# Controller Verification and Parametrization Subject to Quantitative and Qualitative Requirements

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Abstract: Verifying if a process controller achieves a desired goal regarding safety specifications or performance is an important task in practice. This work presents a method for controller verification and parametrization of uncertain polynomial discrete-time systems with closed-loop requirements. Apart from quantitative constraints, also qualitative requirements, which are not directly linked to a specific time or amplitude, are considered. For formalizing these constraints, we employ linear temporal logic formulas and polynomial inequalities. Uncertainties can be considered in the input, the output, the initial conditions and the model parameters to account e.g. for model plant mismatch and noise, described as unknown-but-bounded variables. We combine the requirements and the system dynamics into a nonlinear feasibility problem to verify the controller and determine admissible controller parametrization. This problem is solved by relaxing it to a mixed-integer linear program. The relaxation procedure guarantees that the derived set of possible parametrization fulfill the quantitative and qualitative requirements of the closed-loop behavior despite the present uncertainties. The proposed method is illustrated by verifying and parametrizing a controller for a two tank system.

*Keywords:* set-based method, controller tuning, inner approximation, uncertain systems, qualitative and quantitative requirements

## 1. INTRODUCTION

Verifying the correct behavior or determining the parametrization of a controller that meets design and quality requirements on the closed-loop system are practically important yet challenging tasks for industrial applications. These tasks are especially important for safety critical systems such as guaranteeing a maximal temperature or pressure. Although, tuning controllers for linear systems is rather well studied, see e.g. (Skogestad and Postlethwaite, 2005; Levine, 2010), for nonlinear systems only few systematic approaches are known.

In this work, a method to derive a set of controller parametrizations that guarantee the fulfillment of the desired qualitative and quantitative closed-loop behavior of uncertain polynomial discrete-time systems is proposed. Verification of the controller corresponds then to showing that this set is non-empty.

The considered quantitative requirements are related to transient response characteristics like overshoot, steady state error, etc. While the term qualitative requirements corresponds to specifications that are not directly linked to specific time or amplitude like offset free control, logical constraints, and similar requirements. To express qualitative requirements, we employ linear temporal logic, see e.g. (Bemporad and Morari, 1999; Tabuada and Pappas, 2006; Karaman, 2008; Rumschinski et al., 2012; Wolff et al., 2014).

We consider bounded uncertainties in the input, output, parameters and initial conditions as unknown-butbounded variables, i.e. the exact value of variables is unknown, however, they are located inside a known compact set, see e.g. (Borchers et al., 2009; Savchenko et al., 2014).

In this contribution, we show that quantitative and qualitative requirements on the closed-loop behavior as well as uncertainties can be considered in controller verification and tuning. To do so, we formulate a nonlinear feasibility problem that contains the system dynamics, the requirements, and the uncertainties. The solution space of the feasibility problem describes the set of controller parametrization that guarantee satisfaction of the posed requirements for some combination of the inputs, outputs and states within the present uncertainty bounds. However, this problem is typically non-convex and, therefore, hard to address. Following (Borchers et al., 2009; Rumschinski et al., 2012; Savchenko et al., 2014), we relax the feasibility problem into a mixed-integer linear program.

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Moreover, for deriving a robust controller parametrization we propose strengthening the feasibility formulation to obtain a set of parameters, that guarantee a robust fulfillment of the requirements for any combination of allowed uncertainties. To guarantee that the obtained set contains only solutions that fulfill the requirements, we employ guaranteed inner approximations (Streif et al., 2013). We illustrate the proposed controller verification and parametrization method with a two tank system.

This paper is organized as follows. Section 2 presents the system and controller description. Furthermore, it is illustrated how uncertainties, quantitative and qualitative requirements can be posed as linear inequalities and linear temporal logic formulas. Furthermore, the formulas are also rephrased as linear inequalities. In Section 3, a short overview of nonlinear feasibility problems, outer approximations and inner approximations is given. The section is concluded with an algorithm for obtaining the set of controller parameters that guarantee the qualitative and quantitative requirements. In Section 4 the results of the proposed method are demonstrated.

