Stackelberg Game Approach of H_2/H_∞ Control for Weakly Coupled Stochastic Systems*

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***Abstract*— In this paper, H_2/H_∞ control problem for linear weakly coupled stochastic system with state and external disturbance dependent noise involving multiple decision makers is investigated. After defining strategy set that is based on Stackelberg game, the existence conditions of the proposed strategy set that consists of cross-coupled algebraic nonlinear matrix equations (CANMEs) with small parameter are established. The asymptotic structure of CANMEs is shown for the first time. Particularly, the leader's strategy set are chosen as Pareto strategy set. In order to obtain a solution set, a numerical algorithm is discussed. A simple numerical example is given to demonstrate the efficiency of the proposed method.**

I. INTRODUCTION

The stochastic dynamic games for both continuous-time and discrete-time systems have been widely studied [1], [2], [3], [4]. Recently, it is well known that Stackelberg strategy is important hierarchical strategy [8], [9], [10]. It has been shown that the solution of Stackelberg strategy involves a hierarchical combination of some optimization problems [5], [11]. In order to obtain strategy set, algebraic cross-coupled nonlinear matrix equations should be solved. However, a little attention was paid to numerical approaches for solving such equations [11], [12].

H_2/H_∞ control [6], [7] for a class of stochastic systems against disturbance has attracted considerable attention and is now widely applied to various practical fields. These results are based on the Nash solutions. However, it should be pointed out that limited results are useful for one player case. On the other hand, when multiple decision makers exist, infinite horizon H_2/H_∞ control problems for stochastic and Markov jump linear stochastic systems were considered [11], [12]. Although these results have been contributed to infinite horizon H_2/H_∞ control design with multiple decision makers, weakly-coupled linear systems are not investigated. Although over the past decade, weakly-coupled large scale stochastic systems have been extensively investigated (see [13] and reference therein), the numerical algorithm for solving large-scale equations does not seem enough.

This paper investigates H_2/H_∞ control problem for a class of continuous-time weakly-coupled linear systems with state-dependent noise, which are expressed by the Itô stochastic differential equations. It is noteworthy that earlier studies on the theory for the hierarchical strategy set did not take into consideration the large-scale systems. The main contributions of this paper are as follows. First, stochastic Stackelberg games are introduced by utilizing the existing results [12]. It should be noted that the considered problem is an extension of own result [12] in the sense that weakly-coupled systems are investigated. In order to obtain a strategy set, cross-coupled algebraic nonlinear matrix equations (CANMEs) with small coupling parameter ε are formulated. After introducing an asymptotic structure with positive definiteness for CANMEs solutions, a novel numerical algorithm for solving CANMEs is discussed. Finally, in order to demonstrate the effectiveness of the proposed scheme, a simple numerical example is provided.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\mathbf{E}[\cdot]$ denotes the expectation operator. δ_{ij} denotes the Kronecker delta. $\mathbf{Tr}$ denotes the trace of a matrix. **block diag** denotes a block diagonal matrix. $\otimes$ denotes the Kronecker product. U_{lm} denotes a permutation matrix in the Kronecker matrix sense such that $U_{lm}\mathrm{vec}M = \mathrm{vec}M^T$, $M \in \Re^{l\times m}$. $L^2_F([0,\ \infty),\ \Re^k)$ denotes the space of non-anticipative stochastic processes $u(t) \in \Re^k$ with respect to an increasing σ-algebras F_t, $t \geq 0$ satisfying $\mathbf{E}[\int_0^\infty \|u(t)\|^2 dt] < \infty$.

II. PROBLEM FORMULATION

Consider a linear weakly couple stochastic system governed by Itô differential equation with multiple decision makers defined by

$$dx(t) = \left[A_\varepsilon x(t) + \sum_{j=1}^{2} B_{j\varepsilon} u_j(t) + B_{0\varepsilon} v(t)\right] dt + C_\varepsilon x(t) dw(t), \tag{1a}$$

$$z(t) = E_\varepsilon x(t) + \sum_{j=1}^{2} G_{j\varepsilon} u_j(t), \tag{1b}$$

where $x(t) \in \Re^n$, $x(0) = x^0$ represents the state vector. $u_i(t) \in \Re^{m_i}$, $i = 1,\ 2$ represent the i-th control inputs. $v(t) \in \Re^{n_v}$ represents the external disturbance. $z(t) \in L^2_F([0,\ \infty),\ \Re^{n_z})$ represents the controlled output. $w(t) \in \Re$

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is a one-dimensional standard Wiener process defined in the filtered probability space [6], [7].

Moreover, let us define the following matrices.

$$x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ x_i(t) \in \Re^{n_i},\ n = n_1 + n_2,$$

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{bmatrix},\ C_\varepsilon := \begin{bmatrix} C_{11} & \varepsilon C_{12} \\ \varepsilon C_{21} & C_{22} \end{bmatrix},$$

$$B_{1\varepsilon} := \begin{bmatrix} B_{111} \\ \varepsilon B_{121} \end{bmatrix},\ B_{2\varepsilon} := \begin{bmatrix} \varepsilon B_{212} \\ B_{222} \end{bmatrix},$$

$$B_{0\varepsilon} := \begin{bmatrix} B_{011} & \varepsilon B_{012} \\ \varepsilon B_{021} & B_{022} \end{bmatrix},\ E_\varepsilon := \begin{bmatrix} E_{11} & \varepsilon E_{12} \\ \varepsilon E_{12}^T & \varepsilon E_{22} \end{bmatrix},$$

$$G_{1\varepsilon} := \begin{bmatrix} G_{111} \\ \varepsilon G_{121} \end{bmatrix},\ G_{2\varepsilon} := \begin{bmatrix} \varepsilon G_{212} \\ G_{222} \end{bmatrix}.$$

To simplify our discussion, the following assumption is made.

Assumption 1: **(i)** $E_\varepsilon^T G_{i\varepsilon} = 0$, $i = 1,\ 2$.

(ii) $G_{i\varepsilon}^T G_{j\varepsilon} = 0$, $i \neq j$.

(iii) $G_{i\varepsilon}^T G_{i\varepsilon} = I_{m_i}$, $i = 1,\ 2$.

