Ripple Rejection for a Multirate Control System Including an Integrator

Tomonori Kamiya, Takao Sato, Nozomu Araki and Yasuo Konishi

Abstract— **In the present paper, we discuss a design method for a multirate system, in which the update interval of the control input is shorter than the sampling interval of the plant output. In this multirate system, the intersample output may oscillate between sampled outputs even if the sampled output converges to the reference input because the control input can be changed between sampled outputs. In the conventional method, intersample ripples can be eliminated independently of the sampled response. However, this method is not valid when a controlled plant has an integrator. In the present study, we discuss this problem and propose a method by which to relax the problem constraints.**

I. INTRODUCTION

In the present paper, we discuss a design method for a multirate system [1], in which the sampling interval of the plant output is an integer multiple of the update interval of the control input. In a multirate system, the plant output may oscillate between sampled outputs even if the sampled output settles to the set-point [2]. Intersample ripples can be eliminated by using a generalized holder [3], [4]. However, in the proposed method, both intersample and sampled responses are designed simultaneously, and the sampled response is changed.

For the case in which the discrete-time performance is optimized, the sampled response should be maintained. Therefore, the purpose of the present study is to design the intersample response without changing the pre-designed sampled response. In a design method for a multirate system, intersample ripples can be eliminated in the steady state independently of the sampled response [5]. In the conventional method, a controller is redesigned independently of the sampled response so that the steady-state ripples are eliminated. In the present study, we discuss the condition that invalidates the conventional design method and propose a method by which to resolve this problem.

II. CONTROLLED PLANT

In the present study, the actual controlled plant is a continuous-time system. However, we discuss a design method for a multirate system, in which the control input is updated every step, but the plant output is sampled every two steps, rather than every step. Hence, a single-input single-output single-rate system is converted to a two-input single-output single-rate system using lifting [6].

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Consider the following multirate first-order plus integrator system:

$$\begin{aligned} A[z_2^{-1}]y(k) &= \boldsymbol{B}[z_2^{-1}]^T\boldsymbol{u}(k-2) \\ A[z_2^{-1}] &= (1-z_2^{-1})(1+az_2^{-1}) \\ \boldsymbol{B}[z_2^{-1}] &= \left[\begin{array}{cc} B_1[z_2^{-1}] & B_2[z_2^{-1}] \end{array}\right]^T \\ \boldsymbol{u}(k) &= \left[\begin{array}{cc} u(k) & u(k+1) \end{array}\right]^T \end{aligned} \tag{1}$$

where $y(k)$ is the sampled plant output, $u(k)$ is the control input in discrete time, and z_1^{-1} denotes the one-step backward shift operator, $z_1^{-1}y(k) = y(k-1)$ and $z_j^{-1} = z_1^{-j}$.

III. CONVENTIONAL METHOD AND ITS WEAKNESS

The lifted single-rate system is assumed to be stabilized using the following control law:

$$\begin{aligned} \boldsymbol{Y}[z_2^{-1}]\boldsymbol{u}(k) &= \boldsymbol{K}[z_2^{-1}]r(k) - \boldsymbol{X}[z_2^{-1}]y(k) \\ \boldsymbol{Y}[z_2^{-1}] &= \left[\begin{array}{cc} Y_1[z_2^{-1}] & 0 \\ 0 & Y_2[z_2^{-1}] \end{array}\right] \\ \boldsymbol{K}[z_2^{-1}] &= \left[\begin{array}{c} K_1[z_2^{-1}] \\ K_2[z_2^{-1}] \end{array}\right] \\ \boldsymbol{X}[z_2^{-1}] &= \left[\begin{array}{c} X_1[z_2^{-1}] \\ X_2[z_2^{-1}] \end{array}\right] \end{aligned} \tag{2}$$

where $\boldsymbol{Y}[z_2^{-1}]$ and $\boldsymbol{X}[z_2^{-1}]$ must be designed such that a closed-loop system is stabilized because of the above-mentioned assumption. Furthermore, $\boldsymbol{K}[z_2^{-1}]$ is assumed to be designed to have the plant output converge to the given reference input. The closed-loop system with the control law is given as follows:

$$y(k) = \frac{z_2^{-1}\boldsymbol{Y}_B[z_2^{-1}]^T\boldsymbol{K}[z_2^{-1}]}{T[z_2^{-1}]}r(k) \tag{3}$$

$$\begin{aligned} T[z_2^{-1}] &= A[z_2^{-1}]Y_P[z_2^{-1}] + z_2^{-1}\boldsymbol{Y}_B[z_2^{-1}]^T\boldsymbol{X}[z_2^{-1}] \\ \boldsymbol{Y}_B[z_2^{-1}] &= \left[\begin{array}{cc} B_1[z_2^{-1}]Y_2[z_2^{-1}] & B_2[z_2^{-1}]Y_1[z_2^{-1}] \end{array}\right]^T \\ Y_P[z_2^{-1}] &= Y_1[z_2^{-1}]Y_2[z_2^{-1}] \end{aligned} \tag{4}$$

