Nash Game Approach of H_2/H_∞ Control for Stochastic Discrete-Time Systems with $(x,\ u,\ v)$-Dependent Noise*

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***Abstract*— H_2/H_∞ control problem for a class of stochastic discrete-time linear systems with state, control and external disturbance dependent noise or $(x,\ u,\ v)$-dependent noise involving multiple decision makers, which is very valuable in practice is investigated. It is shown that the strategy set can be obtained by the solutions of cross-coupled stochastic algebraic Riccati equations (CSAREs). In order to solve these equations, some new algorithms are given. Numerical example is given to demonstrate the effectiveness of the novel strategy set.**

I. INTRODUCTION

With the maturity of Nash games, stochastic H_2/H_∞ control problems for a class of discrete-time system have been investigated (see, e.g., [1], [2]). The finite horizon mixed H_2/H_∞ control problem was considered in [1]. These results were extended to the infinite horizon case [2]. It should be noted that the existing results are based on LQ control and H_∞ control technique for continuous and discrete-time stochastic systems [3], [4], [5], [7], [8]. When the strategy set can be designed, cross-coupled stochastic algebraic Riccati equations (CSAREs) have to be solved. However, little attention has been given to a numerical algorithm for solving these nonlinear matrix algebraic equations.

Recently, H_2/H_∞ control problems involving multiple decision makers for discrete-time stochastic systems with state and disturbance dependent noise have been solved [9]. Moreover, the numerical algorithm via linear matrix inequality (LMI) that is based on semidefinite programming (SDP) has been developed. However, the convergence proof has not been investigated up to now even if the proposed algorithm can be worked well in practice. Therefore, it is important to investigate a new numerical algorithm that guarantees a convergence.

In this paper, infinite-horizon H_2/H_∞ control problem with multiple decision makers for a class of discrete-time linear stochastic systems is investigated. As compared with the existing results in [9], state-, control- and disturbance-dependent noise or $(x,\ u,\ v)$-dependent noise is considered. It should be noted that since the existence of control-input dependent noise has been verified [11], the consideration of the control-dependent noise is very important. After establishing the existence conditions that consist of CSAREs, Newton's method is applied to solve these high-order nonlinear matrix equations. As a result, if the initial conditions are appropriately chosen, quadratic convergence is guaranteed. Moreover, in order to avoid the complicated derivation of Newton's method, an algorithm based on LMI is given. Finally, a simple numerical example is given to demonstrate the efficiency of the proposed scheme.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\mathbf{E}[\cdot]$ denotes the expectation operator. δ_{ij} denotes the Kronecker delta. $\mathbf{Tr}$ denotes the trace of a matrix. $\otimes$ denotes the Kronecker product. U_{lm} denotes a permutation matrix in the Kronecker matrix sense such that $U_{lm}\mathrm{vec}M = \mathrm{vec}M^T,\ M \in \Re^{l\times m}$. The l^2-norm of $y(k) \in l^2_w(\mathbf{N},\ \Re^n)$ is defined by $\|y(k)\|^2_{l^2_w(\mathbf{N},\ \Re^n)} := \sum_{k=0}^{\infty} \mathbf{E}[\|y(k)\|^2]$.

II. PRELIMINARY RESULTS

Consider the following discrete-time stochastic system.

$$x(k+1) = Ax(k) + B_1u(k) + [A_px(k) + B_{1p}u(k)]w(k), \quad (1a)$$

$$y(k) = Cx(k), \quad (1b)$$

where $x(k) \in \Re^n$ represents the state vector. $u(k) \in \Re^m$ represents the control input. $y(k) \in \Re^l$ represents the system output. $w(k) \in \Re$ is a one-dimensional sequence of real random process defined in the filtered probability space, which is a wide sense stationary, second-order process with $\mathbf{E}[w(k)] = 0$ and $\mathbf{E}[w(s)w(k)] = \delta_{st}$ [2], [3].

Let us consider the following stochastic linear quadratic (LQ) control problem subject to (1):

$$\text{minimize } J(u) := \sum_{k=0}^{\infty} \mathbf{E}[x^T(k)Qx(k) + u^T(k)Ru(k)],$$
$$Q = Q^T \geq 0,\ R = R^T > 0. \quad (2)$$

The following lemma plays a key technical role in this paper [2], [3].

Lemma 1: Assume that for any $u(k)$, the closed-loop system is mean square stable. Suppose that the following stochastic algebraic Riccati equation (SARE) has a solution $Y = Y^*$.

$$-Y + A^TYA + A_p^TYA_p + Q - \boldsymbol{L}^T\boldsymbol{R}^{-1}\boldsymbol{L} = 0, \quad (3)$$

where $\boldsymbol{R} := R + B_1^TYB_1 + B_{1p}^TYB_{1p}$ and $\boldsymbol{L} := B_1^TYA + B_{1p}^TYA_p$.

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Then, an optimal feedback control is given by

$$u^*(k) = K^* x(k) = -\boldsymbol{R}^{-1}\boldsymbol{L}x(k), \quad (4)$$

where $J(u^*) = x^T(0)Y^*x(0)$.

As another important method, the feedback gain K^* can be obtained by solving the following semidefinite programming (SDP) [6].

$$\text{maximize} \quad \mathbf{Tr}\ [Y], \quad (5a)$$

subject to

$$\begin{bmatrix} -Y + A^TYA + A_p^TYA_p + Q & \boldsymbol{L}^T \\ \boldsymbol{L} & \boldsymbol{R} \end{bmatrix} \geq 0,\ Y > 0. \quad (5b)$$

Moreover, Y^* is a maximal solution, which is the unique optimal solution.

