

# A Positive Realness Based Approach to Design of IIR Low-Pass Differentiators with Prescribed Pole Radius Constraint\*

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**Abstract**— We propose a design method for IIR low-pass differentiators under a specified maximum pole radius constraint. In the proposed method, we express the design problem in a quadratic form with respect to the coefficients of the transfer function. Since the cost function includes a weighting function, the frequency-weighting can be specified in the pass-band. Also, a linear phase property in the pass-band can be achieved similar to band-pass filters. Using the proposed method, stable IIR low-pass differentiators can be easily obtained for approximating the given response. Finally, two numerical examples are given to illustrate the effectiveness of the proposed method by designing IIR low-pass differentiators.

## I. INTRODUCTION

Digital filters [1]–[32] are widely used in the various fields such as signal processing, automatic control, and power system. These can be categorized into two kinds: infinite impulse response (IIR) digital filters [6]–[11], [13]–[26] and finite impulse response (FIR) digital filters [12], [27]–[32]. It is well known that the order of IIR digital filters is usually lower than the one of FIR digital filters to have equivalent approximation; hence, IIR filters are attractive for the hardware realization. Digital differentiators are an important class of digital filters, and several related works and applications have been published in [11], [25], [33]–[43]. Furthermore, there are special differentiators which is referred to as low-pass differentiator in which the low-frequency elements are differentiated and the high-frequency elements are deleted. The low-pass differentiators are well-used for several applications [44]–[50].

The design problems of IIR filters (differentiators) are usually non-linear optimization problems; hence, the stability constraints should be satisfied, i.e., the all poles are placed within a unit circle. It should be noted that some iteration procedure or algorithm is often required in order to solve the design problem. In this paper, we consider to approximate an ideal response of low-pass differentiators by a rational transfer function. Since the design problem of low-pass differentiators is similar to that of IIR band-pass filters, the design problem is formulated by a cost function proposed in [10] with the weighting function. The accuracy

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of the magnitude response and the phase linearity can be adjusted with the frequency-weighting factors. Additionally, we propose an algorithm to compute the coefficients of the transfer function in which the maximum pole radius can be specified.

Filter designers can easily obtain the low-pass differentiators with linear phase property, and the low-pass differentiators are always stable. Since low-pass differentiator is a wider class of full-band differentiator, low-pass differentiators are more useful and applicable in practical use. Of course, the proposed method includes the design of full-band differentiator as a special cases.

Finally, in order to demonstrate the effectiveness of the proposed method, we give two examples for the design of low-pass differentiators. As a result, we confirm the phase linearity in the pass-band, and the all poles are located within the specified circle.

## II. PROBLEM FORMULATION

First of all, we formulate the design problem of a low-pass differentiator. We define a transfer function  $H(z^{-1})$  as

$$H(z^{-1}) = \frac{B(z^{-1})}{A(z^{-1})} = \frac{\sum_{l=0}^n b_l z^{-l}}{\sum_{k=0}^m a_k z^{-k}}. \quad (1)$$

In (1), the denominator polynomial  $A(z^{-1})$  of order  $m$  and the numerical polynomial  $B(z^{-1})$  of order  $n$  are respectively expressed as

$$A(\omega) = \mathbf{a}^T \cdot [1, z^{-1}, \dots, z^{-m}]^T \quad (2)$$

$$B(\omega) = \mathbf{b}^T \cdot [1, z^{-1}, \dots, z^{-n}]^T \quad (3)$$

where the superscript  $T$  indicates transposition of the matrix (vector),  $\mathbf{a}$  and  $\mathbf{b}$  are the denominator and the numerator coefficients vector defined as

$$\mathbf{a} = [a_0 \ a_1 \ \dots \ a_m]^T \quad (4)$$

$$\mathbf{b} = [b_0 \ b_1 \ \dots \ b_n]^T \quad (5)$$

with  $a_0 = 1$ .

Now, the coefficients of  $H(z^{-1})$  can be computed in order to approximate an ideal response  $H_d(z^{-1})$ . Here, we consider a cost function

$$J = \int_0^\pi W(\omega) |B(\omega) - H_d(\omega)A(\omega)|^2 d\omega \quad (6)$$

where  $W(\omega)$  is a weighting function. Based on a change of variable as  $z = e^{j\omega}$  ( $j^2 = -1$ ) on a closed interval  $[0, \pi]$ , we can get  $A(z^{-1})$ ,  $B(z^{-1})$  and  $W(z^{-1})$ . The problem we treat here is formulated as an optimization problem to compute the filter coefficients which minimize (6). Hence, the filter coefficients is obtained as a solution of the problem.