The multirate control law is extended as follows:

$$\begin{aligned} \boldsymbol{Y}_e[z_2^{-1}]\boldsymbol{u}(k) &= \boldsymbol{K}[z_2^{-1}]r(k) - \boldsymbol{X}_e[z_2^{-1}]y(k) \\ \boldsymbol{Y}_e[z_2^{-1}] &= \boldsymbol{Y}[z_2^{-1}] - z_2^{-1}\boldsymbol{U}_u[z_2^{-1}]\boldsymbol{B}[z_2^{-1}]^T \\ \boldsymbol{X}_e[z_2^{-1}] &= \boldsymbol{X}[z_2^{-1}] + \boldsymbol{U}_y[z_2^{-1}]A[z_2^{-1}] \end{aligned} \tag{5}$$

where $\boldsymbol{U}_u[z_2^{-1}]$ and $\boldsymbol{U}_y[z_2^{-1}]$ are design parameters. Using the extended control law, the closed-loop system is given as follows:

$$y(k) = \frac{z_2^{-1}\boldsymbol{Y}_B[z_2^{-1}]^T\boldsymbol{K}[z_2^{-1}]}{T_e[z_2^{-1}]}r(k) \tag{6}$$

$$T_e[z_2^{-1}] = T[z_2^{-1}] + \bar{T}_e[z_2^{-1}]$$

$$\bar{T}_e[z_2^{-1}] = z_2^{-1}A[z_2^{-1}]\boldsymbol{Y}_B[z_2^{-1}]^T(\boldsymbol{U}_y[z_2^{-1}] - \boldsymbol{U}_u[z_2^{-1}])$$

The extended closed-loop system is changed from the original closed-loop system. Here, $\boldsymbol{U}_u[z_2^{-1}]$ and $\boldsymbol{U}_y[z_2^{-1}]$ are designed as follows:

$$\boldsymbol{U}_u[z_2^{-1}] = \begin{bmatrix} U_1[z_2^{-1}]B_2[z_2^{-1}]Y_1[z_2^{-1}] \\ U_2[z_2^{-1}]B_1[z_2^{-1}]Y_2[z_2^{-1}] \end{bmatrix} \tag{7}$$

$$\boldsymbol{U}_y[z_2^{-1}] = \begin{bmatrix} U_2[z_2^{-1}]B_2[z_2^{-1}]Y_1[z_2^{-1}] \\ U_1[z_2^{-1}]B_1[z_2^{-1}]Y_2[z_2^{-1}] \end{bmatrix} \tag{8}$$

Using Eqs. (7) and (8), the extended closed-loop system is the same as the original system because $\bar{T}_e[z_2^{-1}] = 0$ can be achieved independently of the selection of $U_1[z_2^{-1}]$ and $U_2[z_2^{-1}]$. Therefore, the reference response in discrete time is maintained [5].

Using the extended control law, the transfer function from the reference input to the control input is given as follows:

$$\frac{\boldsymbol{u}(k)}{r(k)} = (A[z_2^{-1}]\boldsymbol{Y}_e[z_2^{-1}] + z_2^{-1}\boldsymbol{X}_e[z_2^{-1}]\boldsymbol{B}[z_2^{-1}]^T)^{-1} \cdot A[z_2^{-1}]\boldsymbol{K}[z_2^{-1}] \tag{9}$$

For the case in which the steady-stage gains in (9) are equivalent, intersample ripples can be eliminated in the steady state [7]. In the extended control system, if the design parameters $U_1[z_2^{-1}]$ and $U_2[z_2^{-1}]$ are designed such that the following equation is satisfied, the intersample ripples can be eliminated in the steady state:

$$\begin{aligned}&(A[1]B_1[1]B_2[1](Y_2[1] - Y_1[1]) \\ &+ A[1](Y_1[1]B_2[1]^2 + Y_2[1]B_1[1]^2))(U_1[1] - U_2[1]) \\ &= A[1](Y_1[1] - 1) + X_1[1](B_1[1] + B_2[1]) \\ &- B_1[1] + B_2[1]\end{aligned} \tag{10}$$

However, the above equation cannot always be satisfied if an integrator is included in the plant model. Equation (10) cannot be satisfied using $U_1[z_2^{-1}]$ and $U_2[z_2^{-1}]$ because the left-hand side of Eq. ((10)) will be 0.