To this end, we consider the following system

$$x(k+1) = Ax(k) + Bv(k) + \Big[A_px(k) + B_pv(k)\Big]w(k), \quad (6a)$$

$$z(k) = Cx(k),\ x(0) = x^0, \quad (6b)$$

where $v(k) \in \Re^{n_v}$ represents the external disturbance. $z(k) \in \Re^{n_z}$ represents the controlled output.

Definition 1: [2] Suppose that for any given $0 < T \in \mathbf{N}$, there exists a unique solution $x(k,\ 0,\ v) \in l_w^2(\mathbf{N}_{T+1},\ \Re^n)$ of (6) with initial value $x(0) = 0$. In the system (6), if the disturbance input $v(k) \in l_w^2(\mathbf{N},\ \Re^{n_v})$ and the controlled output $z(k) \in l_w^2(\mathbf{N},\ \Re^{n_z})$, then the perturbed operator $L : l_w^2(\mathbf{N},\ \Re^{n_v}) \to l_w^2(\mathbf{N},\ \Re^{n_z})$ is defined by

$$Lv(k) := Cx(k,\ 0,\ v),\ \forall v(k) \in l_w^2(\mathbf{N},\ \Re^{n_v}),\ x(0) = 0$$

with its norm

$$\|L\|^2 := \sup_{\substack{v(k)\,\in\, l_w^2(\mathbf{N},\ \Re^{n_v}),\\ v(k)\neq 0,\ x^0 = 0}} \frac{\|z(k)\|^2_{l_w^2(\mathbf{N},\ \Re^{n_z})}}{\|v(k)\|^2_{l_w^2(\mathbf{N},\ \Re^{n_v})}}$$
$$= \sup_{\substack{v(k)\,\in\, l_w^2(\mathbf{N},\ \Re^{n_v}),\\ v(k)\neq 0,\ x^0 = 0}} \frac{\mathbf{E}[\|Cx(k)\|^2_{l_w^2(\mathbf{N},\ \Re^{n_z})}]}{\mathbf{E}[\|v(k)\|^2_{l_w^2(\mathbf{N},\ \Re^{n_v})}]}. \quad (7)$$

The following lemma can be viewed as the discrete version.

Lemma 2: [2] If the stochastic system (6) is internally stable and $\|L\| < \gamma$ for given $\gamma > 0$, then there exists a stabilizing solution $X \leq 0$ to the following SARE

$$-X + A^TXA + A_p^TXA_p - C^TC - \boldsymbol{L}_\gamma^T\boldsymbol{R}_\gamma^{-1}\boldsymbol{L}_\gamma = 0, \quad (8)$$

where $\boldsymbol{R}_\gamma := \gamma^2 I_{n_v} + B^TXB + B_p^TXB_p$, $\boldsymbol{L}_\gamma := B^TXA + B_p^TXA_p$, $(A + BF_\gamma,\ A_p + B_pF_\gamma)$ is stable with

$$F_\gamma = -\boldsymbol{R}_\gamma^{-1}\boldsymbol{L}_\gamma. \quad (9)$$

Conversely, if (6) is internally stable and (8) has a stabilizing solution $X \leq 0$, then $\|L\| < \gamma$.

On the other hand, the gain F^* can also be obtained by solving the following SDP.

$$\text{maximize} \quad \mathbf{Tr}\ [X], \quad (10a)$$

subject to

$$\begin{bmatrix} -X + A^TXA + A_p^TXA_p - C^TC & \boldsymbol{L}_\gamma^T \\ \boldsymbol{L}_\gamma & \boldsymbol{R}_\gamma \end{bmatrix} \geq 0,\ X < 0. \quad (10b)$$

Moreover, X^* is a minimal solution, which is the unique optimal solution.

III. H_2/H_∞ CONTROL WITH MULTIPLE DECISION MAKERS

A. PROBLEM FORMULATION

Consider the stochastic linear discrete-time system with state-dependent noises, which involve N-decision makers

$$x(k+1) = Ax(k) + Bv(k) + \sum_{j=1}^{N} B_ju_j(k)$$
$$+ \Big[A_px(k) + B_pv(k) + \sum_{j=1}^{N} B_{pj}u_j(k)\Big]w(k),$$
$$x(0) = x^0, \quad (11a)$$

$$z_i(k) = \begin{bmatrix} C_ix(k) \\ D_iu_i(k) \end{bmatrix},\ z(k) = \begin{bmatrix} Cx(k) \\ D_1u_1(k) \\ \vdots \\ D_Nu_N(k) \end{bmatrix}, \quad (11b)$$

where $D_i^TD_i = I_{m_i}$, $C = \begin{bmatrix} C_1^T & \cdots & C_N^T \end{bmatrix}^T$, $u_i(k) \in l_w^2(\mathbf{N},\ \Re^{m_i})$, $i = 1,\ \dots,N$ represents the i-th control input. It should be noted that as compared with the existing result of [9], the controls u_i-dependent noise are considered.