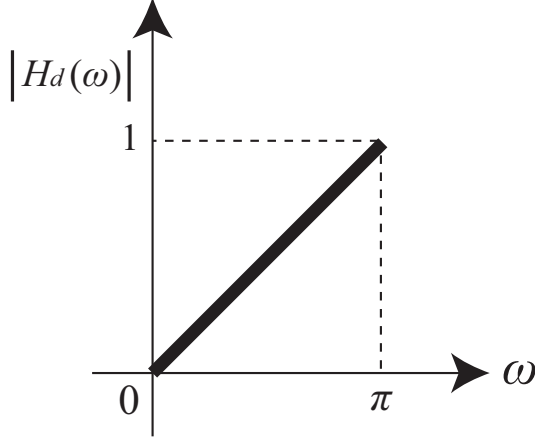


Fig. 1. Ideal response of full-band differentiator.

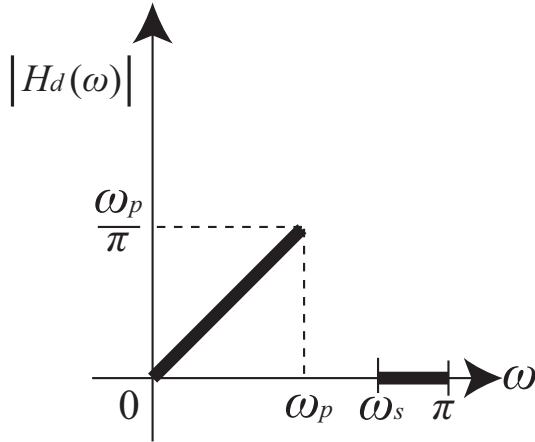


Fig. 2. Ideal response of low-pass differentiator.

#### A. Ideal response of low-pass differentiator

Fig. 1 shows the ideal response of full-band differentiator, and Fig. 2 indicates the ideal response of low-pass differentiator.

With linear phase characteristic, the ideal response of low-pass differentiator  $H_d(\omega)$  is expressed as

$$H_d(\omega) = \begin{cases} \frac{\omega}{\pi} e^{j(0.5\pi - \tau_d\omega)} & 0 \leq \omega \leq \omega_p \\ 0 & \omega_s \leq \omega \leq \pi \end{cases} \quad (7)$$

where  $\omega_p$  is a cut-off frequency of low-pass differentiator. Here, the characteristic in  $[0, \omega_p]$  is a differentiation, on the other hand, a cutoff characteristic in  $[\omega_s, \pi]$  is desired. The group delay  $\tau_d$  is

$$\tau_d = \tau_s + 0.5 \quad (8)$$

where  $\tau_s$  is an integer.

The group delay  $\tau_d$  indicates the delay of each frequency component, and it should be a constant in the pass-band. Also a weighting function is set as

$$W(\omega) = \begin{cases} W_p & 0 \leq \omega \leq \omega_p \\ W_s & \omega_s \leq \omega \leq \pi. \end{cases} \quad (9)$$

It should be noted that, since  $W_p$  is a weighting of the pass-band, the accuracy in the pass-band is improved as  $W_p$  is large. However, as a trade-off, then the accuracy in the stop-band will be degenerated. Also, the closed interval  $[\omega_p, \omega_s]$  is a transition zone. If the transition zone is not employed, we have  $\omega_p = \omega_s$ .

#### B. Quadratic form

Substituting (7) and (9) into (6), we have

$$J = W_p \int_0^{\omega_p} \Phi(\omega) d\omega + W_s \int_{\omega_s}^{\pi} \Psi(\omega) d\omega \quad (10)$$

where

$$\Phi(\omega) = \left| \sum_{l=0}^n b_l e^{-jl\omega} - j \frac{\omega}{\pi} \sum_{k=0}^m a_k e^{-j(k+\tau_d)\omega} \right|^2 \quad (11)$$