The cost performances are defined by

$$\begin{aligned} &J_0(u_1,\ u_2,\ v,\ x^0) \\ &= \mathbf{E}\left[\int_0^\infty [\gamma^2\|v(t)\|^2 - \|z(t)\|^2]dt\right], \quad (2a) \\ &J_i(u_1,\ u_2,\ v,\ x^0) \\ &= \mathbf{E}\left[\int_0^\infty [x^T(t)Q_{i\varepsilon}x(t) + \|u_i(t)\|^2]dt\right], \quad (2b) \end{aligned}$$

where

$$Q_{1\varepsilon} := \begin{bmatrix} Q_{111} & \varepsilon Q_{112} \\ \varepsilon Q_{112}^T & \varepsilon Q_{122} \end{bmatrix},\ Q_{2\varepsilon} := \begin{bmatrix} \varepsilon Q_{211} & \varepsilon Q_{212} \\ \varepsilon Q_{212}^T & Q_{222} \end{bmatrix}.$$

It should be noted that a given γ is chosen by the controller designer.

We consider the hierarchical optimization problem that is based on Pareto optimality.

Each player wants to minimize his own cost described in (2b). As the definition of Pareto efficient solution [4], let us combine the individual cost functions in (2b) into a team cost function according to the following.

$$\begin{aligned} &J_\rho(u_1,\ u_2,\ v,\ x^0) := \sum_{j=1}^2 \rho_j J_j(u_1,\ u_2,\ v,\ x^0) \\ &\rho_1 + \rho_2 = 1,\ 0 < \rho_i < 1,\ i = 1,\ 2. \quad (3) \end{aligned}$$

It is well known that a Pareto solution is a set $(u_1,\ u_2)$, which minimizes $J_\rho(u_1,\ u_2,\ v,\ x^0)$ for any $v = v(t)$. Moreover, the following inequality holds.

$$\begin{aligned} &J_0(u_1,\ u_2,\ v^0(u_1,\ u_2),\ x^0) \\ &= \min_v\ J_0(u_1,\ u_2,\ v,\ x^0) \quad (4) \end{aligned}$$

and

$$v^* = v^0(u_1^*,\ u_2^*). \quad (5)$$

We assume that Pareto optimal solution based on closed-loop Stackelberg strategy has the following form.

$$u_i(x,\ t) = F_{i\varepsilon}x(t). \quad (6)$$

It is shown that the gain $F_{i\varepsilon}$ is dependent on the initial state of the systems $x(0)$. To eliminate this dependence on $x(0)$, it is assumed that $\mathbf{E}[x(0)] = 0$, $\mathbf{E}[x(0)x^T(0)] = I_n$.

Lemma 1: [12] Suppose that a set of cross-coupled algebraic nonlinear matrix equations (CANMEs) (7) has solutions $M_{i\varepsilon} \geq 0$, $N_{i\varepsilon}$, $i = 0,\ 1$ and $F_{i\varepsilon}$, $i = 1,\ 2$.

$$\begin{aligned} \boldsymbol{F}_1 &= A_{U\varepsilon}^T M_{0\varepsilon} + M_{0\varepsilon}A_{U\varepsilon} + C_\varepsilon^T M_{0\varepsilon}C_\varepsilon \\ &\quad -\gamma^{-2}M_{0\varepsilon}S_{0\varepsilon}M_{0\varepsilon} + Q_{F\varepsilon} = 0, \quad (7a) \\ \boldsymbol{F}_2 &= A_{U\varepsilon}^T M_{\rho\varepsilon} + M_{\rho\varepsilon}A_{U\varepsilon} + C_\varepsilon^T M_{\rho\varepsilon}C_\varepsilon + Q_{U\varepsilon} = 0, \quad (7b) \\ \boldsymbol{F}_3 &= A_{U\varepsilon}N_{0\varepsilon} + N_{0\varepsilon}A_{U\varepsilon}^T + C_\varepsilon N_{0\varepsilon}C_\varepsilon^T \\ &\quad +\gamma^{-2}(S_{0\varepsilon}M_{\rho\varepsilon}N_{\rho\varepsilon} + N_{\rho\varepsilon}M_{\rho\varepsilon}S_{0\varepsilon}) = 0, \quad (7c) \\ \boldsymbol{F}_4 &= A_{U\varepsilon}N_{\rho\varepsilon} + N_{\rho\varepsilon}A_{U\varepsilon}^T + C_\varepsilon N_{\rho\varepsilon}C_\varepsilon^T + I_n = 0, \quad (7d) \\ \boldsymbol{F}_5^i &= F_{i\varepsilon}(N_{0\varepsilon}+\rho_i N_{\rho\varepsilon})+B_i^T(M_{0\varepsilon}N_{0\varepsilon}+M_{\rho\varepsilon}N_{\rho\varepsilon}) = 0, \quad (7e) \end{aligned}$$

where

$$Q_{F\varepsilon} := E_\varepsilon^T E_\varepsilon + \sum_{j=1}^2 F_{j\varepsilon}^T F_{j\varepsilon},$$

$$A_{U\varepsilon} = A_\varepsilon + \sum_{j=1}^2 B_{j\varepsilon}F_{j\varepsilon} + \gamma^{-2}S_{0\varepsilon}M_{0\varepsilon},$$

$$Q_{U\varepsilon} := \sum_{j=1}^2 \rho_j[Q_{j\varepsilon} + F_{j\varepsilon}^T F_{j\varepsilon}],\ S_{0\varepsilon} = B_{0\varepsilon}B_{0\varepsilon}^T.$$

Then, this strategy set is called the Nash-Pareto hierarchical strategy.

III. Asymptotic Structure for CANMEs

In this section, asymptotic structures for CANMEs (7) are investigated. Without loss of generality, the following analysis requires a basic assumption [7].

Assumption 2: $(A_{ii},\ B_{iii} \mid C_{ii})$, $i = 1,\ 2$ are stabilizable and $(A_{ii},\ C_{ii} \mid \sqrt{Q_{iii}})$, $i = 1,\ 2$ are exactly observable.