IV. SIMPLIFICATION OF THE CONTROLLED OBJECTIVE

In the conventional method [5], the steady-state intersample ripples cannot be eliminated using $U_1[z_2^{-1}]$ and $U_2[z_2^{-1}]$ when an integrator is included in the controlled plant.

For the case in which the reference input is a ramp function, even if the plant output does not converge to the ramp-type reference input, the difference or differential of the plant output follows the gradient of the reference input, and a control objective may be achieved. Therefore, in the present study, the controlled variable is changed to the difference or differential of the plant output, and a control system is designed.

In the present study, $\Delta y(k)$ is defined as a new controlled variable, and $\Delta(= 1 - z_2^{-1})$ in $A[z_2^{-1}]$ is eliminated as follows:

$$A[z_2^{-1}]y(k) = \bar{A}[z_2^{-1}]\Delta y(k) \tag{11}$$

$$\bar{A}[z_2^{-1}] = 1 + az_2^{-1}$$

$$\Delta y(k) = (1 - z_2^{-1})y(k)$$

Hence, the plant model is rewritten as follows:

$$\bar{A}[z_2^{-1}]\Delta y(k) = \boldsymbol{B}[z_2^{-1}]^T\boldsymbol{u}(k-2) \tag{12}$$

In this case, the control objective is to have the difference of the plant output, $\Delta y(k)$, follow the reference input, which is the gradient of the ramp-type reference input. Furthermore, in the extended control system, $\bar{A}[z_2^{-1}]$ is used instead of $A[z_2^{-1}]$, and the design parameters $U_1[z_2^{-1}]$ and $U_2[z_2^{-1}]$ can be designed such that the intersample ripples are eliminated in the steady state.

V. NUMERICAL EXAMPLES

Consider the following transfer function.

$$G(s) = \frac{1}{s(s+1.8)} \tag{13}$$

The control input is assumed to be updated at intervals of 1 [s], but the plant output is sampled at intervals of 2 [s]. The control objective is to have the differential plant output converge to the set point.

In this case, a multirate system is obtained as follows:

$$\begin{aligned}&(1 - 0.027z_2^{-1})(1 - z_2^{-1})y(k) \\ &= [0.51 + 0.025z_2^{-1} \quad 0.30 + 0.24z_2^{-1}]\boldsymbol{u}(k-2)\end{aligned} \tag{14}$$

Using the following multirate control law, the sampled output $\Delta y(k)$ can follow the step-type reference input:

$$\begin{aligned}&\begin{bmatrix} 1.0 + 0.43z_2^{-1} & 0 \\ 0 & 1.0 \end{bmatrix}\boldsymbol{u}(k) \\ &= \begin{bmatrix} 0.37 \\ 0.37 \end{bmatrix} r_{\Delta_y}(k) - \begin{bmatrix} -1.8 - 0.38z_2^{-1} \\ 1.0 \end{bmatrix}\Delta y(k)\end{aligned} \tag{15}$$

where $\Delta y(k)$ denotes the reference input to be followed by $\Delta y(k)$ and is set to 1. The simulation results obtained using the control law given by Eq. (15) are shown in Figs. 1 through 4. The output and input responses are shown in Figs. 1 and 4, respectively. Fig. 2 shows the trajectory of $\Delta y(k)$, and the sampled output $\Delta y(k)$ converges to the set point. Fig. 3 shows the differential of the continuous-time output response, $\frac{dy(t)}{dt}$, and so the intersample output oscillates (see Fig. 1). This is because the control input oscillates between sampled outputs (see Fig. 4).

In order to compensate the intersample oscillation without changing the sampled response, the control law given by Eq. (15) is extended to Eq. (5), where $U_1[z_2^{-1}]$ and $U_2[z_2^{-1}]$ are set to be -3.0 and 0, respectively. The control results obtained using the extended control law is shown in Figs. 5 through 8. The trajectory of the sampled output $\Delta y(k)$ is shown in Fig. 6 and is the same as that given by Eq. (15). However, the differential of the continuous-time output does