$$\Psi(\omega) = \left| \sum_{l=0}^n b_l e^{-jl\omega} \right|^2. \quad (12)$$

Using Euler's formula, (11) is changed to

$$\begin{aligned} \Phi(\omega) &= \left\{ \sum_{l=0}^n b_l e^{-jl\omega} - j \frac{\omega}{\pi} \sum_{k=0}^m a_k e^{-j(k+\tau_d)\omega} \right\} \\ &\quad \cdot \left\{ \sum_{l=0}^n b_l e^{-jl\omega} - j \frac{\omega}{\pi} \sum_{k=0}^m a_k e^{-j(k+\tau_d)\omega} \right\}^* \\ &= \frac{\omega^2}{\pi^2} \sum_{k=0}^m \sum_{k'=0}^m a_k a_{k'} \cos[(k - k')\omega] \\ &\quad - \frac{2\omega}{\pi} \sum_{k=0}^m \sum_{l=0}^n a_k b_l \sin[(k - l + \tau_d)\omega] \\ &\quad + \sum_{l=0}^n \sum_{l'=0}^n b_l b_{l'} \cos[(l - l')\omega] \end{aligned} \quad (13)$$

where the superscript  $*$  is a complex conjugate. Also, (12) can be expressed as

$$\Psi(\omega) = \sum_{l=0}^n \sum_{l'=0}^n b_l b_{l'} \cos[(l - l')\omega]. \quad (14)$$

It follows from (13) and (14) that (10) can be expressed as

$$\begin{aligned} J &= \sum_{k=0}^m \sum_{k'=0}^m a_k a_{k'} P_{k,k'} + 2 \sum_{k=0}^m \sum_{l=0}^n a_k b_l Q_{k,l} \\ &\quad + \sum_{l=0}^n \sum_{l'=0}^n b_l b_{l'} R_{l,l'} \end{aligned} \quad (15)$$

where

$$P_{k,k'} = \frac{W_p}{\pi^2} \int_0^{\omega_p} \omega^2 \cos[(k-k')\omega] d\omega \quad (16)$$

$$Q_{k,l} = -\frac{W_p}{\pi} \int_0^{\omega_p} \omega \sin[(k-l+\tau_d)\omega] d\omega \quad (17)$$

$$R_{l,l'} = W_p \int_0^{\omega_p} \cos[(l-l')\omega] d\omega + W_s \int_{\omega_s}^{\pi} \cos[(l-l')\omega] d\omega. \quad (18)$$

Furthermore, we can compute (16)-(18) without any integral as

$$P_{k,k'} = \begin{cases} \frac{W_p \omega^3}{3\pi^2}, & \text{if } k-k' = 0 \\ \frac{W_p}{P_{k,k'}}, & \text{if } k-k' \neq 0 \end{cases} \quad (19)$$

$$Q_{k,l} = \frac{W_p}{\pi} \left\{ \frac{\omega_p \cos[(k-l+\tau_d)\omega_p]}{k-l+\tau_d} - \frac{\sin[(k-l+\tau_d)\omega_p]}{(k-l+\tau_d)^2} \right\} \quad (20)$$

$$R_{l,l'} = \begin{cases} W_p \omega_p + W_s(\pi - \omega_s), & \text{if } l-l' = 0 \\ \bar{R}_{l,l'}, & \text{if } l-l' \neq 0 \end{cases} \quad (21)$$

where

$$\bar{P}_{k,k'} = \frac{W_p}{\pi^2} \left\{ \frac{\omega_p^2 \sin[(k-k')\omega_p]}{k-k'} + \frac{2\omega_p \cos[(k-k')\omega_p]}{(k-k')^2} - \frac{2 \sin[(k-k')\omega_p]}{(k-k')^3} \right\} \quad (22)$$

$$\bar{R}_{l,l'} = \frac{W_p \sin[(l-l')\omega_p] - W_s \sin[(l-l')\omega_s]}{l-l'}. \quad (23)$$

Now, let  $P_{k,k'}$ ,  $Q_{k,l}$  and  $R_{l,l'}$  be the elements of  $\mathbf{P}$ ,  $\mathbf{Q}$  and  $\mathbf{R}$ , respectively, i.e.,

$$\mathbf{P} := \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0,m} \\ P_{10} & P_{11} & \cdots & P_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m,0} & P_{m,1} & \cdots & P_{m,m} \end{bmatrix} \quad (24)$$

$$\mathbf{Q} := \begin{bmatrix} Q_{00} & Q_{01} & \cdots & Q_{0,n} \\ Q_{10} & Q_{11} & \cdots & Q_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{m,0} & Q_{m,1} & \cdots & Q_{m,n} \end{bmatrix} \quad (25)$$