Given a disturbance attenuation level $\gamma > 0$, define performance functions

$$J_0(u_1,\ \dots,u_N,\ v) := \sum_{k=0}^{\infty} \mathbf{E}[\gamma^2\|v(k)\|^2 - \|z(k)\|^2] \quad (12)$$

and

$$J_i(u_1,\ \dots,u_N,\ v) := \sum_{k=0}^{\infty} \mathbf{E}[\|z_i(k)\|^2],\ i = 1,\ \dots,N. \quad (13)$$

The infinite horizon stochastic H_2/H_∞ control with multiple decision makers of system (11) is stated as follows:

Given $\gamma > 0$, find if possible strategies $u_i^*(k) \in l_w^2(\mathbf{N},\ \Re^{m_i})$, $i = 1,\ \dots,N$ such that

i) $u_i^*(k)$ stabilizes system (11) internally.

ii) $\|L_{u_i^*}\|^2$

$$= \sup_{\substack{v(k)\,\in\, l_w^2(\mathbf{N},\ \Re^{n_v}),\\ v(k)\neq 0,\ x^0 = 0}} \frac{\sum_{k=0}^{\infty}\mathbf{E}\left[\|Cx(k)\|^2 + \sum_{j=1}^{N}\|u_j^*(k)\|^2\right]}{\sum_{k=0}^{\infty}\mathbf{E}[\|v(k)\|^2]}$$
$$< \gamma^2. \quad (14)$$

iii) When the worst case disturbance $v^*(k) \in l_w^2(\mathbf{N},\ \Re^{n_v})$, if exists, is implemented in (11), $u_i^*(k)$ minimizes the output energy

$$J_i(u_1,\ \dots,u_N,\ v^*) := \sum_{k=0}^{\infty}\mathbf{E}[\|z_i(k)\|^2]$$
$$= \sum_{k=0}^{\infty}\mathbf{E}[\|C_ix(k)\|^2 + \|u_i(k)\|^2],\ i = 1,\ \dots,N. \quad (15)$$

If the above $(u_1^*, \dots, u_N^*, v^*)$ exist, we say that the infinite horizon stochastic H_2/H_∞ control with multiple decision makers is solvable. Obviously, $(u_1^*, \dots, u_N^*, v^*)$ are the Nash equilibria of the two functionals (12) and (13), which satisfy

$$J_0(u_1^*, \dots, u_N^*, v^*) \le J_0(u_1^*, \dots, u_N^*, v), \quad (16a)$$
$$J_i(u_1^*, \dots, u_N^*, v^*) \le J_i(u_1^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_N^*, v^*), \quad i = 1, \dots, N. \quad (16b)$$

These equilibria are based on the Nash solutions and applied to the cases of multiple decision makers [9].

B. SOLUTION TO THE MULTI-OBJECTIVE MIXED H_2/H_∞ PROBLEM

First, a solution to the stochastic H_2/H_∞ control by means of cross-coupled stochastic algebraic Riccati equations (CSAREs) is given.

Theorem 1: For the discrete-time stochastic perturbed systems (11), suppose that the following CSAREs have solutions $(X, Y_1, \dots, Y_N, F, K_1, \dots, K_N)$ with $X < 0$ and $Y_i > 0$, $i = 1, \dots, N$.

$$-X + \boldsymbol{A}_{-F}^T X \boldsymbol{A}_{-F} + \boldsymbol{A}_{-pF}^T X \boldsymbol{A}_{-pF} - C^T C - \sum_{j=1}^{N} K_j^T K_j - \bar{\boldsymbol{L}}_\gamma^T \bar{\boldsymbol{R}}_\gamma^{-1} \bar{\boldsymbol{L}}_\gamma = 0, \quad (17a)$$
$$F = -\bar{\boldsymbol{R}}_\gamma^{-1} \bar{\boldsymbol{L}}_\gamma, \quad (17b)$$
$$-Y_i + \boldsymbol{A}_{-i}^T Y_i \boldsymbol{A}_{-i} + \boldsymbol{A}_{-pi}^T Y_i \boldsymbol{A}_{-pi} + C_i^T C_i - \hat{\boldsymbol{L}}_{-i}^T \hat{\boldsymbol{R}}_i^{-1} \hat{\boldsymbol{L}}_{-i} = 0, \quad (17c)$$
$$K_i = -\hat{\boldsymbol{R}}_i^{-1} \hat{\boldsymbol{L}}_{-i}, \; i = 1, \dots, N, \quad (17d)$$

where $\boldsymbol{A}_{-F} := A + \sum_{j=1}^N B_j K_j$, $\boldsymbol{A}_{-pF} := A_p + \sum_{j=1}^N B_{pj} K_j$, $\bar{\boldsymbol{L}}_\gamma := B^T X \boldsymbol{A}_{-F} + B_p^T X \boldsymbol{A}_{-pF}$, $\bar{\boldsymbol{R}}_\gamma := \gamma^2 I_{n_v} + B^T X B + B_p^T X B_p$, $\boldsymbol{A}_{-i} := A + BF + \sum_{j=1,\, j\neq i}^N B_j K_j$, $\boldsymbol{A}_{-pi} := A_p + B_p F + \sum_{j=1,\, j\neq i}^N B_{pj} K_j$, $\hat{\boldsymbol{L}}_{-i} := B_i^T Y_i \boldsymbol{A}_{-i} + B_{pi}^T Y_i \boldsymbol{A}_{-pi}$, $\hat{\boldsymbol{R}}_i := I_{m_i} + B_i^T Y_i B_i + B_{pi}^T Y_i B_{pi}$.