#### 2. PROBLEM SETUP

#### 2.1 Considered system class

We consider processes that can be described by polynomial or rational discrete-time dynamics:

$$\begin{aligned} x(k+1) &= g(x(k), u(k), q, w(k)), \\ y(k) &= h(x(k), q), \end{aligned}$$
 (1)

where  $x \in \mathbb{R}^{n_x}$ ,  $u \in \mathbb{R}^{n_u}$ ,  $y \in \mathbb{R}^{n_y}$ ,  $q \in \mathbb{R}^{n_q}$  and  $w \subset \mathbb{R}^{n_u+n_x}$ are the states, inputs, outputs, system parameters and disturbances acting on the system. Time is indexed by  $k \in \mathcal{T} = \{1, ..., n_k\}$ ,  $n_k \in \mathbb{N}$ , i.e. we consider a finite time horizon. The variable  $n_x \in \mathbb{N}$  (resp.  $n_y$ ,  $n_q$ ,  $n_u$ ) denotes the dimensions of x (resp. y, q, u).

The process is controlled in closed-loop, where the controller is also given in a polynomial or rational form:

$$u(k+1) = c(u(k), r(k), y(k), p, w(k)),$$
(2)

where  $r \in \mathbb{R}^{n_r}$  and  $p \in \mathcal{P} \in \mathbb{R}^{n_p}$  are the exogenous set-point references and controller parameters. To simplify notation we write  $x = (x(1), \dots, x(k))$ , denoting the states within the time horizon  $\mathcal{T}$ . Equivalently we write y, u and w in place of the other time varying variables.

We assume that every variable of the closed-loop system is subject to constraints, i. e.  $x \in \mathcal{X}$ ,  $u \in \mathcal{U}$ ,  $y \in \mathcal{Y}$ ,  $w \in \mathcal{W}$ , and  $q \in \mathcal{Q}$ , where the compact sets are described by polynomial inequalities. Such constraints represent uncertainties in problem formulation (1), possible input and state constraints, physical limitations, or safety specifications. For shorthand of notation we define the Cartesian product of these constraints as  $\mathcal{F}_0$ :

$$\mathcal{F}_0 = \mathcal{X} \times \mathcal{U} \times \mathcal{Y} \times \mathcal{P} \times \mathcal{Q} \times \mathcal{W} \tag{3}$$

The controller (2) should perform according to additional requirements, that we can pose in form of quantitative or qualitative constraints as specified in Section 2.2, as summarized in the following problem.

Problem 1. (Controller parameter estimation) For a closedloop system (1)-(2) determine a set of admissible controller parameter values  $p \in \mathcal{P}$  that guarantees satisfaction of quantitative and qualitative controller requirements. Note that a controller is verified in case the set  ${\mathcal P}$  is non-empty.

## 2.2 Quantitative and qualitative requirements

We consider two conceptually different classes of requirements: qualitative and quantitative. We provide the definitions and examples for each class.

We refer to quantitative requirements as constraints that limit a variable to particular range of values at a specific moment in time, e.g. in form  $x(k) \in \mathcal{X}(k)$ .

The quantitative requirements are closely related to transient response characteristics, which are used to assure quality of a closed-loop system for a step input. Examples of such characteristics are: error margins at steady state, overshoot, noise attenuation, or rise time, as illustrated in Fig 1. For more on this topic we refer to (Levine, 2010; Franklin et al., 2002).



Fig. 1. An illustration of transient response characteristics.

Characteristics as in Fig. 1 are fixed to specific values at specific time points and therefore can be directly formulated as time-variant constraints. Several examples for qualitative requirements are:

- the maximal *overshoot* has to be less than  $\overline{\eta}$ , which can be formulated as  $x(k) \leq \overline{\eta}$  for all  $k \in \mathcal{T}$ ,
- the delay time has a duration of 10 time steps, which can be formulated in form  $x(k) \leq 10\% \cdot x_{ref}, k \in \{1, \dots, k_{10}\}$
- the settling time of the process has to be  $k_{settle}$  and the error margin is smaller than  $2\epsilon$ , which can be formulated as  $x(k) \leq x_{ref} + \epsilon$  and  $x(k) \geq x_{ref} - \epsilon$  for  $k \in \{k_{settle}, k_{settle} + 1, \dots, k_{end}\}$

In contrast to quantitative requirements, we are also interested in guaranteeing a desired qualitative system behavior.

An example of *qualitative* behavior is Offset Free Control (OFC). The requirements for a system to have OFC is: after a change in the system reference signal the system has to reach eventually the new reference value, i. e. the error term e(k) = r(k) - y(k) goes to zero (see Fig. 2).

