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Robust output feedback model predictive control using reduced order models

Markus Kögel * **Rolf Findeisen** *

* *Institute for Automation Engineering, Otto-von-Guericke-University Magdeburg, Germany. e-mail: {markus.koegel, rolf.findeisen}@ovgu.de.*

Abstract: We consider the robust output feedback control of uncertain systems utilizing a robust predictive control scheme based on reduced order models. In detail, we assume that for the plant an uncertain, linear full order model is available. For this model a reduced order model is derived. The proposed control scheme utilizes an estimator based on the reduced order model, which is combined with a robust model predictive controller to robustly stabilize the system. The proposed framework allows to guarantee robust satisfaction of input and output constraints as well as robust stability. An efficient implementation is possible, because online only the solution of a standard model predictive control problem and a simple state estimation problem are needed, which both involve the reduced model. The applicability of the control scheme is outlined using simulation examples.

Keywords: Model predictive control, Robust model predictive control, Large-scale linear systems, Model order reduction, Robustness stability.

1. INTRODUCTION

Model predictive control (MPC) is an often used advanced control strategy in process control due to its ability to take constraints into account and its applicability to handle multi-variable systems, see e.g. Garcia et al. (1989); Maciejowski (2002); Qin and Badgwell (2003); Rawlings and Mayne (2009); Mayne (2014). The feedback in MPC is based on repeated optimal control and requires at every time instance the solution of an optimal control problem. This can be challenging for large-scale systems such as large plants, systems described by distributed parameters or if only low-cost hardware is available. The last is the case for example for the low cost control of small size distillation column (see e.g. Lima et al. (2013)) or the management of Lithium-ion batteries (see Klein et al. (2011); Suthar et al. (2013); Rausch et al. (2014)). One possibility to decrease the computational burden is to utilize for MPC reduced order models instead of full order models, which is the focus of this paper.

Note that also other approaches to reduce the computation time of MPC are possible such as e.g. tailored solution methods for MPC (see for example Rao et al. (1998); Ferreau et al. (2008); Wang and Boyd (2010); Kögel and Findeisen (2011, 2013); Patrinos and Bemporad (2014) and the references therein) or the distributed solution for interconnected systems (see e.g. Stewart et al. (2010); Kögel and Findeisen (2012)). Note that the proposed approach is no replacement for these methods, in fact it can be combined with these methods to further decrease the computational burden.

One source of large scale systems arise from the control of distributed parameter systems, where typically the partial differential equations are discretized leading to a large set of linear differential equations. These systems are often further reduced using model reduction techniques (see e.g. Antoulas (2005); Benner et al. (2005)) to facilitate the controller design and implementation.

Other examples are high-purity fractionation distillation columns consisting of many stages (see e.g. Skogestad and Morari (1988)) where the size of the arising state space system depends on the number of trays. Although the arising state-space model is rather large, for the control often a less accurate description of the input output behavior is sufficient, see Findeisen and Allgöwer (2000).

Several approaches considering MPC utilizing reduced order models exist. In Hovland et al. (2006) a reduced order model is used to enable the utilization of an explicit model predictive controller (Bemporad et al. (2002)) for the control of systems arising from the heat equation, but the reduction error is not taken into account. In Hovland et al. (2008) conditions given by linear matrix inequalities are presented for the stability of MPC based on reduced order models using hard constraint for the inputs and soft constraints for the output. Sopasakis et al. (2013) considers the control of a discrete-time linear, constrained system without external noise using a reduced order model and assuming that all constrained states are measured. Continuous-time, linear, constrained systems are considered in Löhning et al. (2014). Therein, the reduction error is bounded using an error bounding approach to guarantee stability using a continuous time MPC approach for the case that measurements without noise are available and no external disturbances affect the system.

The contribution of this work is to a control scheme guaranteeing robust constraint satisfaction and robust stability using an output feedback controller based on the reduced order model. In detail, we propose to combine a linear estimator with a tailored robust model predictive control scheme. The main idea is to derive bounds on the estimation error to design an appropriate robust predictive controller.

We outline an efficient approach based on convex optimization for the offline computation of the estimation error bounds. Online, the estimator based MPC can be efficiently be implemented, because at each sampling instance only a convex optimal control problem and the evaluation of the simple estimator are necessary, which both involve only the reduced order dynamics.

The remainder of the paper is structured as follows. The next section outlines the problem setup. In Section III we discuss the state estimation of the reduced order model and present methods to bound the arising estimation error. Section IV investigates the use of robust MPC techniques to guarantee robust constraint satisfaction and robust stability. Section V illustrates the proposed approach using simulations.

2. PROBLEM SETUP

This section outlines the problem setup and proposes an output feedback MPC scheme utilizing reduced order models.