Define the set $(u_1^*, \dots, u_N^*)$ by

$$u_i^*(k) := K_i^* x(k) = -\hat{\boldsymbol{R}}_i^{-1} \hat{\boldsymbol{L}}_{-i} x(k), \; i = 1, \dots, N. \quad (18)$$

Then, this strategy set denotes the finite horizon H_2/H_∞ control.

Proof: Set $Z = -X$ and the equation (17a) yields:

$$-Z + \boldsymbol{A}_{-F}^T Z \boldsymbol{A}_{-F} + \boldsymbol{A}_{-pF}^T Z \boldsymbol{A}_{-pF} + C^T C + \sum_{j=1}^{N} K_j^T K_j + \check{\boldsymbol{L}}_\gamma^T \check{\boldsymbol{R}}_\gamma^{-1} \check{\boldsymbol{L}}_\gamma = 0, \quad (19)$$

where $\check{\boldsymbol{L}}_\gamma = B^T Z \boldsymbol{A}_{-F} + B_p^T Z \boldsymbol{A}_{-pF} = -\bar{\boldsymbol{L}}_\gamma$, $\check{\boldsymbol{R}}_\gamma = \gamma^2 I_{n_v} - B^T Z B - B_p^T Z B_p > 0$.

We rewrite (19) in the form

$$-Z + \boldsymbol{A}_{-F}^T Z \boldsymbol{A}_{-F} + \boldsymbol{A}_{-pF}^T Z \boldsymbol{A}_{-pF} + \boldsymbol{C}^T \boldsymbol{C} + \boldsymbol{C}_p^T \boldsymbol{C}_p = 0, \quad (20)$$

where $\boldsymbol{C} = \begin{pmatrix} U \\ \frac{1}{\sqrt{2}} \check{\boldsymbol{R}}_\gamma^{\frac{1}{2}} \check{F} \\ O_{n_v \times n} \end{pmatrix}$, $\boldsymbol{C}_p = \begin{pmatrix} O_{\rho \times n} \\ O_{n_v \times n} \\ \frac{1}{\sqrt{2}} \check{\boldsymbol{R}}_\gamma^{\frac{1}{2}} \check{F} \end{pmatrix}$ where $\rho = \mathbf{range}(C^T C + \sum_{j=1}^N K_j^T K_j)$, $U \in \Re^{\rho \times n}$ is obtained from the factorization $U^T U = C^T C + \sum_{j=1}^N K_j^T K_j$ and $\check{F} = \check{\boldsymbol{R}}_\gamma^{-1} \check{\boldsymbol{L}}_\gamma = -\bar{\boldsymbol{R}}_\gamma^{-1} \bar{\boldsymbol{L}}_\gamma = F$. By using the similar technique in [10], it is easy to prove that under the considered assumptions the system $(\boldsymbol{A}_{-F}, \boldsymbol{A}_{-pF} \mid \boldsymbol{C}, \boldsymbol{C}_p)$ is detectable. Hence, it is omitted.

Now, let us consider the following problem in which the cost function (21) is minimal at $K_i = K_i^*$.

$$\phi(F) := \sup_{v(k) \in l_w^2(\mathbf{N},\, \Re^{n_v})} \sum_{k=0}^{\infty} \mathbf{E}[\gamma^2 \|v(k)\|^2 - \|\hat{z}(k)\|^2], \quad (21)$$
$$\hat{z}(k) = \bar{\boldsymbol{C}} x(k) = \begin{bmatrix} C^T & (D_1 K_1^*)^T & \cdots & (D_N K_N^*)^T \end{bmatrix}^T x(k),$$

where $x(k)$ follows from

$$x(k+1) = \left(A + \sum_{j=1}^{N} B_j K_j^*\right) x(k) + B v(k) + \left[\left(A_p + \sum_{j=1}^{N} B_{pj} K_j^*\right) x(k) + B_p v(k)\right] w(k). \quad (22)$$

Note that the function ϕ coincides with function J_0 in Lemma 2. Applying Lemma 2 to this optimization problem as $X \Rightarrow P$, $\boldsymbol{A}_{-F} \Rightarrow A$, $\boldsymbol{A}_{-pF} \Rightarrow A_p$ and $\bar{\boldsymbol{C}} \Rightarrow C$, yields the fact that the function ϕ is minimal at

$$F_\gamma^* = -\boldsymbol{R}_\gamma^{-1} \boldsymbol{L}_\gamma \Rightarrow F^* = -\bar{\boldsymbol{R}}_\gamma^{-1} \bar{\boldsymbol{L}}_\gamma. \quad (23)$$

On the other hand, consider the following LQ problem.

$$\psi(K_i) := \min_{u_i(k) \in l_w^2(\mathbf{N},\, \Re^{m_i})} \sum_{k=0}^{\infty} \mathbf{E} \|z_i(k)\|^2 \quad (24)$$

and $x(k)$ follows from

$$x(k+1) = \left(A + BF^* + \sum_{j=1,\, j\neq i}^{N} B_j K_j\right) x(k) + B_i u_i(k) + \left[\left(A_p + B_p F^* + \sum_{j=1,\, j\neq i}^{N} B_{pj} K_j\right) x(k) + B_{pi} u_i(k)\right] w(k). \quad (25)$$

The function ψ coincides with function J_i in Lemma 1. Applying Lemma 1 to this optimization problem as $Y_i \Rightarrow P$, $\boldsymbol{A}_{-i} \Rightarrow A$, $\boldsymbol{A}_{-pi} \Rightarrow A_p$ and $\boldsymbol{A}_{-pi} \Rightarrow A_p$ yields the fact that the function ψ is minimal at

$$K^* = -\boldsymbol{R}^{-1} \boldsymbol{L} \Rightarrow K_i^* = -\hat{\boldsymbol{R}}_i^{-1} \hat{\boldsymbol{L}}_{-i}. \quad (26)$$

So $(u_1^*, \dots, u_N^*, v^*)$ solve the finite horizon H_2/H_∞ control problem of stochastic system (11). ■

IV. NUMERICAL ALGORITHM

In this section, two numerical algorithms are given. The first one is Newton's method and the other one is SDP algorithm.