Since A_ε, C_ε, $B_{i\varepsilon}$, $B_{0\varepsilon}$, E_ε and $G_{i\varepsilon}$ include ε, suppose that solutions $M_{0\varepsilon}$, $M_{\rho\varepsilon}$, $N_{0\varepsilon}$, $N_{\rho\varepsilon}$ and $F_{i\varepsilon}$, $i = 1,\ 2$ of CANMEs (7) have the following structure [12], [13].

$$M_{0\varepsilon} := \begin{bmatrix} M_{011} & \varepsilon M_{012} \\ \varepsilon M_{012}^T & M_{022} \end{bmatrix},\ M_{\rho\varepsilon} := \begin{bmatrix} M_{\rho 11} & \varepsilon M_{\rho 12} \\ \varepsilon M_{\rho 12}^T & M_{\rho 22} \end{bmatrix},$$

$$N_{0\varepsilon} := \begin{bmatrix} N_{011} & \varepsilon N_{012} \\ \varepsilon N_{012}^T & N_{022} \end{bmatrix},\ N_{\rho\varepsilon} := \begin{bmatrix} N_{\rho 11} & \varepsilon N_{\rho 12} \\ \varepsilon N_{\rho 12}^T & N_{\rho 22} \end{bmatrix},$$

$$F_{1\varepsilon} := \begin{bmatrix} F_{111} & \varepsilon F_{122} \end{bmatrix},\ F_{2\varepsilon} := \begin{bmatrix} \varepsilon F_{211} & F_{222} \end{bmatrix}.$$

By substituting these matrices into CANMEs (7), setting $\varepsilon = 0$, and partitioning CANMEs (7), the following reduced-order cross-coupled stochastic algebraic Lyapunov and Riccati equations (RCSALREs) are obtained. It should be noted that $\bar{M}_{0ii}$, $\bar{N}_{0ii}$, $\bar{M}_{\rho ii}$, $\bar{N}_{\rho ii}$ and $\bar{F}_{iii}$, $i = 1,\ 2$ are the limiting solutions of CANMEs (7) as $\varepsilon \to +0$.

$$\begin{aligned}
&A_{Uii}^T \bar{M}_{0ii} + \bar{M}_{0ii} A_{Uii} + C_{ii}^T \bar{M}_{0ii} C_{ii} \\
&\quad -\gamma^{-2} \bar{M}_{0ii} B_{0ii} B_{0ii}^T \bar{M}_{0ii} + E_{ii}^T E_{ii} + F_{iii}^T F_{iii} = 0, \quad \text{(8a)} \\
&A_{Uii}^T \bar{M}_{\rho ii} + \bar{M}_{\rho ii} A_{Uii} + C_{ii}^T \bar{M}_{\rho ii} C_{ii} \\
&\quad +\rho_i (Q_{iii} + F_{iii}^T F_{iii}) = 0, \quad \text{(8b)} \\
&A_{Uii} \bar{N}_{0ii} + \bar{N}_{0ii} A_{Uii}^T + C_{ii} \bar{N}_{0ii} C_{ii}^T \\
&\quad +\gamma^{-2} (B_{0ii} B_{0ii}^T \bar{M}_{\rho ii} \bar{N}_{\rho ii} + \bar{N}_{\rho ii} \bar{M}_{\rho ii} B_{0ii} B_{0ii}^T) = 0, \quad \text{(8c)} \\
&A_{Uii} \bar{N}_{\rho ii} + \bar{N}_{\rho ii} A_{Uii}^T + C_{ii} \bar{N}_{\rho ii} C_{ii}^T + I_{n_i} = 0, \quad \text{(8d)} \\
&F_{iii} (\bar{N}_{0ii} + \rho_i \bar{N}_{\rho ii}) \\
&\quad +B_{iii}^T (\bar{M}_{0ii} \bar{N}_{0ii} + \bar{M}_{\rho ii} \bar{N}_{\rho ii}) = 0, \quad \text{(8e)}
\end{aligned}$$

where $i = 1,\ 2$, $A_{Uii} := A_{ii} + B_{iii} F_{iii} + \gamma^{-2} B_{0ii} B_{0ii}^T M_{0ii}$.

Theorem 1: Under Assumption 2, suppose that the following matrix is nonsingular.

$$\bar{\boldsymbol{J}} := \begin{bmatrix} \bar{J}_{11} & 0 & 0 & 0 & \bar{J}_{15} & \bar{J}_{16} \\ \bar{J}_{21} & \bar{J}_{11} & 0 & 0 & \bar{J}_{25} & \bar{J}_{26} \\ \bar{J}_{31} & \bar{J}_{32} & \bar{J}_{33} & \bar{J}_{34} & \bar{J}_{35} & \bar{J}_{36} \\ \bar{J}_{41} & 0 & 0 & \bar{J}_{33} & \bar{J}_{45} & \bar{J}_{46} \\ \bar{J}_{51} & \bar{J}_{52} & \bar{J}_{53} & \bar{J}_{54} & \bar{J}_{55} & 0 \\ \bar{J}_{61} & \bar{J}_{62} & \bar{J}_{63} & \bar{J}_{64} & 0 & \bar{J}_{66} \end{bmatrix}, \quad (9)$$