There is no general solution how to achieve OFC for the considered class of systems (1) - (3), i. e. nonlinear, multiinput multi-output, perturbed systems subject to uncertainties and constraints. One particular reason for this is the general difficulty to pose qualitative requirements



Fig. 2. Example of offset free control for step input.

given as "plain text" in a formal way that can be used in the controller tuning. One approach that allows for such a formulation is Linear Temporal Logic (LTL). We present next three illustrative examples of LTL reformulations, and we refer the readers to Karaman (2008) and Wolff et al. (2014) for more details and formal definitions of the involved operators.

The first example is a control requirement: "eventually the system has to reach the new reference after a reference change":

$$\omega \to \Diamond \psi,$$

where  $\omega \in \{False, True\}$  is the *atomic proposition* indicating whether the reference change has occurred.  $\psi \in \{False, True\}$  is the atomic proposition indicating whether the output has reached the new reference. The operator  $\diamond$  stands for *eventually*, i.e.  $\diamond \psi$  means that  $\psi$  becomes True some time after the current time point. Further,  $\rightarrow$  is the *implication* operator, so the whole LTL formula means that *if* at a certain time point *k* the reference has changed, *then* the output has to reach the new reference point at some time point  $k^*$ , where  $k^* \geq k$ .

The second example is as follows: "if the tank is not empty then start emptying procedure (proposition  $\beta$ ) and switch on the heating (proposition  $\gamma$ )". Proposition  $\alpha$  refers to "tank is empty":

$$\neg \alpha \to \bigcirc (\beta \land \gamma),$$

where  $\neg$  is the *negation* operator,  $\land$  is the *conjunction* operator (logical "AND") and  $\bigcirc$  is the operator *next*. The formula reads *if* the tank is *not* empty at time point *k*, *then* start emptying *and* heating at time point k + 1.

The third example is: "an item has to be processed either by machines 1, 2 and 3, or to be processed by machines 1 and 4 but not by machine 3". Proposition  $z_i$  refers to being processed by machine *i*:

$$(\Diamond z_1) \land (((\Diamond z_2) \land (\Diamond z_3)) \lor ((\Box \neg z_3) \land (\Diamond z_4))),$$

where  $\lor$  is the *disjunction* operator (logical "OR") and  $\Box$  is the *always* operator.

Qualitative constraints in LTL formulation can be also represented in form of mixed-integer linear inequalities with addition of binary variables (cf. also (Bemporad and Morari, 1999; Rumschinski et al., 2012)). In the most abstract way we represent these constraints in the following form:

$$b_j(u, x, y, z, k) \le 0, \forall k \in \mathcal{T}_j, \tag{4}$$

where  $z = \{0,1\}^{n_j}, n_j \in \mathbb{N}$  are the additional binary variables that help transform the qualitative requirements

## to inequalities, $j = \{1, ..., n_b\}$ and $n_b \in \mathbb{N}$ is the number of qualitative constraints. In general, $b_j$ can be a vector function, since some of the LTL operators require multiple mixed-integer constraints to be formulated in this setting. The time set $\mathcal{T}_j \subseteq \mathcal{T}$ consists only of time steps when the constraint is required.

Note that the quantitative constraints described above can also be represented in form (4). Therefore, for brevity of notation, we refer to (4) for both quantitative and qualitative specifications on the controller.

Controller verification and parametrization, as stated in Problem 1, require a guarantee that the quantitative and qualitative behavior of the closed-loop system is achieved by the controller. We present next a solution to Problem 1 by formulating a feasibility problem consisting of the system (1)-(2) as well as the quantitative and qualitative requirements (4).

## 3. SET APPROXIMATIONS

#### 3.1 Feasibility problem formulation

To solve Problem 1, first we combine the system and controller dynamics into the following initial feasibility problem.

$$\operatorname{FP}_{init}: \begin{cases} \operatorname{find} \ (x, u, y, p, q, w) \\ \text{s.t.} \ x(k+1) = g(x(k), u(k), q, w(k)), \\ y(k) = h(x(k), q), \\ u(k+1) = c(u(k), r(k), y(k), p, w(k)), \\ (x, u, y, p, q, w) \in \mathcal{F}_{0}. \end{cases}$$

We denote the set of all solutions of  $FP_{init}$  as  $\mathcal{F}$ . This set represents the possible combinations of the involved variables, that realize the closed-loop system *without* any qualitative and quantitative requirements (4).

