,I,D$ with $K_P^i,K_I^i,K_D^i\in\mathbb{R}^{m_{2i}\times p_{2i}}$ (i=1,2,…,L), that internally stabilizes the plant (1) and ensures $\|T_{zw}\|_\Omega<\gamma$, where $T_{zw}=P_{11}+P_{12}(I-KP_{22})^{-1}KP_{21}$. Throughout this paper we assume that the plant P satisfies the following assumptions:

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The authors acknowledge the financial support of the National Science Council of Taiwan under the grant NSC 101-2221-E-032-018.

(i) There is no unstable fixed mode with respect to the triplet (C_2, A, B_2) (see e.g., [9]).

(ii) $P_{12}^{\sim}P_{12}(j\omega) > 0 \quad \forall\omega \in \mathbb{R} \cup \{\infty\}$.

(iii) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

(iv) A has no $j\omega$ axis eigenvalues.

Assumption (i) is standard for decentralized stabilization. Assumptions (ii) and (iii) are required for the construction of an inner function that is useful for deriving conditions for guaranteeing a prescribed performance bound (see [7]). Assumption (ii) is not restrictive because the control input is usually taken to be a part of the regulated signals in many applications (See e.g., Example 1 in Section IV). In such a case Assumption (ii) always holds true. Assumption (iv) is used for ensuring the value (e.g., R_1, R_2 and $\bar{P}_{12}$) involved in (14) not go to infinity.

The following lemmas [7] are given, which are useful for deriving a state-space solution.

Lemma 1 [5, 7]: Given a positive value γ, let H be a transfer function which has a real-valued state-space realization (A,B,C,D) and A has no eigenvalues on the $j\omega$ axis. Then under the assumption $D^TD - \gamma^2 I < 0$, the following statements are equivalent.

(i) $\bar{\sigma}(H(j\omega)) < \gamma \ \forall\omega \in \Omega$.

(ii) There exist real symmetric matrices P and Q, satisfying Q>0 and

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \Xi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0 \tag{5}$$

where $\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, and Ξ is given in Table 1.

(iii) There exist real symmetric matrices P, Q and real matrices G, W, satisfying Q>0 and

$$\begin{bmatrix} He\{G\} + \Delta_{11} & -P - W^T + GA + \Delta_{12} & -GB & 0 \\ * & -He\{WA\} + \Delta_{22} & WB & C^T \\ * & * & -\gamma I & -D^T \\ * & * & * & -\gamma I \end{bmatrix} < 0 \tag{6}$$

where $\Delta_{11}, \Delta_{12}, \Delta_{22}$ are specified in Table 1.

Lemma 2 [5, 7]: Let H be a transfer function which has a real-valued state-space realization (A,B,C,D) and A has no eigenvalues on the $j\omega$ axis. Then under the assumption $He\{D\}$>0, the following statements are equivalent.

(i) $He\{H(j\omega)\} > 0 \ \forall\omega \in \Omega$.

(ii) There exist real symmetric matrices P and Q, satisfying Q>0 and

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \Xi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0 \tag{7}$$

where $\Pi = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$, and Ξ is given in Table 1.

(iii) There exist real symmetric matrices P, Q and real matrices G, W, satisfying Q>0 and

$$\begin{bmatrix} He\{G\} + \Delta_{11} & -P - W^T + GA + \Delta_{12} & -GB \\ * & -He\{WA\} + \Delta_{22} & C^T + WB \\ * & * & -He\{D\} \end{bmatrix} < 0 \tag{8}$$

where $\Delta_{11}, \Delta_{12}, \Delta_{22}$ are specified in Table 1.

Lemma 3 [7, 10]: Let H be a transfer function with all poles in the open left half complex plane and has a real-valued, stabilizable and detectable state-space realization (A,B,C,D). Then the following statements are equivalent.

(i) H is SPR.

(ii) There exist real symmetric matrix P satisfying P>0 and

$$\begin{bmatrix} A^TP + PA & PB - C^T \\ * & -D - D^T \end{bmatrix} < 0 \tag{9}$$

(iii) There exist real symmetric matrix P and real matrices G, W satisfying P>0 and

$$\begin{bmatrix} G^T + G & -P - W^T + GA & -GB \\ * & -He\{WA\} & C^T + WB \\ * & * & -He\{D\} \end{bmatrix} < 0 \tag{10}$$

III. MAIN RESULTS

In this section, we present solvability conditions of the finite frequency decentralized PID control problem as stated in Section II.

A. Frequency Domain Solvability Conditions

A popular approach to the underlying problem relies on transforming it into a static output feedback (SOF) problem , and solves by SOF techniques in the sequel [2]. However, in this way the integrator of the PID controller should be absorbed into the plant P, leading to an open-loop pole of the resultant function located at the origin. This prevents direct application of the method of [7] to the problem. To circumvent this difficulty, we seek to solve the problem in another way. In view of (4), the PID controller can be alternatively expressed as follows:

$$K(s) = K_P + K_I s^{-1} + K_D s = [K_P \quad K_I s^{-1} \quad K_D] \times [I \quad I \quad sI]^T \tag{11}$$

define $\bar{y}(t) = [y(t) \quad y(t) \quad d\,y(t)/dt]^T$.

Table 1. Parameters for Lemmas 1, 2. $\varpi_c := (\varpi_1 + \varpi_2)/2$.