2.1 Full order model

We assume that the full order model of the plant to be controlled is given by the following dynamics (the superscript ^{f} is used for the full-order model)

$$x_{k+1}^f = A^f x_k^f + B^f u_k + w_k^f, \tag{1a}$$

$$y_k = C^f x_k^f + v_k^f, \tag{1b}$$

$$z_k = D^f x_k^f + E^f u_k, \tag{1c}$$

where $x^f \in \mathbb{R}^n$ denotes the plant's state, $u \in \mathbb{R}^p$ the controlled input (also called manipulated variables), $w^f \in \mathbb{R}^n$ bounded, unmeasured disturbances (disturbance variables), $y \in \mathbb{R}^q$ the measured output (measured variables) corrupted by the additive, bounded measurement noise v^f. In (1c) $z \in \mathbb{R}^r$ denotes a performance output (also known as controlled variables), which is controlled in order to guarantee constraint satisfaction and to achieve a specific performance. In general, $p \ll n$ and $q \ll n$.

The process noise w_k^f and measurement noise v_k^f are restricted to the convex polytopes $\mathbb{W}$ and $\mathbb{V}$, respectively,

$$w_k^f \in \mathbb{W}, \qquad v_k^f \in \mathbb{V}. \tag{2}$$

The input u_k and the performance output z_k are restricted to the convex polytopes, $\mathbb{U}$ and $\mathbb{Z}$, respectively:[1]

$$u_k \in \mathbb{U}, \ \ z_k \in \mathbb{Z}. \tag{3}$$

2.2 Output feedback based on reduced order models

We want to control the plant (1) via an output feedback. For this we combine an estimator with a MPC as illustrated in Fig. 1. However for large system dimensions utilizing the full order model for MPC results is in general challenging, since:

[1] Rate constraints can be added by extending the state and output.

1) it might be difficult to estimate all states reliable
2) the optimization problem resulting from MPC might be large, i.e. cannot be solved in real-time.

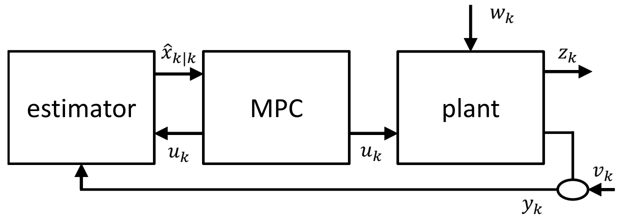

Fig. 1. Proposed output feedback control scheme.

Instead of the full order model (1) we propose to use reduced order models with a robust predictive controller and a state estimator that provides error bounds. The reduced order model is assumed to have the state $x \in \mathbb{R}^m$. The nominal reduced order model is of the form

$$x_{k+1} = Ax_k + Bu_k, \tag{4a}$$

$$y_k = Cx_k, \tag{4b}$$

$$z_k = Dx_k + Eu_k. \tag{4c}$$

We assume that (A, B) is stabilizable and that (A, C) is observable.

Remark 1. (Model reduction method)
We do not require that the reduced order model (4) *is obtained from the full order model* (1) *by a special reduction method. The focus of this work is to derive conditions such that one can guarantee robustness for a given reduced order model; the actual design of the reduced order model is beyond the scope of this work.*

Note that the nominal reduced model (4) does not take the effects of the process noise, the measurement noise and the model reduction error into account. Therefore, we first outline in the next section how we can quantify these errors based on the error analysis of a linear state estimator. Afterwards we combine the proposed state estimator with a tailored robust MPC approach based on the reduced order model to obtain the robust output feedback MPC.

3. STATE ESTIMATION AND ERROR BOUNDING

This section presents the state estimator used in this work. First we discuss the structure of the state estimator. Afterwards we show how the estimation error can be bounded. Finally, we discuss the design of an estimator based on Kalman filtering.

3.1 Structure of state estimator

To estimate the state of the reduced model (4) we utilize a linear estimator based on the input u and measured output y during the last M time instances and the current measurement given by:

$$\hat{x}_{k|k} = G\mathbf{y}_k + H\mathbf{u}_k, \tag{5a}$$

$$\mathbf{u}_k = \left(u_{k-M}^T, \ldots, u_{k-1}^T\right)^T, \tag{5b}$$

$$\mathbf{y}_k = \left(y_{k-M}^T, \ldots, y_k^T\right)^T, \tag{5c}$$

where $G \in \mathbb{R}^{m \times pM}$ and $H \in \mathbb{R}^{m \times r(1+M)}$. Note $\hat{x}_{k|k}$ is estimated using only the previous history of inputs and outputs. Estimators of this type are e.g. (unconstrained) moving horizon estimator with zero prior weighting (see e.g. Rawlings and Mayne (2009)) or fixed interval estimators (see Kailath et al. (2000)).