where

$$\begin{aligned}
&\bar{J}_{11} = A_{\bar{U}}^T \otimes I_n + I_n \otimes A_{\bar{U}}^T + \bar{C}^T \otimes \bar{C}^T, \\
&\bar{J}_{15} = (\bar{B}_1^T \bar{M}_0 + \bar{F}_1)^T \otimes I_n + I_n \otimes (\bar{B}_1^T \bar{M}_0 + \bar{F}_1)^T, \\
&\bar{J}_{16} = (\bar{B}_2^T \bar{M}_0 + \bar{F}_2)^T \otimes I_n + I_n \otimes (\bar{B}_2^T \bar{M}_0 + \bar{F}_2)^T, \\
&\bar{J}_{21} = \gamma^{-2} [(\bar{M}_\rho S_0) \otimes I_n + I_n \otimes (\bar{M}_\rho S_0)], \\
&\bar{J}_{25} = (\bar{B}_1^T \bar{M}_\rho + \rho_1 \bar{F}_1)^T \otimes I_n + I_n \otimes (\bar{B}_1^T \bar{M}_\rho + \rho_1 \bar{F}_1)^T, \\
&\bar{J}_{26} = (\bar{B}_2^T \bar{M}_\rho + \rho_2 \bar{F}_2)^T \otimes I_n + I_n \otimes (\bar{B}_2^T \bar{M}_\rho + \rho_2 \bar{F}_2)^T, \\
&\bar{J}_{31} = \gamma^{-2} [S_0 \otimes \bar{N}_0 + \bar{N}_0 \otimes S_0], \\
&\bar{J}_{32} = \gamma^{-2} [S_0 \otimes \bar{N}_\rho + \bar{N}_\rho \otimes S_0], \\
&\bar{J}_{33} = A_{\bar{U}} \otimes I_n + I_n \otimes A_{\bar{U}} + \bar{C} \otimes \bar{C}, \\
&\bar{J}_{34} = \gamma^{-2} [(S_0 \bar{M}_\rho) \otimes I_n + I_n \otimes (S_0 \bar{M}_\rho)], \\
&\bar{J}_{35} = \gamma^{-2} [\bar{B}_1 \otimes \bar{N}_0 + \bar{N}_0 \otimes \bar{B}_1], \\
&\bar{J}_{36} = \gamma^{-2} [\bar{B}_2 \otimes \bar{N}_0 + \bar{N}_0 \otimes \bar{B}_2], \\
&\bar{J}_{41} = \gamma^{-2} [S_0 \otimes \bar{N}_\rho + \bar{N}_\rho \otimes S_0], \\
&\bar{J}_{45} = \bar{B}_1 \otimes \bar{N}_\rho + \bar{N}_\rho \otimes \bar{B}_1,\ \bar{J}_{46} = \bar{B}_2 \otimes \bar{N}_\rho + \bar{N}_\rho \otimes \bar{B}_2, \\
&\bar{J}_{51} = \bar{N}_0 \otimes \bar{B}_1^T,\ \bar{J}_{52} = \bar{N}_\rho \otimes \bar{B}_1^T, \\
&\bar{J}_{53} = I_n \otimes (\bar{F}_1 + \bar{B}_1^T \bar{M}_0),\ \bar{J}_{54} = I_n \otimes (\rho_1 \bar{F}_1 + \bar{B}_1^T \bar{M}_\rho), \\
&\bar{J}_{55} = \bar{N}_0 + \rho_1 \bar{N}_\rho, \\
&\bar{J}_{61} = \bar{N}_0 \otimes \bar{B}_2^T,\ \bar{J}_{62} = \bar{N}_\rho \otimes \bar{B}_2^T, \\
&\bar{J}_{63} = I_n \otimes (\bar{F}_2 + \bar{B}_2^T \bar{M}_0),\ \bar{J}_{64} = I_n \otimes (\rho_2 \bar{F}_2 + \bar{B}_2^T \bar{M}_\rho), \\
&\bar{J}_{66} = \bar{N}_0 + \rho_2 \bar{N}_\rho, \\
&A_{\bar{U}} := \textbf{block diag} \begin{pmatrix} A_{U11} & A_{U22} \end{pmatrix}, \\
&\bar{C} := \textbf{block diag} \begin{pmatrix} C_{111} & C_{222} \end{pmatrix}, \\
&\bar{B}_1 := \begin{bmatrix} B_{111} \\ 0 \end{bmatrix},\ \bar{B}_2 := \begin{bmatrix} 0 \\ B_{222} \end{bmatrix}, \\
&\bar{M}_0 := \textbf{block diag} \begin{pmatrix} \bar{M}_{011} & \bar{M}_{022} \end{pmatrix}, \\
&\bar{M}_\rho := \textbf{block diag} \begin{pmatrix} \bar{M}_{\rho 11} & \bar{M}_{\rho 22} \end{pmatrix}, \\
&\bar{N}_0 := \textbf{block diag} \begin{pmatrix} \bar{N}_{011} & \bar{N}_{022} \end{pmatrix}, \\
&\bar{N}_\rho := \textbf{block diag} \begin{pmatrix} \bar{N}_{\rho 11} & \bar{N}_{\rho 22} \end{pmatrix}, \\
&\bar{F}_1 := \begin{bmatrix} F_{111} & 0 \end{bmatrix},\ \bar{F}_2 := \begin{bmatrix} 0 & F_{222} \end{bmatrix}.
\end{aligned}$$

Then, there exists a small constant ε^* such that for all $\varepsilon \in (0,\ \varepsilon^*)$, CANMEs (7) admit positive definite solutions $M_{0\varepsilon}$, $M_{\rho\varepsilon}$, $N_{0\varepsilon}$, $N_{\rho\varepsilon}$ and feedback gains $F_{i\varepsilon}$ that can be expressed as

$$M_{0\varepsilon} = \bar{M}_0 + O(\varepsilon),\ M_{\rho\varepsilon} = \bar{M}_\rho + O(\varepsilon), \quad \text{(10a)}$$

$$N_{0\varepsilon} = \bar{N}_0 + O(\varepsilon),\ N_{\rho\varepsilon} = \bar{N}_\rho + O(\varepsilon), \quad \text{(10b)}$$

$$F_{i\varepsilon} = \bar{F}_i + O(\varepsilon),\ i = 1,\ 2. \quad \text{(10c)}$$

Proof: The proof can be done by using the implicit function theorem to CANMEs (7). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. First, the derivative of solutions for CANMEs (7) is given below.

$$\boldsymbol{J} := \nabla(M_{0\varepsilon},\ M_{\rho\varepsilon},\ N_{0\varepsilon},\ N_{\rho\varepsilon},\ F_{1\varepsilon},\ F_{2\varepsilon})$$