Problem 1 can thus be formulated requiring satisfaction of the constraints (4) by every solution of  $FP_{init}$ :

$$\operatorname{FP}(\mathcal{S}) : \begin{cases} \operatorname{find} p \\ \mathrm{s.t.} \quad p \in \mathcal{S}, \\ (x, u, y, p, q, w) \in \mathcal{F}, \\ b_j(u, x, y, z, k) \leq 0, \forall k \in \mathcal{T}_j. \end{cases}$$

Here the set  $S \subseteq \mathcal{P}$  is the parameter search space. We denote with  $\mathcal{P}_{\mathcal{F}}$  the set of all parameter values p from the solution space  $\mathcal{F}$  that satisfy (4), i.e.  $\mathcal{P}_{\mathcal{F}} = FP(\mathcal{P})$ .

However, determining  $\mathcal{P}_{\mathcal{F}}$  precisely is in general a difficult problem. We propose instead to approximate the solution set via a relaxed version of the feasibility problem FP( $\mathcal{P}$ ).

#### 3.2 Outer approximation

Relaxing certain constraints of the feasibility formulation  $FP(\mathcal{P})$  will lead to a simpler problem formulation, whose solution set outer approximates the solution set  $\mathcal{P}_{\mathcal{F}}$ . Common methods to relax the nonlinear problem to a semidefinite or linear program can be found in (Lasserre, 2002; Borchers et al., 2009) and the references therein.

This relaxation process is conservative, meaning it guarantees that the obtained set will nevertheless include every true solution. In this work, we employ a (mixed-integer) linear relaxation following Savchenko et al. (2011). If we

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apply the relaxation procedure directly to the problem formulation  $FP(\mathcal{P})$ , it results in an outer approximation of the admissible parameter values  $\mathcal{P}_{\mathcal{O}} \supseteq \mathcal{P}_{\mathcal{F}}$ .

We briefly outline an iterative process of obtaining the outer approximated set  $\mathcal{P}_{\mathcal{O}}$ , following Savchenko et al. (2014). Denoting with LP( $\mathcal{S}$ ) the relaxed formulation of FP( $\mathcal{S}$ ), we introduce its Lagrangian-dual formulation dLP( $\mathcal{S}$ ). Using then the weak-duality theorem we observe, that in case dLP( $\mathcal{S}$ ) is unbounded, the corresponding primary problem LP( $\mathcal{S}$ ) is infeasible. Due to the conservatism of the relaxation procedure this in turn implies that none of the points in  $\mathcal{S}$  belong to the feasible set  $\mathcal{P}_{\mathcal{F}}$ . Hence, by eliminating the subsets of  $\mathcal{P}$  that are infeasible, we end up with an outer approximation  $\mathcal{P}_{\mathcal{O}}$ :

$$\mathcal{P}_{\mathcal{F}} \subseteq \mathcal{P}_{\mathcal{O}} \coloneqq \mathcal{P} \times \bigcup_{i \in \mathcal{I}, \, dLP(\mathcal{S}_i) \to \infty} \mathcal{S}_i.$$

Clearly, if  $\mathcal{P}_{\mathcal{O}}$  is empty then the controller is not able to fulfill the quantitative and qualitative requirements. Though, outer approximations of solution sets are a useful tool to derive such guaranteed statements, they are less suited for controller parametrization. While we guarantee that outside of the set  $\mathcal{P}_{\mathcal{O}}$  there are no parameter values in  $\mathcal{P}$  satisfying the problem  $FP(\mathcal{P})$ , we cannot confirm that an arbitrary value  $p \in \mathcal{P}_{\mathcal{O}}$  really solves Problem 1.

To estimate the set, consisting only of admissible controller parameters, we introduce next an alternative approximation technique, inverting the original problem formulation.

#### 3.3 Inner approximations via constraint inversion

A feasible parameter  $p^* \in \mathcal{P}_{\mathcal{F}}$  implies that there exists a point  $(x^*, u^*, y^*, p^*, q^*, w^*) \in \mathcal{F}$  for which the constraints (4) are satisfied. To obtain only admissible solutions as in Problem 1, we pose the following definition.

Definition 2. We define a set of guaranteed solutions of FP( $\mathcal{P}$ ) as  $\mathcal{P}_{\mathcal{G}}$ , if for every  $p^* \in \mathcal{P}_{\mathcal{G}}$  the following holds:  $(x, u, y, p^*, q, w) \in \mathcal{F} \Rightarrow b_j(u, x, y, z, k) \leq 0, \forall k \in \mathcal{T}_j$ .