	Low frequency range	Middle frequency range	High frequency range
Ω	$\{\omega : \lvert\omega\rvert \le \varpi_l\}$	$\{\omega : \varpi_1 \le \omega \le \varpi_2\}$	$\{\omega : \lvert\omega\rvert \ge \varpi_h\}$
Ξ	$\begin{bmatrix} -Q & P \\ P & \varpi_l^2 Q \end{bmatrix}$	$\begin{bmatrix} -Q & P + j\varpi_c Q \\ P - j\varpi_c Q & -\varpi_1\varpi_2 Q \end{bmatrix}$	$\begin{bmatrix} Q & P \\ P & -\varpi_h^2 Q \end{bmatrix}$
$\Delta_{11}, \Delta_{12}, \Delta_{22}$	$\Delta_{11} = -Q, \Delta_{12} = 0, \Delta_{22} = \varpi_l^2 Q$	$\Delta_{11} = -Q, \Delta_{12} = -j\varpi_c Q, \Delta_{22} = -\varpi_1\varpi_2 Q$	$\Delta_{11} = Q, \Delta_{12} = 0, \Delta_{22} = -\varpi_h^2 Q$

Absorbing the term $[I \quad I \quad sI]^T$ into the plant (1) yields the new generalized plant $\bar{P}$

$$\bar{P}(s)=\begin{bmatrix}\bar{P}_{11}(s) & \bar{P}_{12}(s)\\ \bar{P}_{21}(s) & \bar{P}_{22}(s)\end{bmatrix}=\left[\begin{array}{c|cc} A & B_1 & B_2\\ \hline C_1 & D_{11} & D_{12}\\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22}\end{array}\right] \tag{12}$$

where

$$\bar{C}_2=\begin{bmatrix}C_2\\ C_2\\ C_2A\end{bmatrix}, \bar{D}_{21}=\begin{bmatrix}0\\ 0\\ C_2B_1\end{bmatrix}, \bar{D}_{22}=\begin{bmatrix}0\\ 0\\ C_2B_2\end{bmatrix} \tag{13}$$

On the other hand, denote $[K_P \quad K_I s^{-1} \quad K_D]$ by $\bar{K}(s)$, which can now be considered as a controller that is in feedback connection with the plant $\bar{P}$. Hence Problem 1 is equivalently transformed into the problem of designing $\bar{K}$.

Under Assumptions (i)-(iii), there exists a right coprime factorization $[\bar{P}_{12}^T \quad \bar{P}_{22}^T]^T=[N_{12}^T \quad N_{22}^T]^T M_{22}^{-1}$ for $[\bar{P}_{12}^T \quad \bar{P}_{22}^T]^T$ with N_{12} being inner [7]. Notice that implicit in Assumptions (ii) is the restriction that $p_1 \ge m_2$. For the case $p_1 > m_2$, since N_{12} is inner, there exists a CIF of N_{12} such that $\Psi:=[N_{12} \quad N_{12}^{\perp}]$ is square and is inner [8]. In this case, we define the notation $[R_1^T \quad R_2^T]^T=\Psi^{\sim}\bar{P}_{11}$, where R_1 and R_2 are $m_2\times m_1$ and $(p_1-m_2)\times m_1$ real-rational proper transfer functions, respectively. As for the case $p_1=m_2$, please see Remark 1. Assume that $\bar{K}$ has the following coprime fractional representation $\bar{K}=\tilde{M}_K^{-1}\tilde{N}_K$ where $\tilde{M}_K,\tilde{N}_K\in RH_\infty$, we may apply Theorem 1 of [7] to yield a solvability condition of Problem 1 as stated as follows .

Theorem 1: Assume $p_1 > m_2$. With notations of N_{22}, M_{22}, R_1, and R_2 defined above for the case, let γ be given positive value and Ω be an interval in $\mathbb{R}$. Suppose that there exist a positive value α, and real-rational proper transfer functions $V,\tilde{N}_K,\tilde{M}_K$, satisfying $\tilde{M}_K^{-1}\tilde{N}_K=[K_P \quad K_I s^{-1} \quad K_D]$ and the following conditions:

(i) $\bar{\sigma}\left(\begin{bmatrix}S_{cl}R_1+\tilde{N}_K\bar{P}_{21}\\ VR_2\end{bmatrix}(j\omega)\right)<\alpha\gamma \quad \forall\omega\in\Omega$. (14)

(ii) $He\{S_{cl}(j\omega)-\alpha I\}>0 \quad \forall\omega\in\Omega$. (15)

(iii) $He\{V(j\omega)-\alpha I\}>0 \quad \forall\omega\in\Omega$. (16)

(iv) S_{cl} is SPR. (17)

where $S_{cl}=\tilde{M}_K M_{22}-\tilde{N}_K N_{22}$ .Then the decentralized PID controller is determined by $K=\tilde{M}_K^{-1}\tilde{N}_K[I_{p_2} \quad I_{p_2} \quad sI_{p_2}]^T$, which internally stabilizes plant (1) and ensures $\|T_{zw}\|_\Omega<\gamma$.

Note that condition (iv) is a well-known sufficient condition that ensures well-posedness and closed-loop stability [8].

B. State-Space Solutions

For the purpose of efficiently computing the PID gains, we'd like to convert the frequency-domain conditions of Theorem 1 into a state-space form. The main difficulties come from two technical points: First, the mismatch in order between the generalized plants and the functions to be determined; Second, the decentralized structural constraint of the PID controllers. Specifically, for the order mismatch problem it can be verified that each generalized plant involved in conditions (i), (ii) and (iv) of Theorem 1 has order *3n*, *n*, and *n*, respectively. However, $\tilde{N}_K,\tilde{M}_K$ has order only m_2 (which is usually less than the order of the plant). This is a new challenge to be dealt with.