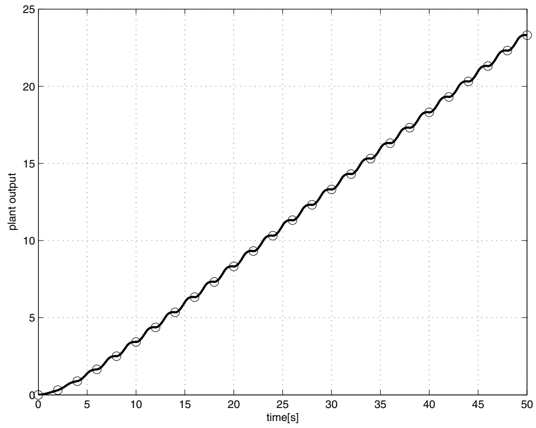

Fig. 1. Output response obtained using the conventional method

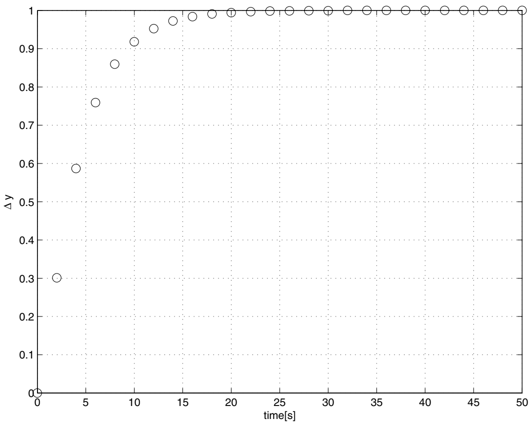

Fig. 2. Sampled output response obtained using the conventional method

not oscillate in the steady state (see Fig. 7). Therefore, Fig. 5 shows that the intersample ripples are eliminated.

The trajectories of the plant output obtained by both the conventional and proposed multirate control laws are shown in Fig. 9. An enlarged view of Fig. 9 is shown in Fig. 10. From these figures, the sampled response is maintained, and the intersample ripples are eliminated using the extended control law.

VI. CONCLUSIONS

In the present paper, we discussed a design method for a multirate system, in which the sampling interval of the plant output is an integer multiple of the update interval of the control input. In a multirate system, intersample ripples may occur between sampled outputs even if the sampled output converges to the reference input. In the conventional method [5], intersample ripples can be eliminated independently of the sampled reference response. However, the conventional method is not valid if the controlled plant has an integrator. In the present study, the controlled variable is changed from $y(k)$ to $\Delta y(k)$, and, as a result, this restriction is relaxed.

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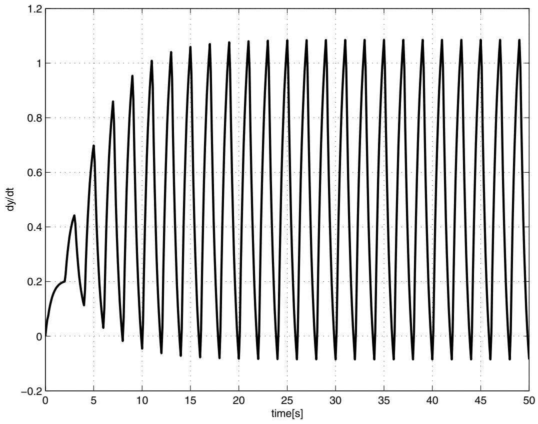