In plain words, the solution  $p^*$  is called *guaranteed* if the quantitative and qualitative requirements of the controller are satisfied with the value  $p^*$  regardless of the uncertainties in the other problem variables. From the definition it is also clear that the inclusion  $\mathcal{P}_{\mathcal{G}} \subseteq \mathcal{P}_{\mathcal{F}}$  holds.

To find an inverse problem formulation we introduce a collection of binary variables  $d_j$ , each associated with a single qualitative constraint (4). We then generate a set of logical equivalence constraints, setting  $d_j$  to *true* if and only if the corresponding constraint (4) is satisfied. Formulating initial problem FP(S) in terms of such constraints can be done via requiring  $\sum_{j=1}^{n_b} d_j = n_b$ . Hence, inversion of this formulation is achieved if this constraint is not satisfied (cf. also Streif et al. (2013)).

We formally introduce the inverse problem as follows.

$$\operatorname{IFP}(\mathcal{S}): \begin{cases} \operatorname{find} & p \\ \operatorname{s.t.} & p \in \mathcal{S}, \\ & (x, u, y, p, q, w) \in \mathcal{F}, \\ & d_j = 1 \Leftrightarrow b_j(u, x, y, z, k) \leq 0, \forall k \in \mathcal{T}_j, \\ & \sum_{j=1}^{n_b} d_j \leq n_b - 1. \end{cases}$$

From the construction of IFP(S) and Definition 2 it is clear, that its solution set contains parameter values, that

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are not in  $\mathcal{P}_{\mathcal{G}}$ . To simplify further reasoning we introduce the following assumptions related to the solution sets.

Assumption 3. (Existence of solutions) For every  $p \in \mathcal{P}$  there exists a combination of x, u, y and w that satisfy constraints (1)-(2) and are included in the domain of function  $b_j$  for every  $k \in \mathcal{T}_j$ .

This assumption simply guarantees that the system is well defined, which in the current formulation means that the domains of functions g, h, c and  $b_j$  are non-empty and can include the sequence of variables for the considered time horizon.

Assumption 4. (Parameter space bounds) The projection of  $\mathcal{F}$  onto the subspace of variables p is equal to the set  $\mathcal{P}$ .

This assumption requires that for each point of  $\mathcal{P}$  there exists a solution of (1) and (2) for every point in  $\mathcal{F}_0$ . This assumption is made for brevity, otherwise the projection of the set  $\mathcal{F}$  onto the subspace of parameters has to be considered in place of  $\mathcal{P}$  throughout the rest of the article.

We can now apply the technique described in Section 3.2 to the formulation IFP(S) instead, achieving the following result.

Theorem 5. (Inner approximation) Under Assumptions 3 and 4 the Lagrangian-dual dILP(S) of the mixed-integer relaxation of the problem IFP(S) is unbounded for  $S \subseteq \mathcal{P}$ if and only if  $S \subseteq \mathcal{P}_{\mathcal{G}}$ .

**Proof.** The weak-duality theorem and employed relaxation procedure guarantee that if the Lagrangian-dual of IFP is unbounded, then it does not admit a solution. Using Assumption 4 and the fact that  $S \subseteq \mathcal{P}$  we conclude that the constraint  $\sum_{j=1}^{n_b} d_j \leq n_b - 1$  cannot be satisfied. This, in turn, means that for each point  $p \in S$  the corresponding points of  $\mathcal{F}$  satisfy the constraint (4).

Algorithm 1 employs Theorem 5 together with the technique presented in Section 3.2, to obtain both the outer approximation  $\mathcal{P}_{\mathcal{O}}$  of the feasible set  $\mathcal{P}_{\mathcal{F}}$ , and the inner approximation  $\mathcal{P}_{\mathcal{I}}$  of the set of guaranteed solutions  $\mathcal{P}_{\mathcal{G}}$ .

Application of the Algorithm 1 leads to the following result

$$(\mathcal{P}_{\mathcal{O}} \smallsetminus \mathcal{P}_{\mathcal{I}}, \mathcal{P}_{\mathcal{I}}, \mathcal{P}_{\varnothing}) \coloneqq \mathrm{IA}(\mathcal{P}),$$

where the set  $\mathcal{P}_{\emptyset}$  denotes the subset of  $\mathcal{P}$ , that does not contain any solutions of  $FP(\mathcal{P})$ .