Larger values of M can improve the accuracy of the estimate $\hat{x}_{k|k}$. Furthermore, below a specific value of M one cannot guarantee that the estimate of $\hat{x}_{k|k}$ is consistent even in the absence of any noise or model reduction error.

In Section 3.3 we discuss the design of (5) using least-square estimation by adding some fictitious process/ measurement noise.

3.2 Error bounds

Due to the process noise w^f, measurement noise v^f and model reduction two types of error arise: First, the estimated performance output $\hat{z}_{k|k}$ might be different from the real performance output z_k. We denote this error by

$$\Delta z_k = \hat{z}_{k|k} - z_k. \tag{6}$$

Second, we have some disturbance d_k on the dynamics of the estimated state of the reduced order model given by

$$d_k = \hat{x}_{k+1|k+1} - A\hat{x}_{k|k} - Bu_k. \tag{7}$$

The idea of this work is to bound both errors and explicitly consider them in the controller. Therefore, we aim to derive convex, polytopic sets $\Delta\mathbb{Z}$ and $\mathbb{D}$ such that

$$\Delta z_k \in \Delta\mathbb{Z}, \qquad d_k \in \mathbb{D}, \tag{8}$$

for any admissible state, input, process noise and measurement noise.

Utilizing the sets $\mathbb{D}$ and $\Delta\mathbb{Z}$ allows to utilize tailored robust MPC setups to robustly control the system (1) and guarantee constraint satisfaction, while utilizing only the reduced order model (4) for estimation and prediction.

In the following, we assume for a parameter P that:

Assumption 2. (Consistency of past behavior) *Over the last $P + M$ time instances the trajectory of the full-order system (1) and input sequence satisfy the constraints (3) and are consistent with the dynamics (1), (2). In detail, $\overline{\mathbf{x}}_k = \{x_{k-M-P}, \dots x_k\}$ and $\overline{\mathbf{u}}_k = \{u_{k-M-P}, \dots u_{k-1}\}$ satisfy*

$$u_i \in \mathbb{U},\ i = k - M - P, \dots, k - 1, \tag{9a}$$
$$z_j = D^f x_j + E^f u_j \in \mathbb{Z},\ j = k - M - P, \dots, k, \tag{9b}$$

and $\overline{\mathbf{x}}_k$, $\overline{\mathbf{u}}_k$ satisfy (1) for some, admissible realizations of the process and measurement noise.

Note that this assumption is often satisfied in process control, since the system is started / the set-point is changed only after staying long enough in or near a well known (steady) state. Observe that if the Assumption 2 is satisfied at k and a control scheme guaranteeing constraint satisfaction is used, then it is also satisfied at $k + 1$.

Assumption 2 allows to explicitly characterize the sets $\Delta\mathbb{Z}$ and $\mathbb{D}$ due to the special form of the estimator (5):

Theorem 3. (Characterizing the set $\Delta\mathbb{Z}$) *Let Assumption 2 hold. The set $\Delta\mathbb{Z}$ is given by all Δz_k with*

$$\Delta z_k = \hat{z}_{k|k} - z_k, \tag{10a}$$

such that $u_i \in \mathbb{U}$, $w_i^f \in \mathbb{W}$, $v_i^f \in \mathbb{V}$, $z_j \in \mathbb{Z}$ and for $i = k - M - P, \dots, k - 1$ and $j = k - M - P, \dots, k - 1$

$$x_{i+1}^f = A^f x_i^f + B^f u_i + w_i^f, \tag{10b}$$
$$y_i = C^f x_i^f + v_i^f, \tag{10c}$$
$$z_j = D^f x_j^f + E^f u_j, \tag{10d}$$

and

$$\hat{x}_{k|k} = G\mathbf{y}_k + H\mathbf{u}_k, \qquad \hat{z}_{k|k} = D\hat{x}_{k|k} + Eu_k, \tag{10e}$$

with $\mathbf{u}_k$ and $\mathbf{y}_k$ as in (5).

The theorem directly follows from the problem definition, the definition of the estimator (5) and Assumption 2. Note that all appearing equalities and inequalities are affine. Therefore, it is easily possible to compute off-line an outer approximation of $\Delta\mathbb{Z}$ in form of a box by solving a sequence of linear programs (LPs). This is can be done by minimizing/maximizing for all basis vectors e_i, i.e. for $e_1 = (1, 0, \dots)^T$, $e_2 = (0, 1, 0, \dots)^T$ etc., the linear cost function $J_i = e_i(\hat{z}_{k|k} - z_k)$. Note that the size of these LPs is influenced by the size of the full and reduced order model as well as the choice of the design parameter P and the estimation interval M. Fortunately, these LPs need to be solved only once off-line.