A. Newton's method

First, CSAREs can be changed as follows.

$$
\begin{aligned}
&\boldsymbol{F}_X(X,\ F,\ K_1,\ \dots,K_N)\\
&= -X+\boldsymbol{A}^TX\boldsymbol{A}+\boldsymbol{A}_p^TX\boldsymbol{A}_p\\
&\quad +\gamma^2F^TF-C^TC-\sum_{j=1}^{N}K_j^TK_j=0, \qquad (27a)\\
&\boldsymbol{F}_{Y_i}(Y_i,\ F,\ K_1,\ \dots,K_N)\\
&= -Y_i+\boldsymbol{A}^TY_i\boldsymbol{A}+\boldsymbol{A}_p^TY_i\boldsymbol{A}_p\\
&\quad +C_i^TC_i+K_i^TK_i=0,\ i=1,\ \dots,N, \qquad (27b)\\
&\boldsymbol{F}_F(X,\ F,\ K_1,\ \dots,K_N)\\
&= \gamma^2F+B^TX\boldsymbol{A}+B_p^TX\boldsymbol{A}_p=0, \qquad (27c)\\
&\boldsymbol{F}_{K_i}(Y_i,\ F,\ K_1,\ \dots,K_N)\\
&= K_i+B_i^TY_i\boldsymbol{A}+B_{pi}^TY_i\boldsymbol{A}_p=0,\ i=1,\ \dots,N, \qquad (27d)
\end{aligned}
$$

where $\boldsymbol{A} := A+BF+\sum_{j=1}^{N}B_jK_j$, $\boldsymbol{A}_p := A_p+BF+\sum_{j=1}^{N}B_{pj}K_j$.

Let us define Newton's method as follows:

$$
\boldsymbol{x}^{(n+1)}=\boldsymbol{x}^{(n)}-[\boldsymbol{J}^{(n)}]^{-1}\boldsymbol{f}(\boldsymbol{x}^{(n)}), \qquad (28)
$$

where

$$
\begin{aligned}
\boldsymbol{f}(\boldsymbol{x}^{(n)}) &:= \Big[[\mathrm{vec}\boldsymbol{F}_X^{(n)}]^T\ \ [\mathrm{vec}\boldsymbol{F}_{Y_1}^{(n)}]^T\ \cdots\ [\mathrm{vec}\boldsymbol{F}_{Y_N}^{(n)}]^T\\
&\qquad [\mathrm{vec}\boldsymbol{F}_F^{(n)}]^T\ \ [\mathrm{vec}\boldsymbol{F}_{K_1}^{(n)}]^T\ \cdots\ [\mathrm{vec}\boldsymbol{F}_{K_N}^{(n)}]^T\Big]^T,\\
\boldsymbol{x}^{(n)} &:= \Big[[\mathrm{vec}X^{(n)}]^T\ \ [\mathrm{vec}Y_1^{(n)}]^T\ \cdots\ [\mathrm{vec}Y_N^{(n)}]^T\\
&\qquad [\mathrm{vec}F^{(n)}]^T\ \ [\mathrm{vec}K_1^{(n)}]^T\ \cdots\ [\mathrm{vec}K_N^{(n)}]^T\Big]^T,\\
\boldsymbol{J}^{(n)} &:= \boldsymbol{J}(X^{(n)},Y_1^{(n)},\ \dots,Y_N^{(n)},\ F^{(n)},\ K_1^{(n)},\ \dots,K_N^{(n)})
\end{aligned}
$$

$$
\boldsymbol{J}(X,\ Y_1,\ \dots,Y_N,\ F,\ K_1,\ \dots,K_N)
$$

$$
=\begin{bmatrix}
\bar{\boldsymbol{J}}_{00} & O & \cdots & O & \bar{\boldsymbol{J}}_{00}^F & \bar{\boldsymbol{J}}_{01}^{K_1} & \cdots & \bar{\boldsymbol{J}}_{0N}^{K_N}\\
O & \bar{\boldsymbol{J}}_{00} & \cdots & O & \bar{\boldsymbol{J}}_{10}^F & \bar{\boldsymbol{J}}_{11}^{K_1} & \cdots & \bar{\boldsymbol{J}}_{1N}^{K_N}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
O & O & \cdots & \bar{\boldsymbol{J}}_{00} & \bar{\boldsymbol{J}}_{N0}^F & \bar{\boldsymbol{J}}_{N1}^{K_1} & \cdots & \bar{\boldsymbol{J}}_{NN}^{K_N}\\
\hat{\boldsymbol{J}}_{00} & O & \cdots & O & \hat{\boldsymbol{J}}_{00}^F & \hat{\boldsymbol{J}}_{01}^{K_1} & \cdots & \hat{\boldsymbol{J}}_{0N}^{K_N}\\
O & \hat{\boldsymbol{J}}_{11} & \cdots & O & \hat{\boldsymbol{J}}_{10}^F & \hat{\boldsymbol{J}}_{11}^{K_1} & \cdots & \hat{\boldsymbol{J}}_{1N}^{K_N}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
O & O & \cdots & \hat{\boldsymbol{J}}_{NN} & \hat{\boldsymbol{J}}_{N0}^F & \hat{\boldsymbol{J}}_{N1}^{K_1} & \cdots & \hat{\boldsymbol{J}}_{NN}^{K_N}
\end{bmatrix},
$$