Instead of employing any existing reduced-order design methodology, we seek to cope with this problem via introducing some functions into the design procedure. First, for condition (i) of Theorem 1 we may take advantage of the function V by assuming it to be of order $3n-m_2$. Hence we see the advantages brought by the free function V, which not only is useful in reducing performance bound, but also in coping with the order problem. As for conditions (ii) and (iv) we introduce an additional function U to cope with the problem, i.e., the conditions (ii) and (iv) are replaced with the new conditions (ii') and (iv') respectively as follows:

(ii') $He\begin{Bmatrix}S_{cl}(j\omega)-\alpha I & 0\\ 0 & U(j\omega)\end{Bmatrix}>0 \quad \forall\omega\in\Omega$ (18)

(iv') $\begin{pmatrix}S_{cl} & 0\\ 0 & U\end{pmatrix}$ is SPR. (19)

where U is of order $n-m_2$. It is noteworthy that the new conditions (ii') and (iv') are equivalent to the original ones (i.e., Theorem 1 conditions (ii) and (iv)).

Next, we proceed by recasting the whole set of modified solvability conditions (14), (16), (18) and (19) in terms of linear fractional transformation [8]:

Condition (15): $\Pi^{(a)}:=\begin{bmatrix}S_{cl}R_1+\tilde{N}_K\bar{P}_{21}\\ VR_2\end{bmatrix}=F_l(P^{(a)},K^{(a)})$ where

$$P^{(a)}=\left[\begin{array}{c|cc} 0 & I & 0\\ 0 & 0 & I\\ \hline M_{22}R_1 & 0 & 0\\ \bar{P}_{21}-N_{22}R_1 & 0 & 0\\ R_2 & 0 & 0\end{array}\right]=\left[\begin{array}{c|c|c} A^{(a)} & B_1^{(a)} & 0\\ \hline 0 & 0 & D_{12}^{(a)}\\ \hline C_2^{(a)} & D_{21}^{(a)} & 0\end{array}\right], \tag{20}$$

$$K^{(a)}=\begin{bmatrix}\begin{bmatrix}\tilde{M}_K & \tilde{N}_K\end{bmatrix} & 0\\ 0 & V\end{bmatrix}=\left[\begin{array}{c|c} A_K^{(a)} & B_K^{(a)}\\ \hline C_K^{(a)} & D_K^{(a)}\end{array}\right].$$

Conditions (18) and (19): $\Pi^{(b)}:=S_{cl}-\alpha I=F_l(P^{(b)},K^{(b)})$ and $\Pi^{(d)}:=S_{cl}=F_l(P^{(d)},K^{(d)})$, where for $i=b,d$,

$$P^{(i)}=\left[\begin{array}{cc|cc} \eta^{(i)} & 0 & I & 0\\ 0 & 0 & 0 & I\\ \hline \begin{bmatrix}M_{22}\\ -N_{22}\end{bmatrix} & 0 & 0 & 0\\ 0 & I & 0 & 0\end{array}\right]=\left[\begin{array}{c|c|c} A^{(i)} & B_1^{(i)} & 0\\ \hline 0 & D_{11}^{(i)} & D_{12}^{(i)}\\ \hline C_2^{(i)} & D_{21}^{(i)} & 0\end{array}\right], \tag{21}$$

$$K^{(i)}=\begin{bmatrix}\begin{bmatrix}\tilde{M}_K & \tilde{N}_K\end{bmatrix} & 0\\ 0 & U\end{bmatrix}$$

with $\eta^{(b)}=-\alpha I, \eta^{(d)}=0$

Condition (16): $\Pi^{(c)} := V - \alpha I = F_l(P^{(c)}, K^{(c)})$ where

$$P^{(c)} = \left[\begin{array}{c:c} -\alpha I & I \\ \hdashline I & 0 \end{array}\right] = \left[\begin{array}{c:c:c} A^{(c)} & B_1^{(c)} & 0 \\ \hline 0 & D_{11}^{(c)} & D_{12}^{(c)} \\ \hdashline C_2^{(c)} & D_{21}^{(c)} & 0 \end{array}\right], K^{(c)} = V. \quad (22)$$

where $B_1^{(c)} = 0, C_2^{(c)} = 0,$ and $A^{(c)}$ can be set to be any square matrix of dimension $(3n-p_2)\times(3n-p_2)$; $T^{(a)}$, $T^{(b)}$, $T^{(c)}$ and $T^{(d)}$ are nonsingular matrices which play the role of coordinate transformation. With these, the conditions (iii) of Lemmas 1, 2, and 3 are invoked to convert the aforementioned frequency-domain solvability conditions into LMI-based conditions. For ease of exposition, define notation $S = \{diag(S_1,\cdots,S_L) : S_i \in \mathbb{R}^{m_{2i}\times m_{2i}}, i = 1,\cdots,L\}$.