Fig. 3. Differential plant output obtained using the conventional method

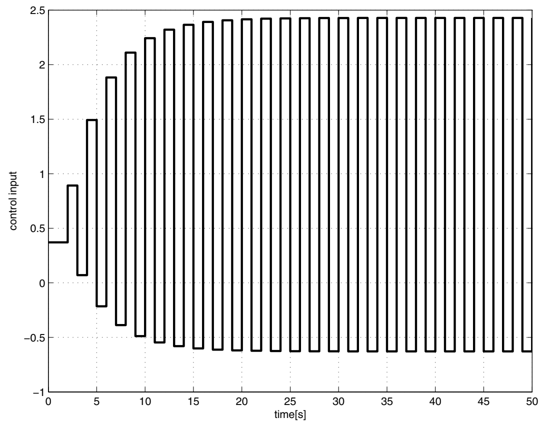

Fig. 4. Input response obtained using the conventional method

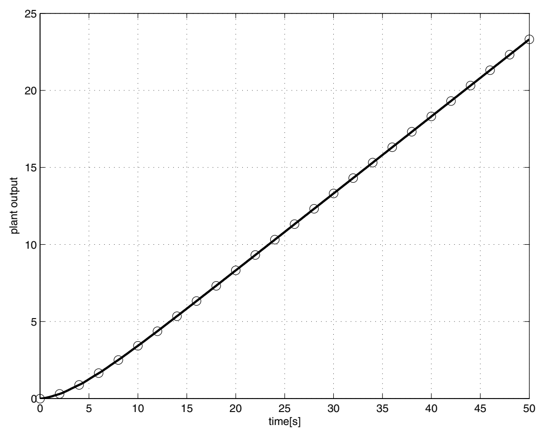

Fig. 5. Output response obtained using the proposed method

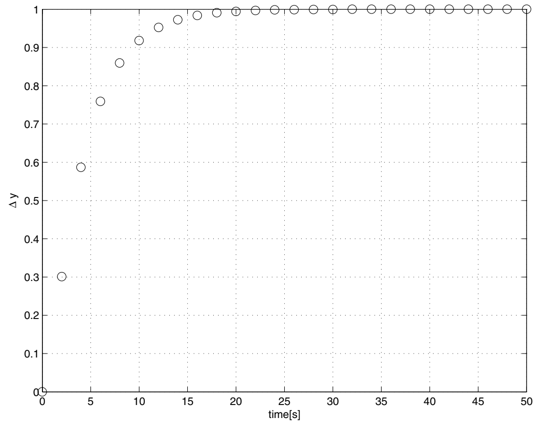

Fig. 6. Sampled output response obtained using the proposed method

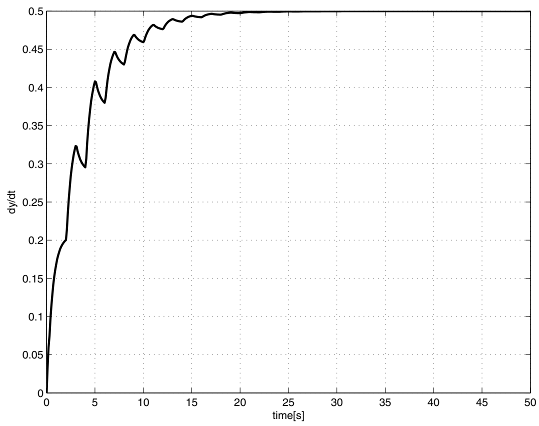

Fig. 7. Differential plant output obtained using the proposed method

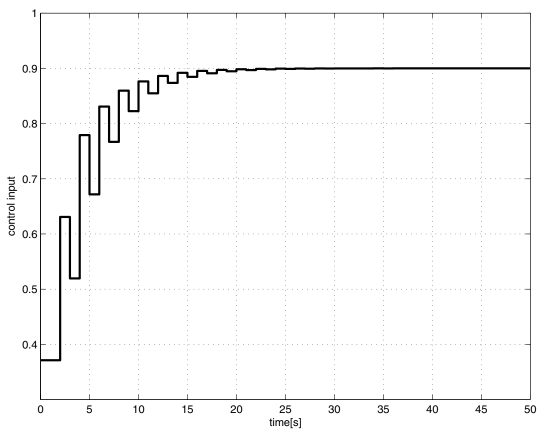

Fig. 8. Input response obtained using the proposed method

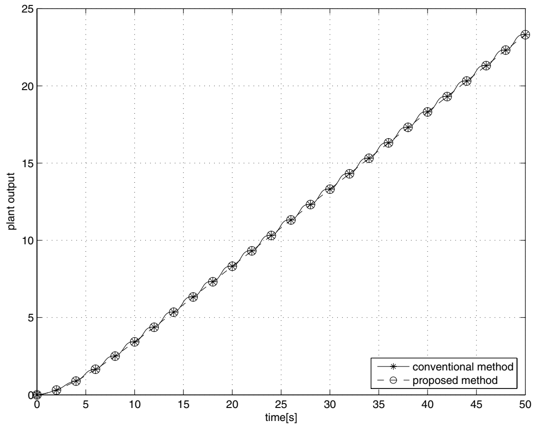

Fig. 9. Merged output figure

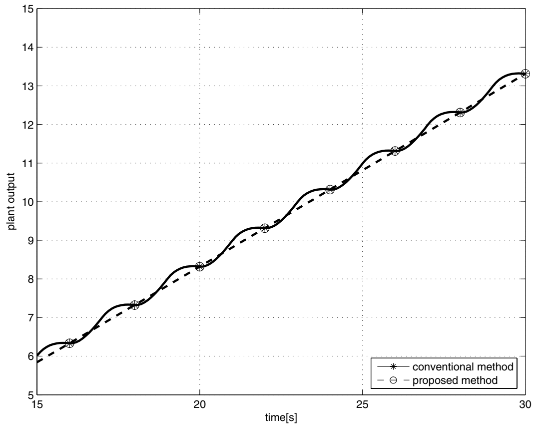

Fig. 10. Enlarged view of Fig. 9

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