$$
\begin{aligned}
\boldsymbol{J}_{00} &:= -I_n\otimes I_n+\boldsymbol{A}^T\otimes\boldsymbol{A}^T+\boldsymbol{A}_p^T\otimes\boldsymbol{A}_p^T,\\
\bar{\boldsymbol{J}}_{00}^F &:= I_n\otimes(\boldsymbol{A}^TXB)+[(\boldsymbol{A}^TXB)\otimes I_n]U_{nn_v}\\
&\quad +I_n\otimes(\boldsymbol{A}_p^TXB_p)+[(\boldsymbol{A}_p^TXB_p)\otimes I_n]U_{nn_v}\\
&\quad +\gamma^2(I_n\otimes F^T+[F^T\otimes I_n]U_{nn_v}),\\
\bar{\boldsymbol{J}}_{0i}^{K_i} &:= I_n\otimes(\boldsymbol{A}^TXB_i)+[(\boldsymbol{A}^TXB_i)\otimes I_n]U_{nm_i}\\
&\quad +I_n\otimes(\boldsymbol{A}_p^TXB_{pi})+[(\boldsymbol{A}_p^TXB_{pi})\otimes I_n]U_{nm_i}\\
&\quad -(I_n\otimes K_i^T+[K_i^T\otimes I_n]U_{nm_i}),\\
\bar{\boldsymbol{J}}_{i0}^F &:= I_n\otimes(\boldsymbol{A}^TY_iB)+[(\boldsymbol{A}^TY_iB)\otimes I_n]U_{nn_v}\\
&\quad +I_n\otimes(\boldsymbol{A}_p^TY_iB_p)+[(\boldsymbol{A}_p^TY_iB_p)\otimes I_n]U_{nn_v},\\
\bar{\boldsymbol{J}}_{ii}^{K_i} &:= I_n\otimes(\boldsymbol{A}^TY_iB_i)+[(\boldsymbol{A}^TY_iB_i)\otimes I_n]U_{nm_i}\\
&\quad +I_n\otimes(\boldsymbol{A}_p^TY_iB_{pi})+[(\boldsymbol{A}_p^TY_iB_{pi})\otimes I_n]U_{nm_i}\\
&\quad +I_n\otimes K_i^T+[K_i^T\otimes I_n]U_{nm_i},\\
\bar{\boldsymbol{J}}_{ij}^{K_j} &:= I_n\otimes(\boldsymbol{A}^TY_iB_j)+[(\boldsymbol{A}^TY_iB_j)\otimes I_n]U_{nm_i}\\
&\quad +I_n\otimes(\boldsymbol{A}_p^TY_iB_{pj})+[(\boldsymbol{A}_p^TY_iB_{pj})\otimes I_n]U_{nm_i},\\
\hat{\boldsymbol{J}}_{00} &:= \boldsymbol{A}^T\otimes B^T+\boldsymbol{A}^T\otimes B_p^T,\ \ \hat{\boldsymbol{J}}_{00}^F=I_n\otimes\bar{\boldsymbol{R}}_\gamma,\\
\hat{\boldsymbol{J}}_{0i}^{K_i} &:= I_n\otimes(B^TXB_i)+I_n\otimes(B_p^TXB_{pi}),\\
\hat{\boldsymbol{J}}_{ii} &:= \boldsymbol{A}^T\otimes B_i^T+\boldsymbol{A}_p^T\otimes B_{pi}^T,\\
\hat{\boldsymbol{J}}_{i0}^F &:= I_n\otimes(B_i^TY_iB+B_{pi}^TY_iB_p),\\
\hat{\boldsymbol{J}}_{ii}^{K_i} &:= I_n\otimes(I_{m_i}+B_i^TY_iB_i+B_{pi}^TY_iB_{pi}),\\
\hat{\boldsymbol{J}}_{ij}^{K_i} &:= I_n\otimes(B_i^TY_iB_j+B_{pi}^TY_iB_{pj}),\ i\neq j.
\end{aligned}
$$

It should be noted that the initial conditions are chosen such that the closed loop stochastic systems are internally stable. On the other hand, it should be noted that the choice of initial conditions is very important because the inappropriate ones yield other solutions or divergence. In order to obtain the appropriate initial conditions, the inverse-time computing [1] would be available. Furthermore, if $[\boldsymbol{J}^{(n)}]^{-1}$ does not exist, the inverse-time computing can also be used instead of Newton's method.

B. SDP ALGORITHM

The SDP algorithm for obtaining the strategy set is given below.