$$:= \begin{bmatrix}
\frac{\partial \boldsymbol{F}_1}{\partial M_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_1}{\partial M_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_1}{\partial N_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_1}{\partial N_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_1}{\partial F_{1\varepsilon}} & \frac{\partial \boldsymbol{F}_1}{\partial F_{2\varepsilon}} \\
\frac{\partial \boldsymbol{F}_2}{\partial M_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_2}{\partial M_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_2}{\partial N_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_2}{\partial N_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_2}{\partial F_{1\varepsilon}} & \frac{\partial \boldsymbol{F}_2}{\partial F_{2\varepsilon}} \\
\frac{\partial \boldsymbol{F}_3}{\partial M_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_3}{\partial M_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_3}{\partial N_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_3}{\partial N_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_3}{\partial F_{1\varepsilon}} & \frac{\partial \boldsymbol{F}_3}{\partial F_{2\varepsilon}} \\
\frac{\partial \boldsymbol{F}_4}{\partial M_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_4}{\partial M_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_4}{\partial N_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_4}{\partial N_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_4}{\partial F_{1\varepsilon}} & \frac{\partial \boldsymbol{F}_4}{\partial F_{2\varepsilon}} \\
\frac{\partial \boldsymbol{F}_5^1}{\partial M_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_5^1}{\partial M_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_5^1}{\partial N_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_5^1}{\partial N_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_5^1}{\partial F_{1\varepsilon}} & \frac{\partial \boldsymbol{F}_5^1}{\partial F_{2\varepsilon}} \\
\frac{\partial \boldsymbol{F}_5^2}{\partial M_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_5^2}{\partial M_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_5^2}{\partial N_{0\varepsilon}} & \frac{\partial \boldsymbol{F}_5^2}{\partial N_{\rho\varepsilon}} & \frac{\partial \boldsymbol{F}_5^2}{\partial F_{1\varepsilon}} & \frac{\partial \boldsymbol{F}_5^2}{\partial F_{2\varepsilon}}
\end{bmatrix}. \quad (11)$$

Hence, the Jacobian at $\varepsilon = 0$ is given by (9). That is, $\bar{\boldsymbol{J}} = \boldsymbol{J}\big|_{\varepsilon=0}$. By using the assumption that the Jacobian is nonsingular, it is proved that the asymptotic structure has the form of (10) from the implicit function theorem. ■

IV. COMPUTATIONAL ALGORITHM

In order to obtain the solutions of CANMEs (7), the following numerical computation that is based on the Newton's method is given:

$$\begin{aligned}
&A_{U\varepsilon}^{(n)T} M_{0\varepsilon}^{(n+1)} + M_{0\varepsilon}^{(n+1)} A_{U\varepsilon}^{(n)} + C^T M_{0\varepsilon}^{(n+1)} C \\
&\quad + \sum_{j=1}^{2} \Bigg[\left(B_{j\varepsilon}^T M_{0\varepsilon}^{(n)} + F_{j\varepsilon}^{(n)} \right)^T F_{j\varepsilon}^{(n+1)} \\
&\quad + F_{j\varepsilon}^{(n+1)T} \left(B_{j\varepsilon}^T M_{0\varepsilon}^{(n)} + F_{j\varepsilon}^{(n)} \right) \Bigg] - \boldsymbol{L}_{1\varepsilon}^{(n)} = 0, \quad \text{(12a)} \\
&\gamma^{-2} \left(M_{\rho\varepsilon}^{(n)} S_{0\varepsilon} M_{0\varepsilon}^{(n+1)} + M_{0\varepsilon}^{(n+1)} S_{0\varepsilon} M_{\rho\varepsilon}^{(n)} \right) \\
&\quad + A_{U\varepsilon}^{(n)T} M_{\rho\varepsilon}^{(n+1)} + M_{\rho\varepsilon}^{(n+1)} A_{U\varepsilon}^{(n)} + C^T M_{\rho\varepsilon}^{(n+1)} C \\
&\quad + \sum_{j=1}^{2} \Bigg[\left(B_{j\varepsilon}^T M_{\rho\varepsilon}^{(n)} + \rho_j F_{j\varepsilon}^{(n)} \right)^T F_{j\varepsilon}^{(n+1)}
\end{aligned}$$

$$+F_{j\varepsilon}^{(n+1)T}\left(B_{j\varepsilon}^T M_{\rho\varepsilon}^{(n)}+\rho_j F_{j\varepsilon}^{(n)}\right)\Big]-\boldsymbol{L}_{2\varepsilon}^{(n)}=0, \quad (12b)$$

$$\gamma^{-2}\Big(N_{0\varepsilon}^{(n)}M_{0\varepsilon}^{(n+1)}S_{0\varepsilon}+S_{0\varepsilon}M_{0\varepsilon}^{(n+1)}N_{0\varepsilon}^{(n)}\Big)$$
$$+\gamma^{-2}\Big(N_{\rho\varepsilon}^{(n)}M_{\rho\varepsilon}^{(n+1)}S_{0\varepsilon}+S_{0\varepsilon}M_{\rho\varepsilon}^{(n+1)}N_{\rho\varepsilon}^{(n)}\Big)$$
$$+\sum_{j=1}^{2}\Big(N_{0\varepsilon}^{(n)}F_{j\varepsilon}^{(n+1)T}B_{j\varepsilon}^T+B_{j\varepsilon}F_{j\varepsilon}^{(n+1)}N_{0\varepsilon}^{(n)}\Big)$$
$$+A_{U\varepsilon}^{(n)}N_{0\varepsilon}^{(n+1)}+N_{0\varepsilon}^{(n+1)}A_{U\varepsilon}^{(n)T}+CN_{0\varepsilon}^{(n+1)}C^T$$
$$+\gamma^{-2}\Big(S_{0\varepsilon}M_{\rho\varepsilon}^{(n)}N_{\rho\varepsilon}^{(n+1)}+N_{\rho\varepsilon}^{(n+1)}M_{\rho\varepsilon}^{(n)}S_{0\varepsilon}\Big)$$
$$-\boldsymbol{L}_{3\varepsilon}^{(n)}=0, \quad (12c)$$

$$\gamma^{-2}\Big(N_{\rho\varepsilon}^{(n)}M_{0\varepsilon}^{(n+1)}S_{0\varepsilon}+S_{0\varepsilon}M_{0\varepsilon}^{(n+1)}N_{\rho\varepsilon}^{(n)}\Big)$$
$$+\sum_{j=1}^{2}\Big(N_{\rho\varepsilon}^{(n)}F_{j\varepsilon}^{(n+1)T}B_{j\varepsilon}^T+B_{j\varepsilon}F_{j\varepsilon}^{(n+1)}N_{\rho\varepsilon}^{(n)}\Big)$$
$$+A_{U\varepsilon}^{(n)}N_{\rho\varepsilon}^{(n+1)}+N_{\rho\varepsilon}^{(n+1)}A_{U\varepsilon}^{(n)T}$$
$$+CN_{\rho\varepsilon}^{(n+1)}C^T-\boldsymbol{L}_{4\varepsilon}^{(n)}=0, \quad (12d)$$