$$\mathbf{R} := \begin{bmatrix} R_{00} & R_{01} & \cdots & R_{0,n} \\ R_{10} & R_{11} & \cdots & R_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n,0} & R_{n,1} & \cdots & R_{n,n} \end{bmatrix} \quad (26)$$

where  $\mathbf{P}$  is  $(m+1) \times (m+1)$  matrix,  $\mathbf{Q}$  is  $(m+1) \times (n+1)$  matrix and  $\mathbf{R}$  is  $(n+1) \times (n+1)$  matrix. Then, (15) can be expressed as a quadratic form:

$$J = \mathbf{a}^T \mathbf{P} \mathbf{a} + 2\mathbf{a}^T \mathbf{Q} \mathbf{b} + \mathbf{b}^T \mathbf{R} \mathbf{b}. \quad (27)$$

The feature of the proposed method is the cost function  $J$  is formulated as the quadratic form without any integration. Hence, we can achieve a good performance since the computation of  $\mathbf{P}$ ,  $\mathbf{Q}$  and  $\mathbf{R}$  do not require any approximate calculation.

### III. POSITIVE REALNESS

The positive realness of denominator polynomial is expressed as [11]

$$\text{Re}[A(e^{j\omega})] > 0, \quad 0 \leq \omega \leq \pi \quad (28)$$

where  $\text{Re}[A(e^{j\omega})]$  indicates the real part of  $A(e^{j\omega})$ . Eq. (28) is the positive realness condition in the case of  $r_m = 1$  where  $r_m$  is a maximum pole radius. Now, let us consider the general case that the maximum pole radius is  $r_m (\leq 1)$ , then, we have

$$\text{Re}[A(r_m e^{j\omega})] > 0, \quad 0 \leq \omega \leq \pi. \quad (29)$$

We discretize  $\omega$  in  $[0, \pi]$  as  $\Omega_i, i = 1, 2, \dots, L$ . Next, based on  $\Omega_i, i = 1, 2, \dots, L$ , we consider  $L$  constraints. Then, the constraints are expressed as

$$\mathbf{C} \mathbf{v} \leq \mathbf{e} \quad (30)$$

with

$$\mathbf{C} = - \begin{bmatrix} \frac{\cos(\Omega_1)}{r_m} & \frac{\cos(2\Omega_1)}{r_m^2} & \cdots & \frac{\cos(m\Omega_1)}{r_m^m} \\ \frac{\cos(\Omega_2)}{r_m} & \frac{\cos(2\Omega_2)}{r_m^2} & \cdots & \frac{\cos(m\Omega_2)}{r_m^m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\cos(\Omega_L)}{r_m} & \frac{\cos(2\Omega_L)}{r_m^2} & \cdots & \frac{\cos(m\Omega_L)}{r_m^m} \end{bmatrix} \mathbf{O} \quad (31)$$

$$\mathbf{v} = [a_1, a_2, \dots, a_m, b_0, b_1, \dots, b_n]^T \quad (32)$$

$$\mathbf{e} = [1, 1, \dots, 1]^T \quad (33)$$

where  $\mathbf{C}$  is  $L \times (m+n+1)$  matrix,  $\mathbf{v}$  is  $(m+n+1) \times 1$  matrix, and  $\mathbf{e}$  is  $L \times 1$  matrix. This is a positive realness constraint when the maximum pole radius is  $r_m (\leq 1)$ . In this paper, we use this constraint in order to compute the IIR low-pass differentiators.

### IV. DESIGN ALGORITHM

Eq. (27) can be converted to

$$\begin{aligned} J &= \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}^T \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{Q}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}^T \begin{bmatrix} P_{0,0} & \bar{\mathbf{q}} \\ \bar{\mathbf{q}}^T & \bar{\mathbf{R}} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \\ &= P_{0,0} + 2\bar{\mathbf{q}} \mathbf{v} + \mathbf{v}^T \bar{\mathbf{R}} \mathbf{v} \end{aligned} \quad (34)$$

where  $P_{0,0}$  is a scalar,  $\bar{\mathbf{q}}$  is  $(m+n+1) \times 1$  matrix,  $\bar{\mathbf{R}}$  is  $(m+n+1) \times (m+n+1)$  matrix.

Hence, the problem becomes a constraint quadratic programming problem which is well-structured. The design algorithm is summarized as follows.