Theorem 2: Assume $p_1 > m_2$ and $n > m_2$. Let $\gamma > 0$, and ϖ_1, ϖ_2 be given values with $0 \le \varpi_1 < \varpi_2$. There is a solution to Problem 1 with $\Omega = [\varpi_1, \varpi_2]$ and $\varpi_c := (\varpi_1 + \varpi_2)/2$ if there exist $\alpha \in \mathbb{R}$, real symmetric matrices $\bar{P}^{(j)} = \begin{bmatrix} \bar{P}_{11}^{(j)} & \bar{P}_{21}^{(j)} \\ \bar{P}_{12}^{(j)T} & \bar{P}_{22}^{(j)} \end{bmatrix}$, $\bar{Q}^{(j)} = \begin{bmatrix} \bar{Q}_{11}^{(j)} & \bar{Q}_{12}^{(j)} \\ \bar{Q}_{12}^{(j)T} & \bar{Q}_{22}^{(j)} \end{bmatrix} > 0\ (j = a,b,c), \bar{P}^{(d)} = \begin{bmatrix} \bar{P}_{11}^{(d)} & \bar{P}_{12}^{(d)} \\ \bar{P}_{12}^{(d)T} & \bar{P}_{22}^{(d)} \end{bmatrix} > 0, \bar{Q}_{11}^{(d)} = 0,$ $\bar{Q}_{12}^{(d)} = 0, \bar{Q}_{22}^{(d)} = 0,$ and real matrices $Z_i^{(MN)}, Z_i^{(V)}, Z_i^{(U)}$ $(i = 1,2,3,4); \Sigma_2^{(MN)}, \Sigma_2^{(V)}, \Sigma_2^{(U)}, R^{(j)}, \bar{R}^{(j)}, \Sigma_1^{(j)}, \bar{\Sigma}_1^{(j)} (j = a,b,c,d)$ with the constraints $Z_1^{(MN)} = Z_3^{(MN)} \in S, Z_2^{(MN)} = [I_{m_2} \;\vdots\; \hat{K}_P \;\; \hat{K}_I \;\; \hat{K}_D],$ $Z_4^{(MN)} = [I_{m_2} \;\vdots\; \hat{K}_P \;\; 0 \;\; \hat{K}_D], \Sigma_2^{(MN)} \in S,$ where $\hat{K}_\chi = diag(\hat{K}_\chi^1, \cdots, \hat{K}_\chi^L),$ $\chi = P,I,D$ with $\hat{K}_P^i, \hat{K}_I^i, \hat{K}_D^i \in \mathbb{R}^{m_{2i}\times p_{2i}}$ (i=1,2,…,L), satisfying the following linear matrix inequalities:

$$\begin{bmatrix} \Theta_{11}^{(a)} & \Theta_{12}^{(a)} & \Theta_{13}^{(a)} & \Theta_{14}^{(a)} & -\Sigma_1^{(a)}B_1^{(a)} & 0 \\ * & \Theta_{22}^{(a)} & \Theta_{23}^{(a)} & \Theta_{24}^{(a)} & \Gamma_{25} & 0 \\ * & * & \Theta_{33}^{(a)} & \Theta_{34}^{(a)} & \bar{\Sigma}_1^{(a)}B_1^{(a)} & \Gamma_{36} \\ * & * & * & \Theta_{44}^{(a)} & \Gamma_{45} & C_2^{(a)T}Z_4^{(a)T}D_{12}^{(a)T} \\ * & * & * & * & -\alpha\gamma I & -D_{21}^{(a)T}Z_4^{(a)T}D_{12}^{(a)T} \\ * & * & * & * & * & -\alpha\gamma I \end{bmatrix} < 0 \quad (23)$$

$$\begin{bmatrix} \Theta_{11}^{(b)} & \Theta_{12}^{(b)} & \Theta_{13}^{(b)} & \Theta_{14}^{(b)} & -\Sigma_1^{(b)}B_1^{(b)} \\ * & \Theta_{22}^{(b)} & \Theta_{23}^{(b)} & \Theta_{24}^{(b)} & -R^{(b)}B_1^{(b)} - Z_2^{(b)}D_{21}^{(b)} \\ * & * & \Theta_{33}^{(b)} & \Theta_{34}^{(b)} & C_2^{(b)T}Z_4^{(b)T}D_{12}^{(b)T} + \Phi_{35}^{(b)} \\ * & * & * & \Theta_{44}^{(b)} & C_2^{(b)T}Z_4^{(b)T}D_{12}^{(b)T} + \Phi_{45}^{(b)} \\ * & * & * & * & -He\left\{D_{11}^{(b)} + D_{12}^{(b)}Z_4^{(b)}D_{21}^{(b)}\right\} \end{bmatrix} < 0 \quad (24)$$

$$\begin{bmatrix} \Theta_{11}^{(c)} & \Theta_{12}^{(c)} & \Theta_{13}^{(c)} & \Theta_{14}^{(c)} & -\Sigma_1^{(c)}B_1^{(c)} \\ * & \Theta_{22}^{(c)} & \Theta_{23}^{(c)} & \Theta_{24}^{(c)} & -R^{(c)}B_1^{(c)} - Z_2^{(c)}D_{21}^{(c)} \\ * & * & \Theta_{33}^{(c)} & \Theta_{34}^{(c)} & C_2^{(c)T}Z_4^{(c)T}D_{12}^{(c)T} + \Phi_{35}^{(c)} \\ * & * & * & \Theta_{44}^{(b)} & C_2^{(c)T}Z_4^{(c)T}D_{12}^{(c)T} + \Phi_{45}^{(c)} \\ * & * & * & * & -He\left\{D_{11}^{(c)} + D_{12}^{(c)}Z_4^{(c)}D_{21}^{(c)}\right\} \end{bmatrix} < 0 \quad (25)$$