$$B_i^T\Big(M_{0\varepsilon}^{(n+1)}N_{0\varepsilon}^{(n)}+M_{\rho\varepsilon}^{(n+1)}N_{\rho\varepsilon}^{(n)}\Big)$$
$$+F_{i\varepsilon}^{(n+1)}\Big(N_{0\varepsilon}^{(n)}+\rho_i N_{\rho\varepsilon}^{(n)}\Big)+\Big(F_{i\varepsilon}^{(n)}+B_i^T M_{0\varepsilon}^{(n)}\Big)N_{0\varepsilon}^{(n+1)}$$
$$+\Big(\rho_i F_{i\varepsilon}^{(n)}+B_i^T M_{\rho\varepsilon}^{(n)}\Big)N_{\rho\varepsilon}^{(n+1)}-\boldsymbol{L}_{5\varepsilon}^{i(n)}=0, \quad (12e)$$

where $i=1,\ 2,$

$$A_{U\varepsilon}^{(n)}=A+\sum_{j=1}^{2}B_{j\varepsilon}F_{j\varepsilon}^{(n)}+\gamma^{-2}S_{0\varepsilon}M_{0\varepsilon}^{(n)},$$

$$\boldsymbol{L}_{1\varepsilon}^{(n)}=\sum_{j=1}^{2}\Big(M_{0\varepsilon}^{(n)}B_{j\varepsilon}F_{j\varepsilon}^{(n)}+F_{j\varepsilon}^{(n)T}B_{j\varepsilon}^T M_{0\varepsilon}^{(n)}\Big)$$
$$+\gamma^{-2}M_{0\varepsilon}^{(n)}S_{0\varepsilon}M_{0\varepsilon}^{(n)}-E_\varepsilon^T E_\varepsilon+\sum_{j=1}^{2}F_{j\varepsilon}^{(n)T}F_{j\varepsilon}^{(n)},$$

$$\boldsymbol{L}_{2\varepsilon}^{(n)}=\gamma^{-2}\Big(M_{\rho\varepsilon}^{(n)}S_{0\varepsilon}M_{0\varepsilon}^{(n)}+M_{0\varepsilon}^{(n)}S_{0\varepsilon}M_{\rho\varepsilon}^{(n)}\Big)$$
$$+\sum_{j=1}^{2}\Big(M_{\rho\varepsilon}^{(n)}B_{j\varepsilon}F_{j\varepsilon}^{(n)}+F_{j\varepsilon}^{(n)T}B_{j\varepsilon}^T M_{\rho\varepsilon}^{(n)}\Big)$$
$$-\sum_{j=1}^{2}\rho_i\Big(Q_{j\varepsilon}-F_{j\varepsilon}^{(n)T}F_{j\varepsilon}^{(n)}\Big),$$

$$\boldsymbol{L}_{3\varepsilon}^{(n)}=\gamma^{-2}\Big(N_{0\varepsilon}^{(n)}M_{0\varepsilon}^{(n)}S_{0\varepsilon}+S_{0\varepsilon}M_{0\varepsilon}^{(n)}N_{0\varepsilon}^{(n)}\Big)$$
$$+\gamma^{-2}\Big(N_{\rho\varepsilon}^{(n)}M_{\rho\varepsilon}^{(n)}S_{0\varepsilon}+S_{0\varepsilon}M_{\rho\varepsilon}^{(n)}N_{\rho\varepsilon}^{(n)}\Big)$$
$$+\sum_{j=1}^{2}\Big(N_{0\varepsilon}^{(n)}F_{j\varepsilon}^{(n)T}B_{j\varepsilon}^T+B_{j\varepsilon}F_{j\varepsilon}^{(n)}N_{0\varepsilon}^{(n)}\Big),$$

$$\boldsymbol{L}_{4\varepsilon}^{(n)}=\gamma^{-2}\Big(N_{\rho\varepsilon}^{(n)}M_{0\varepsilon}^{(n)}S_{0\varepsilon}+S_{0\varepsilon}M_{0\varepsilon}^{(n)}N_{\rho\varepsilon}^{(n)}\Big)$$
$$+\sum_{j=1}^{2}\Big(N_{\rho\varepsilon}^{(n)}F_{j\varepsilon}^{(n)T}B_{j\varepsilon}^T+B_{j\varepsilon}F_{j\varepsilon}^{(n)}N_{\rho\varepsilon}^{(n)}\Big)-I_n,$$

$$\boldsymbol{L}_{5\varepsilon}^{i(n)}=\Big(F_{i\varepsilon}^{(n)}+B_i^T M_{0\varepsilon}^{(n)}\Big)N_{0\varepsilon}^{(n)}$$
$$+\Big(\rho_i F_{i\varepsilon}^{(n)}+B_i^T M_{\rho\varepsilon}^{(n)}\Big)N_{\rho\varepsilon}^{(n)},\ i=1,\ 2.$$

Moreover, the initial conditions $M_{0\varepsilon}^{(0)}$, $M_{\rho\varepsilon}^{(0)}$, $N_{0\varepsilon}^{(0)}$, $N_{\rho\varepsilon}^{(0)}$ and $F_{i\varepsilon}^{(0)}$, $i=1,\ 2$ are chosen as given below.

$$M_{0\varepsilon}^{(0)}=\bar{M}_0,\ M_{\rho\varepsilon}^{(0)}=\bar{M}_\rho,$$
$$N_{0\varepsilon}^{(0)}=\bar{N}_0,\ N_{\rho\varepsilon}^{(0)}=\bar{N}_\rho,$$
$$F_{1\varepsilon}^{(0)}=\bar{F}_1,\ F_{2\varepsilon}^{(0)}=\bar{F}_2.$$

The following theorem indicates that the proposed algorithm (12) that is based on the Newton's method attains the quadratic convergence.