- step1 Compute  $\bar{\mathbf{q}}$  and  $\bar{\mathbf{R}}$  according to the given specification.
- step2 Compute  $\mathbf{C}$  by  $r_m$ .

step3 Solve the constraint problem:

$$\begin{aligned} \min_{\mathbf{v}} \quad & 2\bar{q}\mathbf{v} + \mathbf{v}^T \bar{\mathbf{R}}\mathbf{v} \\ \text{subject to} \quad & \mathbf{C}\mathbf{v} \leq \mathbf{e}. \end{aligned} \quad (35)$$

Since (35) is a constraint quadratic programming problem, we employ quadprog function of MATLAB.

## V. DESIGN EXAMPLE

In order to show the effectiveness of the proposed method, we give two examples for the design of the low-pass differentiators. The specifications are  $m = 6$ ,  $n = 6$ ,  $\tau_s = 3$  ( $\tau_d = 3.5$ ),  $W_p = 1$  or 100,  $W_s = 1$ ,  $\omega_p = 0.6\pi$ ,  $\omega_s = 0.7\pi$ ,  $r_m = 0.86$  and  $L = 512$  in Example 1. Also,  $m = 5$ ,  $n = 12$ ,  $\tau_s = 6$  ( $\tau_d = 6.5$ ),  $W_p = 1$  or 100,  $W_s = 1$ ,  $\omega_p = 0.7\pi$ ,  $\omega_s = 0.8\pi$ ,  $r_m = 0.9$ , and  $L = 512$  in Example 2. The results of the design of the low-pass differentiators are shown in Fig. 3 (Example 1) and Fig. 4 (Example 2) where the solid line indicates the case of  $W_p = 1$ , broken line indicates the case of  $W_p = 100$ , and the small circles in Fig. 3 (c) and Fig. 4 (c) show the location of poles which are computed with  $W_p = 100$ .

From Figs. 3 and 4, we can see that the magnitude responses and phase responses are both well-approximated, and the phase linearity can be improved as the weighting function in the pass-band is large. On the other hand, the sharpness of the cutoff frequency is degenerated as the weighting function is enlarged. Hence, we find a tradeoff between the accuracy in the pass-band and the sharpness of the cutoff frequency.

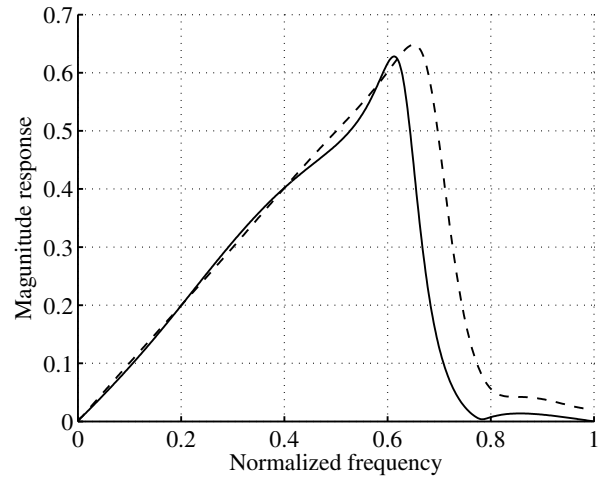
Furthermore, the poles are located within the specified circle due to the constraint of the positive realness with prescribed pole radius constraint. We confirm the proposed method can place the poles within the specified circle.

## VI. CONCLUSION

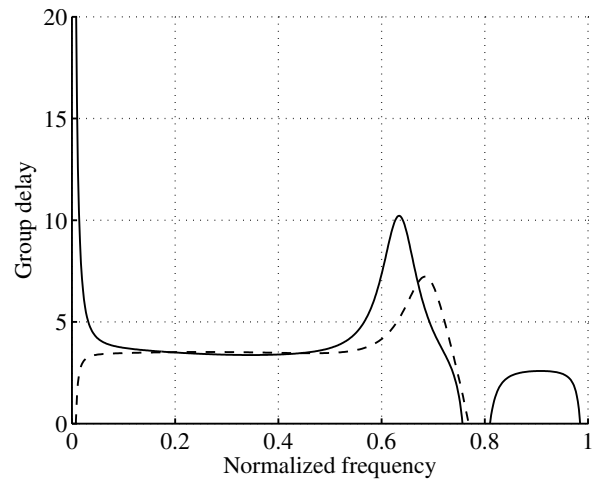
We have proposed the design algorithm for IIR low-pass differentiators based on the positive realness. The coefficients of the low-pass differentiators can be computed by solving the constrained quadratic problem. Since the proposed algorithm can specify the frequency-weighting in the pass-band frequency, the accuracy of the frequency response in the specified frequency can be adjusted. From the numerical examples, we show that the magnitude response and group delay of the low-pass differentiators are both well approximated, and the poles are located within the specified circle. Hence, we confirm the effectiveness of the proposed method.