Step 1. As the initialization procedure, solve the following two types of SDPs independently.

$$
\begin{aligned}
&\text{maximize}\ \ \mathbf{Tr}\ [Y_i^{(0)}], \qquad (29a)\\
&\text{subject to}\ \begin{bmatrix}\Phi^{(0)} & \hat{\boldsymbol{L}}_{-i}^{(0)T}\\ \hat{\boldsymbol{L}}_{-i}^{(0)} & \hat{\boldsymbol{R}}_i^{(0)}\end{bmatrix}\geq 0,\ i=1,\ \dots,N, \qquad (29b)\\
&\text{maximize}\ \ \mathbf{Tr}\ [X^{(0)}], \qquad (30a)\\
&\text{subject to}\ \begin{bmatrix}\Psi^{(0)} & \bar{\boldsymbol{L}}^{(0)T}\\ \bar{\boldsymbol{L}}^{(0)} & \bar{\boldsymbol{R}}_\gamma^{(0)}\end{bmatrix}\geq 0,\ X^{(0)}\leq 0, \qquad (30b)
\end{aligned}
$$

where $\Phi^{(0)} := -Y_i^{(0)}+A^TY_i^{(0)}A+A_p^TY_i^{(0)}A_p+C_i^TC_i$, $\hat{\boldsymbol{L}}_{-i}^{(0)} := B_i^TY_i^{(0)}A+B_{pi}^TY_i^{(0)}A_p$, $\hat{\boldsymbol{R}}_i^{(0)} := I_{m_i}+B_i^TY_i^{(0)}B_i+B_{pi}^TY_i^{(0)}B_{pi}$, $\Psi^{(0)} := -X^{(0)}+A^TX^{(0)}A+A_p^TX^{(0)}A_p-C^TC$, $\bar{\boldsymbol{L}}^{(0)} := B^TX^{(0)}A+B_p^TX^{(0)}A_p$, $\bar{\boldsymbol{R}}_\gamma^{(0)} := \gamma I_{n_v}+B^TX^{(0)}B+B_p^TX^{(0)}B_p$.

Step 2. Set $K_i^{(0)}=-[\hat{\boldsymbol{R}}_i^{(0)}]^{-1}\hat{\boldsymbol{L}}_{-i}^{(0)}$ and $X^{(0)} := -[\bar{\boldsymbol{R}}_\gamma^{(0)}]^{-1}\bar{\boldsymbol{L}}^{(0)}$.

Step 3. Solve the following two SDPs independently.

$$\text{maximize} \quad \mathbf{Tr}\ [Y_i^{(k+1)}], \tag{31a}$$

$$\text{subject to} \begin{bmatrix} \Phi^{(k)} & \hat{\boldsymbol{L}}_{-i}^{(k)T} \\ \hat{\boldsymbol{L}}_{-i}^{(k)} & \hat{\boldsymbol{R}}_i^{(k)} \end{bmatrix} \geq 0,\ i=1,\ \dots,N, \tag{31b}$$

where $\Phi^{(k)} := -Y_i^{(k+1)} + \boldsymbol{A}_{-i}^{(k)T} Y_i^{(k+1)} \boldsymbol{A}_{-i}^{(k)} + \boldsymbol{A}_{-pi}^{(k)T} Y_i^{(k+1)} \boldsymbol{A}_{-pi}^{(k)T} + C_i^T C_i$, $\boldsymbol{A}_{-i}^{(k)} := A + BF^{(k)} + \sum_{j=1,\ j\neq i}^{N} B_j K_j^{(k)}$, $\boldsymbol{A}_{-pi}^{(k)} := A_p + B_p F^{(k)} + \sum_{j=1,\ j\neq i}^{N} B_{pj} K_j^{(k)}$, $\hat{\boldsymbol{L}}_{-i}^{(k)} := B_i^T Y_i^{(k+1)} \boldsymbol{A}_{-i}^{(k)} + B_{pi}^T Y_i^{(k+1)} \boldsymbol{A}_{-pi}^{(k)}$, $\hat{\boldsymbol{R}}_i^{(k)} := I_{m_i} + B_i^T Y_i^{(k+1)} B_i + B_{pi}^T Y_i^{(k+1)} B_{pi}$.

$$\text{maximize} \quad \mathbf{Tr}\ [X^{(k)}], \tag{32a}$$

$$\text{subject to} \begin{bmatrix} \Psi^{(k)} & \bar{\boldsymbol{L}}^{(k)T} \\ \bar{\boldsymbol{L}}^{(k)} & \bar{\boldsymbol{R}}_\gamma^{(k)} \end{bmatrix} \geq 0,\ X^{(k)} \leq 0, \tag{32b}$$

where $\Psi^{(k)} := -X^{(k+1)} + \boldsymbol{A}_{-F}^{(k)T} X^{(k+1)} \boldsymbol{A}_{-F}^{(k)} + \boldsymbol{A}_{-pF}^{(k)T} X^{(k+1)} \boldsymbol{A}_{-pF}^{(k)} - C^T C$, $\boldsymbol{A}_{-F}^{(k)} := A + \sum_{j=1}^{N} B_j K_j^{(k)}$, $\boldsymbol{A}_{-pF}^{(k)} := A_p + \sum_{j=1}^{N} B_{pj} K_j^{(k)}$, $\bar{\boldsymbol{L}}^{(k)} := B^T X^{(k+1)} \boldsymbol{A}_{-F}^{(k)} + B_p^T X^{(k+1)} \boldsymbol{A}_{-pF}^{(k)}$, $\bar{\boldsymbol{R}}_\gamma^{(k)} := \gamma I_{n_v} + B^T X^{(k+1)} B + B_p^T X^{(k+1)} B_p$.