and

$$\begin{bmatrix} \Theta_{11}^{(d)} & \Theta_{12}^{(d)} & \Theta_{13}^{(d)} & \Theta_{14}^{(d)} & -\Sigma_1^{(d)}B_1^{(d)} \\ * & \Theta_{22}^{(d)} & \Theta_{23}^{(d)} & \Theta_{24}^{(d)} & -R^{(d)}B_1^{(d)} - Z_2^{(d)}D_{21}^{(d)} \\ * & * & \Theta_{33}^{(d)} & \Theta_{34}^{(d)} & C_2^{(d)T}Z_4^{(d)T}D_{12}^{(d)T} + \Phi_{35}^{(d)} \\ * & * & * & \Theta_{44}^{(d)} & C_2^{(d)T}Z_4^{(d)T}D_{12}^{(d)T} + \Phi_{45}^{(d)} \\ * & * & * & * & -He\left\{D_{12}^{(d)}Z_4^{(d)}D_{21}^{(d)}\right\} \end{bmatrix} < 0 \quad (26)$$

where

$\Gamma_{25} = -R^{(a)}B_1^{(a)} - Z_2^{(a)}D_{21}^{(a)}, \Gamma_{36} = C_2^{(a)T}Z_4^{(a)T}D_{12}^{(a)T} + Z_3^{(a)T}D_{12}^{(a)T},$

$\Gamma_{45} = \bar{R}^{(a)}B_1^{(a)} + Z_2^{(a)}D_{21}^{(a)},$

and for $j = a,b,c,d$

$\Theta_{11}^{(j)} = -\bar{Q}_{11}^{(j)} + He\left\{\Sigma_1^{(j)}\right\}, \Theta_{12}^{(j)} = -\bar{Q}_{12}^{(j)} + R^{(j)T} + \Sigma_2^{(j)T} + \Sigma_1^{(j)},$

$\Theta_{13}^{(j)} = -\bar{P}_{11}^{(j)} - j\varpi_c\bar{Q}_{11}^{(j)} - \bar{\Sigma}_1^{(j)T} + \Sigma_1^{(j)}A^{(j)},$

$\Theta_{14}^{(j)} = -\bar{P}_{12}^{(j)} - j\varpi_c\bar{Q}_{12}^{(j)} - \bar{R}^{(j)T} - \Sigma_2^{(j)T} + \Sigma_1^{(j)}A^{(j)},$

$\Theta_{22}^{(j)} = -\bar{Q}_{22}^{(j)} + He\left\{R^{(j)T}\right\},$

$\Theta_{23}^{(j)} = -\bar{P}_{12}^{(j)T} - j\varpi_c\bar{Q}_{12}^{(j)T} - \bar{\Sigma}_1^{(j)T} + R^{(j)}A^{(j)} + Z_2^{(j)}C_2^{(j)} + Z_1^{(j)},$

$\Theta_{24}^{(j)} = -\bar{P}_{22}^{(j)} - j\varpi_c\bar{Q}_{22}^{(j)} - \bar{R}^{(j)T} + R^{(j)}A^{(j)} + Z_2^{(j)}C_2^{(j)},$

$\Theta_{33}^{(j)} = -\varpi_1\varpi_2\bar{Q}_{11}^{(j)} - He\left\{\bar{\Sigma}_1^{(j)}A^{(j)}\right\},$

$\Theta_{34}^{(j)} = -\varpi_1\varpi_2\bar{Q}_{12}^{(j)} - \bar{\Sigma}_1^{(j)}A^{(j)} - A^{(j)T}\bar{R}^{(j)T} - C_2^{(j)T}Z_2^{(j)T} - Z_1^{(j)T},$

$\Theta_{44}^{(j)} = -\varpi_1\varpi_2\bar{Q}_{22}^{(j)} - He\left\{\bar{R}^{(j)}A^{(j)} + Z_2^{(j)}C_2^{(j)}\right\},$

and for $j = b,c,d$

$\Phi_{35}^{(j)} = Z_3^{(j)T}D_{12}^{(j)T} + \bar{\Sigma}_1^{(j)}B_1^{(j)}, \Phi_{45}^{(j)} = \bar{R}^{(j)}B_1^{(j)} + Z_2^{(j)}D_{21}^{(j)}$.

with

$Z_i^{(a)} = diag(Z_i^{(MN)}, Z_i^{(V)})\ (i = 1,2,3,4), \Sigma_2^{(a)} = diag(\Sigma_2^{(MN)}, \Sigma_2^{(V)});$

$Z_i^{(b)} = Z_i^{(d)} = diag(Z_i^{(MN)}, Z_i^{(U)})\ (i = 1,2,3,4),$

$\Sigma_2^{(b)} = \Sigma_2^{(d)} = diag(\Sigma_2^{(MN)}, \Sigma_2^{(U)}); Z_i^{(c)} = Z_i^{(V)} (i = 1,2,3,4), \quad \Sigma_2^{(c)} = \Sigma_2^{(V)}.$

In the affirmative case, the decentralized dynamic controller is determined by the formula $K = K_P + K_I s^{-1} + K_D s$ where the gains $K_P = \hat{K}_P$ and $K_D = \hat{K}_D$ can be obtained directly from $Z_2^{(MN)}$:

$$Z_2^{(MN)} = [I_{m_2} \;\vdots\; \hat{K}_P \quad \hat{K}_I \quad \hat{K}_D] \quad (27)$$

and the gain K_I can be computed by the following formula:

$$K_I = Z_1^{(MN)}(\Sigma_2^{(MN)})^{-1}\hat{K}_I \quad (28)$$

In addition, the aforementioned statement hold true for the low- and high-frequency cases subject to the following changes:

1) (low frequency) Set $\varpi_1 = -\varpi_l, \varpi_2 = \varpi_l$.
2) (high frequency) Set $\varpi_1 = -\varpi_h, \varpi_2 = \varpi_h$ and replace $\bar{Q}_{11}^{(j)}, \bar{Q}_{12}^{(j)}, \bar{Q}_{22}^{(j)}$ by $-\bar{Q}_{11}^{(j)}, -\bar{Q}_{12}^{(j)}, -\bar{Q}_{22}^{(j)}$ for $j = a,b,c$.