To summarize, the relations between the different sets described in this section are shown in Fig. 3 and can be formally written as

$$\mathcal{P}_{\mathcal{I}} \subseteq \mathcal{P}_{\mathcal{G}} \subseteq \mathcal{P}_{\mathcal{F}} \subseteq \mathcal{P}_{\mathcal{O}} \subseteq \mathcal{P}.$$

We demonstrate in the following section the results of applying Algorithm 1 to determine the parameters guaranteeing an offset free control of the two tank system.

# 4. EXAMPLE AND DISCUSSION

In this section we present an example of controller verification and parameter tuning for a two tank system. We consider a step change in the reference signal and search for the controller parameters that allow the system to reach the new steady state in finite time.

#### Algorithm 1 Inner approximation (IA)

Input:  $\mathcal{S}$ 

 $\textbf{Output:} \ (\mathcal{S}_{\mathcal{B}}, \mathcal{S}_{\mathcal{I}}, \mathcal{S}_{\varnothing})$ 

```
\mathbf{set}\ \mathcal{S}_{\mathcal{B}}\coloneqq \varnothing,\ \mathcal{S}_{\mathcal{I}}\coloneqq \varnothing,\ \mathcal{S}_{\varnothing}\coloneqq \varnothing
if dLP(S) < \infty then
            if dILP(\mathcal{S}) < \infty then
                       if split depth is not reached then
                                   partition \mathcal{S} \to {\mathcal{S}_1, ..., \mathcal{S}_N}
                                   for i \in \{1, \dots, N\} do
set (\mathcal{S}^*_{\mathcal{I}}, \mathcal{S}^*_{\mathcal{I}}, \mathcal{S}^*_{\emptyset}) \coloneqq IA(\mathcal{S}_i)
                                             S_{\mathcal{B}} \coloneqq S_{\mathcal{B}} \cup S_{\mathcal{B}}^{*}S_{\mathcal{I}} \coloneqq S_{\mathcal{I}} \cup S_{\mathcal{I}}^{*}S_{\mathcal{G}} \coloneqq S_{\mathcal{G}} \cup S_{\mathcal{G}}^{*}
                                   end for
                       else
                                   set \mathcal{S}_{\mathcal{B}} \coloneqq \mathcal{S}
                       end if
           else
                       \mathbf{set}\ \mathcal{S}_\mathcal{I} \coloneqq \mathcal{S}
           end if
else
           set \mathcal{S}_{\varnothing} \coloneqq \mathcal{S}
```

end if 
$$p_i^{\dagger}$$



Fig. 3. Geometric representation of the estimated parameter sets.

## 4.1 Plant model description: Two tanks

We consider the two tank system depicted in Fig. 4 similar to (Ding, 2013). Our setup consists of two tanks connected with a pipe, a regulated inflow into the first tank, and an outflow from the second tank. Torricelli's law for the



Fig. 4. Plant setup.

dynamics of the system and Euler time discretization lead to the following discrete-time description of the system.

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where  $Q_u(k)$ ,  $Q_{12}(k)$  and  $Q_2(k)$  are the input volume flow, the volume flow from tank 1 to tank 2 and the free outflow from tank 2 at time k. Respectively  $h_i(k)$ , i = 1, 2 denote the heights of the water levels in each tank at time k. The sampling time  $T_s$  is set to 2 seconds. For the given setup we can assume that tank 1 has a higher water level, thus the simplified structure of  $Q_{12}(k)$  and  $Q_2(k)$  looks as follows.

$$Q_{12}(k) = a_1 s_{12} \sqrt{2g (h_1(k) - h_2(k))},$$
  

$$Q_2(k) = a_2 s_2 \sqrt{2g h_2(k)}.$$
(6)

From the dynamics of the system we observe, that the steady state is achieved in case  $Q_{12}(k) = Q_2(k)$ , which in turn means  $h_1(k) = 2h_2(k)$ . The scenario we consider has a goal to control the water level in tank 2. We define the two error terms as  $e_1(k) = 2r - h_1(k)$ ,  $e_2(k) = r - h_2(k)$ , where r denotes the reference water level of tank 2.

We consider a PI controller for the discretized system, with the calculated control signal

$$u(k+1) = Q_u(k) + (E(k) - E(k-1)) + K_I T_s E(k),$$

where  $E(k) = K_{P_1} e_1(k) + K_{P_2} e_2(k)$  denotes a cumulative proportional error ("velocity" PI controller).