## ACKNOWLEDGMENT

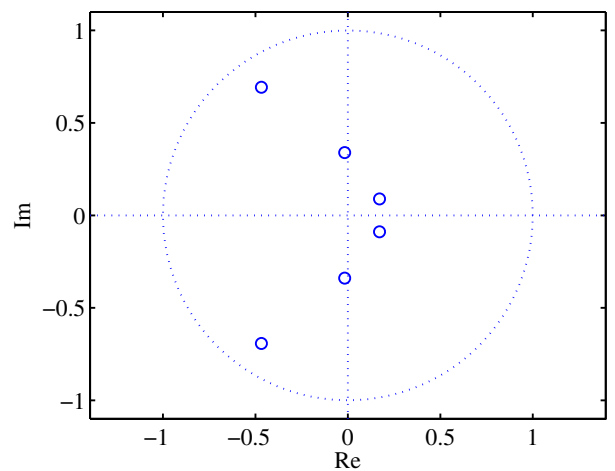
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(a) Magnitude response

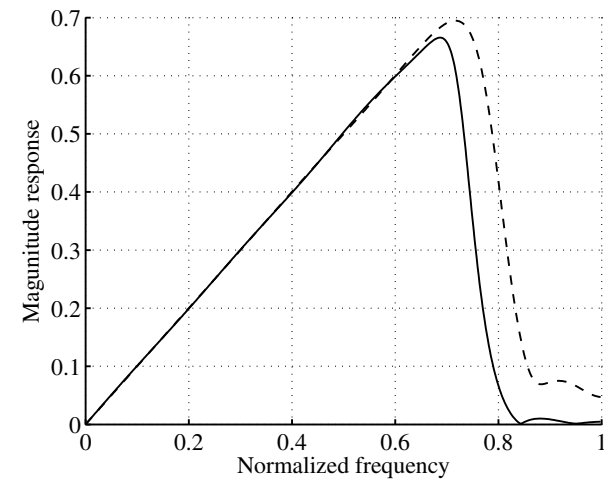


(b) Phase response

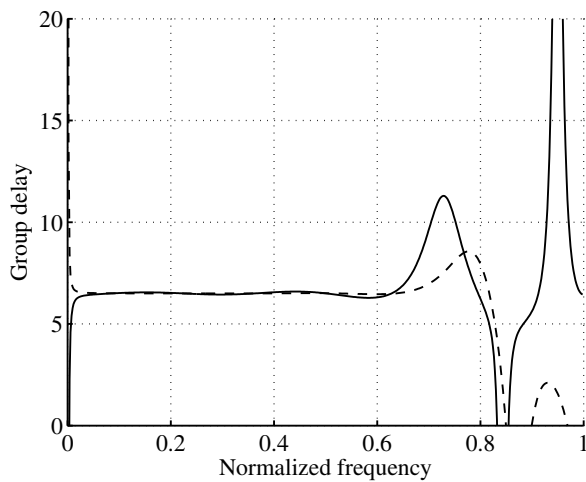


(c) Location of poles ( $W_p = 100$ )

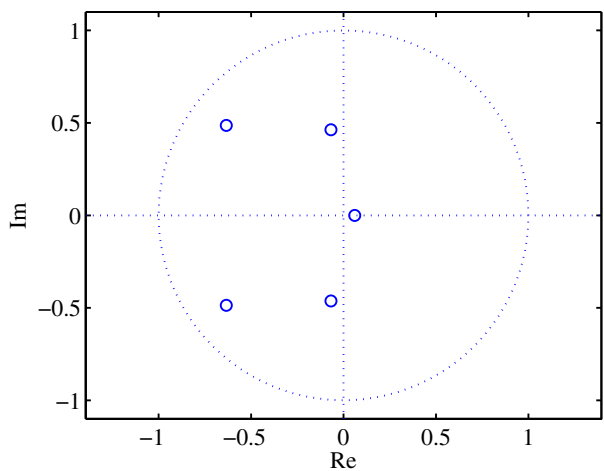
Fig. 3. The characteristics of the low-pass differentiator in Example 1.



(a) Magnitude response



(b) Phase response



(c) Location of poles ( $W_p = 100$ )

Fig. 4. The characteristics of the low-pass differentiator in Example 2.

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