Remark 1. With appropriate minor modifications the aforementioned proposed method (i.e., Theorem 2) can be applied to the other cases, i.e., $p_1 = m_2$ and/or $n \le m_2$. For the case $p_1 = m_2$ one may replace P_{11} and P_{12} by the augmented plants $[P_{11}^T \quad 0_{m_1\times r}]^T$ and $[P_{12}^T \quad 0_{m_2\times r}]^T$, respectively, where r is any positive integer. The trick renders an equivalent problem reformulation to the original problem; furthermore, it circumvents the difficulty of lacking the function V for coping with the incurred order problem, and hence the above proposed method is readily applied. For the case $n=m_2$ (see Section IV), it may simply set the order of function U to be zero. For the case $n<m_2$, we may set U to be a known function of order m_2-n.

Remark 2. With appropriate modifications the proposed method is also applicable to the design of another type of PID controllers, $K(s)=K_P+K_I s^{-1}+(K_D s/1+\varepsilon s)$.

Remark 3. Dual results of Theorems 1, 2 can be derived in the same manner for the case $m_1 \geq p_2$ by applying the property of norm (e.g., the largest singular value) preserving under matrix transpose. In this case the assumptions differ and this extends the results of this work.

IV. Numerical Example

In this section, a numerical example is given for illustration of the proposed method.

Example 1: Consider the classical tracking control system with plant described as follows [11].

$$P(s)=\begin{bmatrix}(s-3)/(s-1)(s-2) & -1/(s-1)(s-2)\\ 2/(s-1)(s-2) & s/(s-1)(s-2)\end{bmatrix}=\left[\begin{array}{c|c}A_o & B_o\\ \hline C_o & D_o\end{array}\right] \quad (29)$$

The objective is to attenuate the effect of the disturbances upon the tracking error signals and the control inputs via designing a two-channel decentralized PID controller. Accordingly, the generalized plant data of can be described as follows:

$$\left[\begin{array}{c|c}P_{11}(s) & P_{12}(s)\\ \hline P_{21}(s) & P_{22}(s)\end{array}\right]=\left[\begin{array}{c|c|c}A_o & B_o & B_o\\ \hline -C_o & -D_o & -D_o\\ 0 & 0 & I\\ \hline -C_o & -D_o & -D_o\end{array}\right] \quad (30)$$

It is assumed that the disturbances whose frequency contents concentrated mainly on the frequency range $[10,\infty)$ enter the control system from the plant's input. Two PID controllers are determined (by Theorem 2) with respect to the frequency ranges: $[10,\infty)$ and $[0,\infty)$, which reads:

$$diag\left(2.0369+\frac{0.4774}{s}+0.1074s,\ 3.3830+\frac{0.2341}{s}+0.1100s\right) \quad (31)$$

$$diag\left(14.1840+\frac{1.0289}{s}+2.9592s,\ 27.0872+\frac{1.0669}{s}+3.6290s\right) \quad (32)$$

More comparative results are shown in Table 2.

For demonstrating the effectiveness of the proposed method, the disturbances are modelled as the output signals of high-pass functions (with passband edge frequency 10 rads/sec) driven by Gaussian noises with zero mean and unit variance. Figures 1,2 show the time responses of the two designs, in which it is observed that the controller (31) outperforms the controller (32) in terms of the range of variation.

Table 2 Results of the designed decentralized PID controllers

Case	$[10,\infty)$	$[0,\infty)$
α	0.9905	3.7132
$\lVert T_{zw}\rVert_{[10,\infty)}$	0.3517	0.8760
order of V/U	4/0	
Closed- loop poles	$-0.9993 \pm 1.2236i$, $-0.0929 \pm 0.1667i$.	$-0.0729, -0.0451$, $-5.0881, -3.5800$.
Controllers	(31)	(32)

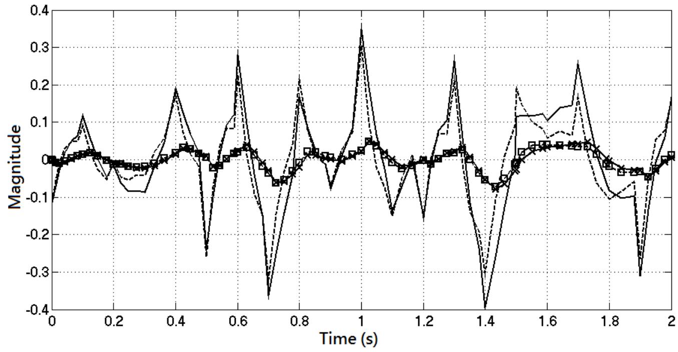

Fig. 1. The performance of the PID controller (31).