Furthermore, we assume the control signal is saturated

$$Q_u(k) = \begin{cases} \overline{u}, & u(k) \ge \overline{u}, \\ u(k), & \underline{u} \le u(k) < \overline{u}, \\ \underline{u}, & u(k) < \underline{u}. \end{cases}$$

The employed plant parameter values for the simulations are represented in Table 1. Reformulation of the dynamics

Table 1. Plant parameters

Parameter	Symbol	Values	Unit
cross section of the tanks	A	28	$cm^2$
cross section of the pipes	$s_{12}, s_2$	0.1578	$\mathrm{cm}^2$
pipe coefficients	$a_1, a_2$	0.45	$\mathrm{cm}^2$
max flow rate of the pump	$\overline{u}$	3	$\rm cm^3/s$
min flow rate of the pump	$\underline{u}$	0	$\rm cm^3/s$
max. height of the tanks	$H_{max}$	50	cm
gravitation constant	g	981	$\rm cm/s^2$

in polynomial form and the mixed-integer linear relaxation are performed as shown in Savchenko et al. (2011).

The saturation constraints for the inflow  $Q_u$  are realized using two additional binary variables as follows.

$$z_1(k) = 1 \Leftrightarrow u(k) \le \overline{u},$$
  

$$z_2(k) = 1 \Leftrightarrow u(k) \ge \underline{u},$$
  

$$Q_u(k) = z_1(k)z_2(k)u(k) + (1 - z_1(k))\overline{u} + (1 - z_2(k))u.$$

The uncertain initial water levels are  $h_1(0) = 18 \pm 0.02$ cm and  $h_2(0) = 9 \pm 0.01$ cm. We consider a time horizon of 7 time steps, i. e.  $k_{end} = 7$ . We require that the system enters the proximity of the new steady state within finite time to approximate the behavior of an offset free controller, which we realize through bounding the error terms at the time point  $k_{end}$  as  $e_2(k_{end}) \le 0.1$  and  $E(k_{end}) \le 1$ .

For illustrative purposes, we have chosen the search regions as  $K_{P_1} = [0.01, 0.51]$  and  $K_{P_2} = [0.85, 1.85]$ . The integral gain is fixed to  $K_I = 0.01$  and the new reference is r = [10, 10.01].



Fig. 5. The result of Algorithm 1 applied to the two tank scenario.

## 4.2 Parameter estimation results

We implemented Algorithm 1 using the Analysis, Design and Model Invalidation Toolbox (ADMIT) Streif et al. (2012) with the IBM ILOG CPLEX (2009) solver. Algorithm 1 leads to the estimation results shown in Fig 5. The region depicted in gray ( $\mathcal{P}_{\emptyset}$ ) represents the parameter values that are guaranteed to not satisfy the posed bounds on the error terms at  $k_{end}$ . The region depicted in red ( $\mathcal{P}_{\mathcal{I}}$ ) shows the parameter combinations that guarantee satisfaction of the approximate offset free control requirements subject to all uncertainties posed on the system variables. The rest (depicted in blue) are the parameter values for which conclusive answers cannot be provided. Note, that one can obtain better estimates of the actual sets of all admissible solutions  $\mathcal{P}_{\mathcal{F}}$  and guaranteed solutions  $\mathcal{P}_{G}$  if one further partitions the blue regions.

In conclusion, with the help of Algorithm 1 we determined the subset of the proportional gain parameters, that allow the two tank system to satisfy the posed requirements. The shape of the obtained region has a complex structure due to the nonlinear plant dynamics and the saturation constraints on the controller.

# 5. CONCLUSIONS

In this contribution, an approach for set-based controller verification and parametrization for uncertain polynomial discrete-time systems was presented. This approach allows to derive a set of controller parameters, that guarantee a closed-loop behavior consistent with quantitative and qualitative specifications.

The proposed approach is based on formulating qualitative and quantitative controller requirements given as linear temporal logic formulas in form of a nonlinear feasibility problem. A conservative relaxation of the solution set is then employed to determine the set of controller parameters consistent with the posed requirements despite the considered uncertainties. To do so, a recursive algorithm was proposed, that derives an inner approximation of this set of guaranteed parameter values.

So far we considered only controllers with a fixed structure. Identifying admissible controller structures as well as the allowed parameter values is a subject of future research.

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