Theorem 2: Suppose that the Jacobian (9) is nonsingular. Under Assumption 2, there exists the small constant $\bar{\varepsilon}$ such that for all $\varepsilon\in(0,\ \bar{\varepsilon})$, $\bar{\varepsilon}\le\varepsilon^*$, the iterative algorithm (12) converges to the exact solutions with the rate of the quadratic convergence. Moreover, the convergence solutions attain a local unique solutions of the CANMEs (7) in the neighborhood of the initial conditions. That is, the following conditions are satisfied.

$$\|M_{0\varepsilon}^{(n)}-M_{0\varepsilon}\|=O(\varepsilon^{2^n}),\ \|M_{\rho\varepsilon}^{(n)}-M_{\rho\varepsilon}\|=O(\varepsilon^{2^n}), \quad (13a)$$
$$\|N_{0\varepsilon}^{(n)}-N_{0\varepsilon}\|=O(\varepsilon^{2^n}),\ \|N_{\rho\varepsilon}^{(n)}-N_{\rho\varepsilon}\|=O(\varepsilon^{2^n}), \quad (13b)$$
$$\|F_{i\varepsilon}^{(n)}-F_{i\varepsilon}\|=O(\varepsilon^{2^n}), \quad (13c)$$

where $i=1,\ 2,\ n=0,\ 1,\ 2,\ \ldots$.

Proof: The proof is given directly by applying the Newton-Kantorovich theorem [14] for CANMEs (7). It is immediately obtained from CANMEs (7) that there exists a positive scalar γ such that for any solutions

$$\|\nabla(M_{0\varepsilon}^a,\ M_{\rho\varepsilon}^a,\ N_{0\varepsilon}^a,\ N_{\rho\varepsilon}^a,\ F_{1\varepsilon}^a,F_{2\varepsilon}^a)$$
$$-\nabla(M_{0\varepsilon}^b,\ M_{\rho\varepsilon}^b,\ N_{0\varepsilon}^b,\ N_{\rho\varepsilon}^b,\ F_{1\varepsilon}^b,F_{2\varepsilon}^b)\|$$
$$\le\gamma\|([\mathrm{vec}M_{0\varepsilon}^a]^T,\ [\mathrm{vec}M_{\rho\varepsilon}^a]^T,\ [\mathrm{vec}N_{0\varepsilon}^a]^T,$$
$$[\mathrm{vec}N_{\rho\varepsilon}^a]^T,\ [\mathrm{vec}F_{1\varepsilon}^a]^T,\ [\mathrm{vec}F_{2\varepsilon}^a]^T)$$
$$-([\mathrm{vec}M_{0\varepsilon}^b]^T,\ [\mathrm{vec}M_{\rho\varepsilon}^b]^T,\ [\mathrm{vec}N_{0\varepsilon}^b]^T,$$
$$[\mathrm{vec}N_{\rho\varepsilon}^b]^T,\ [\mathrm{vec}F_{1\varepsilon}^b]^T,\ [\mathrm{vec}F_{2\varepsilon}^b])\|. \quad (14)$$

Moreover, it is easy to verify that $\nabla(M_{0\varepsilon}^{(0)},\ M_{\rho\varepsilon}^{(0)},\ N_{0\varepsilon}^{(0)},\ N_{\rho\varepsilon}^{(0)},\ F_{1\varepsilon}^{(0)},\ F_{2\varepsilon}^{(0)})=\bar{\boldsymbol{J}}+O(\varepsilon)$ is nonsingular because for small ε, using (9) and $\bar{\boldsymbol{J}}$ is also nonsingular. Therefore, there exists β such that $\beta=\|[\nabla(M_{0\varepsilon}^{(0)},\ M_{\rho\varepsilon}^{(0)},\ N_{0\varepsilon}^{(0)},\ N_{\rho\varepsilon}^{(0)},\ F_{1\varepsilon}^{(0)},\ F_{2\varepsilon}^{(0)})]^{-1}\|$. On the other hand, since $\|\boldsymbol{F}_i(M_{0\varepsilon}^{(0)},\ M_{\rho\varepsilon}^{(0)},\ N_{0\varepsilon}^{(0)},\ N_{\rho\varepsilon}^{(0)},\ F_{1\varepsilon}^{(0)},\ F_{2\varepsilon}^{(0)})\|=O(\varepsilon)$, $i=1,\ \ldots,4$, $\|\boldsymbol{F}_5^i(M_{0\varepsilon}^{(0)},\ M_{\rho\varepsilon}^{(0)},\ N_{0\varepsilon}^{(0)},\ N_{\rho\varepsilon}^{(0)},\ F_{1\varepsilon}^{(0)},\ F_{2\varepsilon}^{(0)})\|=O(\varepsilon)$, $i=1,\ 2$, there exists η such that $\eta\le\|[\nabla(M_{0\varepsilon}^{(0)},\ M_{\rho\varepsilon}^{(0)},\ N_{0\varepsilon}^{(0)},\ N_{\rho\varepsilon}^{(0)},\ F_{1\varepsilon}^{(0)},\ F_{2\varepsilon}^{(0)})]^{-1}\|\cdot(\sum_{j=1}^{4}\|\boldsymbol{F}_j\|+\sum_{j=1}^{2}\|\boldsymbol{F}_5^j\|)=O(\varepsilon)$. Thus, there exists θ

such that $\theta = \beta\eta\gamma < 2^{-1}$ because $\eta = O(\varepsilon)$. Finally, the Newton-Kantorovich theorem results in the desired results (13). ■

It should be noted that the 0-th order algebraic equations (8) are nonlinear matrix equations. Hence, it is hard to solve these equations. However, if γ is sufficiently large, these equations can be changed as follows.

$$\begin{aligned}
&(A_{ii} + B_{iii}F_{iii})^T \hat{M}_{0ii} + \hat{M}_{0ii}(A_{ii} + B_{iii}F_{iii}) \\
&\quad + C_{ii}^T \hat{M}_{0ii} C_{ii} + E_{ii}^T E_{ii} + F_{iii}^T F_{iii} = 0, && \text{(15a)} \\
&A_{ii}^T \hat{M}_{\rho ii} + \hat{M}_{\rho ii} A_{ii} + C_{ii}^T \hat{M}_{\rho ii} C_{ii} \\
&\quad - \rho_i^{-2} \hat{M}_{\rho ii} B_{iii} B_{iii}^T \hat{M}_{\rho ii} + \rho_i Q_{iii} = 0, && \text{(15b)} \\
&\hat{N}_{0ii} = 0, && \text{(15c)} \\
&(A_{ii} + B_{iii}F_{iii}) \hat{N}_{\rho ii} + \hat{N}_{\rho ii}(A_{ii} + B_{iii}F_{iii})^T \\
&\quad + C_{ii} \hat{N}_{\rho ii} C_{ii}^T + I_{n_i} = 0, && \text{(15d)} \\
&\hat{F}_{iii} = -\rho_i^{-1} B_{iii}^T \hat{M}_{\rho ii}. && \text{(15e)}
\end{aligned}$$

It is worth pointing out that these equations can be solved separately. These initial conditions would be useful when the parameter γ is sufficiently large. However, even if this parameter is small, the same initial conditions would be reliable. In fact, this important feature can be verified in numerical section.

V. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the proposed scheme, a simple example is given. The system matrices are given as follows.

$$\varepsilon = 0.01,\ A_\varepsilon = \begin{bmatrix} 0.2 & \varepsilon \\ -\varepsilon & -1 \end{bmatrix},\ C_\varepsilon = \begin{bmatrix} 0.1 & 0.05\varepsilon \\ 0.05\varepsilon & 0.1 \end{bmatrix},$$

$$B_{1\varepsilon} = \begin{bmatrix} -1 \\ \varepsilon \end{bmatrix},\ B_{2\varepsilon} = \begin{bmatrix} 0.5\varepsilon \\ 2 \end{bmatrix},\ B_{0\varepsilon} = \begin{bmatrix} 0.1 & \varepsilon \\ \varepsilon & 0.2 \end{bmatrix},$$

$$E_\varepsilon = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ G_{1\varepsilon} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \varepsilon \end{bmatrix},\ G_{2\varepsilon} = \begin{bmatrix} 0 \\ 0 \\ \varepsilon \\ 0.5 \end{bmatrix},$$

$$Q_{1\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 2\varepsilon \end{bmatrix},\ Q_{2\varepsilon} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \rho_1 = \rho_2 = 0.5.$$

First, in order to obtain initial conditions, 0-th order solutions of (8) is solved. It should be noted that $\gamma = 5$ is chosen. These solutions are given below.

$$\begin{aligned}
M_{0\varepsilon}^{(0)} &= \bar{M}_0 = \textbf{block diag}\left(1.2261 \ \ 9.8110\text{e}{-01}\right), \\
M_{\rho\varepsilon}^{(0)} &= \bar{M}_\rho = \textbf{block diag}\left(6.1319\text{e}{-01} \ \ 1.5496\text{e}{-01}\right), \\
N_{0\varepsilon}^{(0)} &= \bar{N}_0 = \textbf{block diag}\left(1.1767\text{e}{-04} \ \ 2.4846\text{e}{-05}\right), \\
N_{\rho\varepsilon}^{(0)} &= \bar{N}_\rho = \textbf{block diag}\left(4.8977\text{e}{-01} \ \ 2.2384\text{e}{-01}\right), \\
F_{1\varepsilon}^{(0)} &= \bar{F}_1 = \left[1.2264 \ \ 0\right], \\
F_{2\varepsilon}^{(0)} &= \bar{F}_2 = \left[0 \ \ -6.2015\text{e}{-01}\right].
\end{aligned}$$

These solutions were obtained by using the other Newton's method. On the other hand, even if the parameter γ is small for $\gamma = 5$, other initial conditions that are based on (15) are computed separately. These solutions are given below.

$$\begin{aligned}
\hat{M}_0 &= \textbf{block diag}\left(1.2258 \ \ 9.8120\text{e}{-01}\right), \\
\hat{M}_\rho &= \textbf{block diag}\left(6.1290\text{e}{-01} \ \ 1.5485\text{e}{-01}\right), \\
\hat{N}_0 &= 0, \\
\hat{N}_\rho &= \textbf{block diag}\left(4.8981\text{e}{-01} \ \ 2.2383\text{e}{-01}\right), \\
\hat{F}_1 &= \left[1.2258 \ \ 0\right], \\
\hat{F}_2 &= \left[0 \ \ -6.1942\text{e}{-01}\right].
\end{aligned}$$

It is easy to verify that these solutions seem to be very close even if the parameter γ is small. Therefore, this initial condition can also be worked well.

It should be noted that the Newton's method (12) converges to the exact solution with a computational error of the order of e−14 after three iterations. The convergence property is given by Table 1.

Table 1.

n	$\sum_{j=1}^{4} \|\boldsymbol{F}_j\| + \sum_{j=1}^{2} \|\boldsymbol{F}_5^j\|$
0	5.7596455805e−02
1	5.6284043724e−04
2	1.5060512883e−08
3	2.0901004009e−15

It is easy to verify that the quadratic convergence can be attained. The exact strategies F_i, $i = 1,\ 2$ and the solutions of CANMEs (7) are given below.

$$\begin{aligned}
F_{1\varepsilon} &= \begin{bmatrix} 1.2263 & -2.7333\text{e}{-04} \end{bmatrix}, \\
F_{2\varepsilon} &= \begin{bmatrix} -1.1942\text{e}{-02} & -6.2910\text{e}{-01} \end{bmatrix}, \\
M_{0\varepsilon} &= \begin{bmatrix} 1.2260 & 7.6610\text{e}{-03} \\ 7.6610\text{e}{-03} & 9.7583\text{e}{-01} \end{bmatrix}, \\
M_{\rho\varepsilon} &= \begin{bmatrix} 6.1317\text{e}{-01} & 1.4355\text{e}{-03} \\ 1.4355\text{e}{-03} & 1.5720\text{e}{-01} \end{bmatrix}, \\
N_{0\varepsilon} &= \begin{bmatrix} 1.1896\text{e}{-04} & 1.1539\text{e}{-05} \\ 1.1539\text{e}{-05} & 2.4686\text{e}{-05} \end{bmatrix}, \\
N_{\rho\varepsilon} &= \begin{bmatrix} 4.8976\text{e}{-01} & -2.7093\text{e}{-03} \\ -2.7093\text{e}{-03} & 2.2209\text{e}{-01} \end{bmatrix}.
\end{aligned}$$

VI. CONCLUSIONS

In this paper, an H_2/H_∞ control problem for a class of weakly coupled stochastic system governed by Itô differential equation with state and external disturbance dependent noise involving multiple decision makers was discussed by using a Stackelberg game approach. After establishing the asymptotic structure, Newton's method was derived to solve CANMEs. Particularly, the computation of the initial conditions was discussed deeply. As a result, the initial conditions can be obtained separately and the reduced-order computation is only needed. Finally, the effectiveness of the numerical algorithms was confirmed in the numerical example.

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