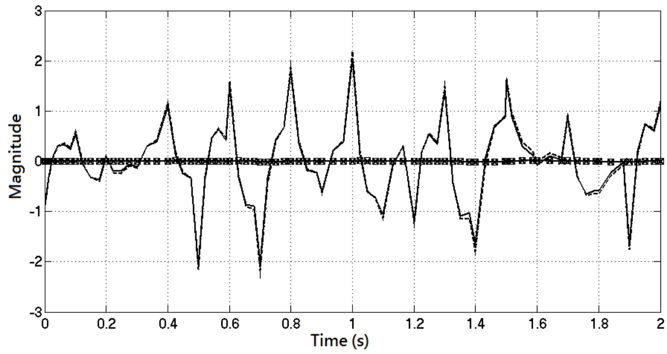

Fig. 2. The performance of the PID controller (32).

V. Conclusion

In this work a novel decentralized PID controller synthesis has been presented, which extends the current results of SISO, H-infinity PID control in finite frequency domain to the MIMO case of decentralized control. Under some mild assumptions, frequency-domain solvability conditions were derived. LMI-based state-space solutions were given that can be efficiently solved via convex programming.

Appendix

Proof of Theorem 2

We prove the claims only for the case of middle frequency range. The results of the other cases can be showed in a similar manner. To get started, the proof proceeds by converting the conditions (14), (16), (18) and (19) into matrix inequalities one by one. A part of the proof is lengthy and much the same as that in [7], therefore we give only a sketch of this part. First, we consider condition (i) of Theorem 1, partition the instrumental matrix variables of Lemma 1(iii) in the following form:

$$P^{(a)} = \begin{bmatrix} P_{11}^{(a)} & P_{12}^{(a)} \\ * & P_{22}^{(a)} \end{bmatrix}, Q^{(a)} = \begin{bmatrix} Q_{11}^{(a)} & Q_{12}^{(a)} \\ * & Q_{22}^{(a)} \end{bmatrix},$$
$$G^{(a)} = \begin{bmatrix} R^{(a)} & X_{12}^{(a)} \\ X_{21}^{(a)} & X_{22}^{(a)} \end{bmatrix}, W^{(a)} = \begin{bmatrix} \bar{R}^{(a)} & X_{12}^{(a)} \\ M_{21}^{(a)} & M_{22}^{(a)} \end{bmatrix} \quad \text{(A1)}$$

Define notations

$$(G^{(a)})^{-1} = \begin{bmatrix} S^{(a)} & Y_{12}^{(a)} \\ Y_{21}^{(a)} & Y_{22}^{(a)} \end{bmatrix}, (W^{(a)})^{-1} = \begin{bmatrix} \bar{S}^{(a)} & \bar{Y}_{12}^{(a)} \\ N_{21}^{(a)} & N_{22}^{(a)} \end{bmatrix}, T_1^{(a)} = \begin{bmatrix} S^{(a)} & Y_{12}^{(a)} \\ I & 0 \end{bmatrix},$$
$$T_2^{(a)} = \begin{bmatrix} I & 0 \\ R^{(a)} & X_{12}^{(a)} \end{bmatrix}, T_3^{(a)} = \begin{bmatrix} \bar{S}^{(a)} & \bar{Y}_{12}^{(a)} \\ I & 0 \end{bmatrix}, T_4^{(a)} = \begin{bmatrix} I & 0 \\ \bar{R}^{(a)} & X_{12}^{(a)} \end{bmatrix}$$

It is easy to verify that $T_1^{(a)}G^{(a)} = T_2^{(a)}$ and $T_3^{(a)}W^{(a)} = T_4^{(a)}$. Without loss of generality we may assume $S^{(a)}$ and $\bar{S}^{(a)}$ are invertible. Furthermore, we assume $(S^{(a)})^{-1}Y_{12}^{(a)} = (\bar{S}^{(a)})^{-1}\bar{Y}_{12}^{(a)}$. Then condition (23) can be obtained via the following steps.

Step 1: get a realization of $F_l(P^{(a)}, K^{(a)})$ from (20), substitute it into Lemma 1(iii), and perform the congruence transformation:

$$diag\left(diag((S^{(a)})^{-1}, I) \times T_1^{(a)}, diag((\bar{S}^{(a)})^{-1}, I) \times T_3^{(a)}, I, I\right).$$

Step 2: apply the following change of variables to the resulting condition obtained in Step 1:

$$\begin{bmatrix} \bar{P}_{11}^{(a)} & \bar{P}_{12}^{(a)} \\ * & \bar{P}_{22}^{(a)} \end{bmatrix} := \begin{bmatrix} I & (S^{(a)})^{-1}Y_{12}^{(a)} \\ I & 0 \end{bmatrix} \begin{bmatrix} P_{11}^{(a)} & P_{12}^{(a)} \\ * & P_{22}^{(a)} \end{bmatrix} \begin{bmatrix} I & (S^{(a)})^{-1}Y_{12}^{(a)} \\ I & 0 \end{bmatrix}^T$$
$$\begin{bmatrix} \bar{Q}_{11}^{(a)} & \bar{Q}_{12}^{(a)} \\ * & \bar{Q}_{22}^{(a)} \end{bmatrix} := \begin{bmatrix} I & (S^{(a)})^{-1}Y_{12}^{(a)} \\ I & 0 \end{bmatrix} \begin{bmatrix} Q_{11}^{(a)} & Q_{12}^{(a)} \\ * & Q_{22}^{(a)} \end{bmatrix} \begin{bmatrix} I & (S^{(a)})^{-1}Y_{12}^{(a)} \\ I & 0 \end{bmatrix}^T$$
$$\Sigma_1^{(a)} := (S^{(a)})^{-1},\ \bar{\Sigma}_1^{(a)} := (\bar{S}^{(a)})^{-1}, \Sigma_2^{(a)} := X_{12}^{(a)}(Y_{12}^{(a)})^T(S^{(a)})^{-T},$$
$$Z_1^{(a)} := X_{12}^{(a)}A_K^{(a)}(Y_{12}^{(a)})^T(S^{(a)})^{-T}, Z_2^{(a)} := X_{12}^{(a)}B_K^{(a)}, \quad \text{(A2)}$$
$$Z_3^{(a)} := C_K^{(a)}(Y_{12}^{(a)})^T(S^{(a)})^{-T}, Z_4^{(a)} := D_K^{(a)}.$$