Step 4. Set $K_i^{(k+1)}$ as follows.

$$K_i^{(k+1)} = -[\hat{\boldsymbol{R}}_i^{(k)}]^{-1} \hat{\boldsymbol{L}}_{-i}^{(k)},\ F^{(k+1)} = -[\bar{\boldsymbol{R}}_\gamma^{(k)}]^{-1} \bar{\boldsymbol{L}}^{(k)}. \tag{33}$$

Step 5. If the algorithm converges, then $X^{(k)} \to X$, $Y_i^{(k)} \to Y_i$ as $k \to \infty$, where Y_i is the solution of CSAREs (17c), STOP. Otherwise, increment $k \to k+1$ and go to Step 3. If the algorithm does not converge, the proposed algorithm would be fail.

V. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of our proposed strategies, a simple numerical example is investigated. The corresponding matrices are given below.

$$\gamma = 2,\ A = \begin{bmatrix} 0.5 & 1 \\ 0 & -0.5 \end{bmatrix},\ A_p = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix},\ B_p = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix},$$
$$B_{p1} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix},\ B_{p2} = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix},\ B_{p3} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix},$$
$$C_1 = I_2,\ C_2 = 0.5 I_2,\ C_3 = 2 I_2,\ D_1 = D_2 = D_3 = 1.$$

By solving the corresponding CSAREs (17), we obtain the linear state feedback strategies and the solutions.

$$F = \begin{bmatrix} 9.5868\text{e}{-3} & -9.4227\text{e}{-3} \end{bmatrix},$$
$$K_1 = \begin{bmatrix} -1.1225\text{e}{-1} & -6.4417\text{e}{-2} \end{bmatrix}.$$
$$K_2 = \begin{bmatrix} -1.1268\text{e}{-2} & 4.4757\text{e}{-2} \end{bmatrix},$$
$$K_3 = \begin{bmatrix} -3.0192\text{e}{-1} & -7.8399\text{e}{-1} \end{bmatrix},$$
$$X = \begin{bmatrix} -1.1137 & -2.5414\text{e}{-1} \\ -2.5414\text{e}{-1} & -1.6519 \end{bmatrix},$$
$$Y_1 = \begin{bmatrix} 1.0209 & 1.8549\text{e}{-2} \\ 1.8549\text{e}{-2} & 1.0349 \end{bmatrix}.$$
$$Y_2 = \begin{bmatrix} 2.5218\text{e}{-1} & 2.2935\text{e}{-3} \\ 2.2935\text{e}{-3} & 2.5970\text{e}{-1} \end{bmatrix},$$
$$Y_3 = \begin{bmatrix} 4.1258 & 2.8051\text{e}{-1} \\ 2.8051\text{e}{-1} & 4.7379 \end{bmatrix}.$$

It is easy to verify that these strategies satisfy the multi-objective control purpose, respectively.

Table 1. Error Per Iterations.

n	$\|\mathcal{E}^{(n)}\|$
0	$1.6843e+01$
1	$1.3845e-01$
2	$9.5408e-04$
3	$2.7064e-09$
4	$2.2025e-15$

Second, the convergence property is confirmed. It is verified that the solution of the CSAREs (17) converges to the exact solution with accuracy of $\mathcal{E} < 10^{-11}$ after four iterations, where the function $\mathcal{E}$ is defined as follows:

$$\begin{aligned} &\mathcal{E}^{(n)} \\ &= \|\boldsymbol{F}_X(X^{(n)},\ F^{(n)},\ K_1^{(n)},\ \dots,K_N^{(n)})\| \\ &\quad + \sum_{j=1}^{3} \|\boldsymbol{F}_{Y_j}(Y_j^{(n)},\ F^{(n)},\ K_1^{(n)},\ \dots,K_N^{(n)})\| \\ &\quad + \|\boldsymbol{F}_F(X^{(n)},\ F^{(n)},\ K_1^{(n)},\ \dots,K_N^{(n)})\| \\ &\quad + \sum_{j=1}^{3} \|\boldsymbol{F}_{K_j}(Y_j^{(n)},\ F^{(n)},\ K_1^{(n)},\ \dots,K_N^{(n)})\|. \end{aligned}$$

In order to verify the exactness of the solution, the remainder per iteration by substituting solutions $X^{(n)}$, $Y_i^{(n)}$, $F^{(n)}$, $K_1^{(n)}$, ... , $K_N^{(n)}$ into the (17) is computed. It can be verified from Table 1 that Newton's method (28) generates the quadratic convergence.

On the other hand, LMI approach that is based on SDP needs 52 iterations. Although the convergence of the SDP algorithm is not guaranteed, the resulting algorithm is also reliable because the required computation work space is small.

VI. CONCLUSION

Infinite-horizon H_2/H_∞ control with multiple decision makers for discrete-time stochastic system has been studied. Particularly, as the extension of the existing H_2/H_∞

control problem that has been investigated in [9], $(x,\ u,\ v)$-dependent noise was considered. Moreover, in order to guarantee the convergence, Newton's method was applied to CSAREs. As a result, it has been shown that the local quadratic convergence was guaranteed. It is well known that the Newton's method seems to be classical. Hence, although the proposed algorithm does not seem novel, it is worth pointing out that fast convergence is attained under assumption that the appropriate initial guesses are chosen. Furthermore, SDP algorithm was also established to reduce the computational work space. A numerical example has shown the validity of the proposed scheme.

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