Note that the condition obtained in Step 2 is a LMI in the variables $Z_i^{(a)}, i = 1,2,3,4$ and $\Sigma_2^{(a)}$. It can be showed by the aforementioned change of variables that the values of $(A_K^{(a)}, B_K^{(a)}, C_K^{(a)}, D_K^{(a)})$ (see (20)) can be recovered from the following formula:

$$\left[\begin{array}{c:c} A_K^{(a)} & B_K^{(a)} \\ \hdashline C_K^{(a)} & D_K^{(a)} \end{array}\right] = \begin{bmatrix} (X_{12}^{(a)})^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_1^{(a)}(\Sigma_2^{(a)})^{-1} & Z_2^{(a)} \\ Z_3^{(a)}(\Sigma_2^{(a)})^{-1} & Z_4^{(a)} \end{bmatrix} \begin{bmatrix} X_{12}^{(a)} & 0 \\ 0 & I \end{bmatrix} \quad \text{(A3)}$$

where $X_{12}^{(a)} = \Sigma_2^{(a)}(\Sigma_1^{(a)})^{-T}(Y_{12}^{(a)})^{-T}$. In view of (20), the structural constraints imposed upon $(A_K^{(a)}, B_K^{(a)}, C_K^{(a)}, D_K^{(a)})$ need to be translated into the variables $Z_i^{(a)}, i = 1,2,3,4$ and $\Sigma_2^{(a)}$. The idea is to restrict the variables to be block-diagonal, specifically, $Z_i^{(a)} = diag(Z_i^{(MN)}, Z_i^{(V)}), i = 1,2,3,4$ and $\Sigma_2^{(a)} = diag(\Sigma_2^{(MN)}, \Sigma_2^{(V)})$ in which $(Z_1^{(j)}(\Sigma_2^{(j)})^{-1}, Z_2^{(j)}, Z_3^{(j)}(\Sigma_2^{(j)})^{-1}, Z_4^{(j)}), j = MN, V$ are required to realize $[\tilde{M}_K \ \ \tilde{N}_K]$ and V, respectively.

To be more specific, note that $[\tilde{M}_K \ \ \tilde{N}_K]$ is a left coprime pair of $\bar{K}$, and that $\bar{K}$ has a realization:

$$\bar{K} = \left[\begin{array}{c|ccc} 0_{m_2} & 0_{m_2 \times p_2} & K_I & 0_{m_2 \times p_2} \\ \hline I_{m_2} & K_P & 0_{m_2 \times p_2} & K_D \end{array}\right] \quad \text{(A4)}$$

By (1) it admits a left coprime factorization $\bar{K} = \tilde{M}_K^{-1}\tilde{N}_K$:

$$\begin{bmatrix} \tilde{M}_K & \tilde{N}_K \end{bmatrix} = \left[\begin{array}{c|cc} L & L & [LK_P \ \ K_I \ \ LK_D] \\ \hline I_{m_2} & I_{m_2} & [K_P \ \ 0_{m_2 \times p_2} \ \ K_D] \end{array}\right] \quad \text{(A5)}$$

where L is a $m_2 \times m_2$ Hurwitz matrix. It is easy to check that equating $(Z_1^{(MN)}(\Sigma_2^{(MN)})^{-1}, Z_2^{(MN)}, Z_3^{(MN)}(\Sigma_2^{(MN)})^{-1}, Z_4^{(MN)})$ with their counterparts in (A5) does not lead to a LMI condition in the variables L, K_P, and K_D. To circumvent this difficulty, another realization of $[\tilde{M}_K \ \ \tilde{N}_K]$ as shown below is considered.

$$\left[\begin{array}{c|cc} L & I_{m_2} & [K_P \ \ L^{-1}K_I \ \ K_D] \\ \hline L & I_{m_2} & [K_P \ \ 0_{m_2 \times p_2} \ \ K_D] \end{array}\right] \quad \text{(A6)}$$

Now let

$$Z_1^{(MN)} = Z_3^{(MN)} \in S, Z_2^{(MN)} = [I_{m_2} \ \vdots\ \hat{K}_P \ \ \hat{K}_I \ \ \hat{K}_D],$$
$$Z_4^{(MN)} = [I_{m_2} \ \vdots\ \hat{K}_P \ \ 0_{m_2 \times p_2} \ \ \hat{K}_D], \Sigma_2^{(MN)} \in S. \quad \text{(A7)}$$

where $\hat{K}_I := L^{-1}K_I$. While the gains of K_P and K_D can be obtained directly from $Z_2^{(MN)}$ (or $Z_4^{(MN)}$), It is readily verified that the gain $K_I = Z_1^{(MN)}(\Sigma_2^{(MN)})^{-1}\hat{K}_I$ where $\hat{K}_I$ can be obtained directly from $Z_2^{(MN)}$. The rest of the conditions of Theorem 2 can be derived in the same manner. This part is omitted due to space limitation. □

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