Also the set $\mathbb{D}$ is described similarly.

Theorem 4. (Characterization of the set $\mathbb{D}$) *Let Assumption 2 hold. The set $\mathbb{D}$ is given by all d_{k-1} satisfying*

$$d_{k-1} = \hat{x}_k - A\hat{x}_{k-1} - Bu_{k-1}, \tag{11a}$$

where

$$\hat{x}_k = G\mathbf{y}_k + H\mathbf{u}_k, \quad \hat{x}_{k-1} = G\mathbf{y}_{k-1} + H\mathbf{u}_{k-1}, \tag{11b}$$

with $\mathbf{u}_l$ and $\mathbf{y}_l$ as in (5) and where $u_i \in \mathbb{U}$, $w_i^f \in \mathbb{W}$, $v_i^f \in \mathbb{V}$, $z_j \in \mathbb{Z}$ with $j = k - M - P, \dots, k$, $i = k - M - P, \dots, k - 1$ and

$$x_{i+1}^f = A^f x_i^f + B^f u_i + w_i^f, \tag{11c}$$
$$y_i = C^f x_i^f + v_i^f, \tag{11d}$$
$$z_j = D^f x_j^f + E^f u_j. \tag{11e}$$

This theorem is an immediate consequence of the definition of $\mathbb{D}$, of the estimator (5) and Assumption 2. As above it is a set of affine equalities and inequalities, which allows to compute a box as an outer bound of $\mathbb{D}$.

In summary, one can compute an outer approximation of $\Delta\mathbb{Z}$ and of $\mathbb{D}$ by solving a number of large, but efficiently solvable optimization problems. Fortunately, these computations can be carried out offline.

3.3 Fixed interval, least square estimator

One possibility to design the linear estimator (5) is to utilize (linear) least square estimation, see e.g. Kailath et al. (2000). To utilize this setup let us add the artificial process noise w_k and measurement noise v_k on the nominal, reduced order model (4) resulting in a model of the form

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{12a}$$
$$y_k = Cx_k + v_k. \tag{12b}$$

Using least square estimation we determine a state sequence $\hat{\mathbf{x}}_k$ and noise sequences $\hat{\mathbf{v}}_k$ and $\hat{\mathbf{w}}_k$ given by

$$\hat{\mathbf{x}}_k = (\hat{x}_{k-M|k}, \dots, \hat{x}_{k|k}), \tag{13a}$$
$$\hat{\mathbf{v}}_k = (\hat{v}_{k-M|k}, \dots, \hat{v}_{k|k}), \tag{13b}$$
$$\hat{\mathbf{w}}_k = (\hat{w}_{k-M|k}, \dots, \hat{w}_{k-1|k}), \tag{13c}$$

such that the convex quadratic cost function

$$J^e = \sum_{j=k-M}^{k} \hat{v}_{j|k}^T V \hat{v}_{j|k} + \sum_{i=k-M}^{k-1} \hat{w}_{i|k}^T W \hat{w}_{i|k}, \tag{14}$$

is minimized and the equality constraints arising from the dynamics and measurement equations are satisfied. V and W are symmetric, positive definite penalty matrices. The equalities to be satisfied are given by

$$\hat{x}_{i+1|k} = A\hat{x}_{i|k} + Bu_k + \hat{w}_{i|k}, \quad y_j = C\hat{x}_{j|k} + \hat{v}_{j|k}, \tag{15}$$

where $i = k-M, \dots, k-1$ and $j = k-M, \dots, k$. The resulting estimate for $\hat{z}_{k|k}$ is given by $\hat{z}_{k|k} = D\hat{x}_{k|k} + Eu_k$.

This means that the estimate is based on the solution of the equality constrained, convex quadratic program

$$\min_{\hat{\mathbf{x}}_k, \hat{\mathbf{v}}_k, \hat{\mathbf{w}}_k} J^e(\hat{\mathbf{v}}_k, \hat{\mathbf{w}}_k) \text{ subject to (15).} \tag{16}$$

Since there are no inequality constraints, one can solve this optimization problem analytically, which results in a solution of the form (5), see e.g. Nocedal and Wright (2000). Note that for certain special cases, e.g. if the matrix A is invertible, then one can solve (16) efficiently, by using Kalman filtering based on the information filter formulation.

Remark 5. (Choice of penalty matrices V and W)
In the cost function J^e, the penalty matrices $V = V^T > 0$ and $W = W^T > 0$ are design parameters, which influence how nonzero $\hat{w}_{i|k}$ and $\hat{v}_{i|k}$ are penalized. Therefore it is important to tune them.

4. ROBUST MODEL PREDICTIVE CONTROL

In this section we present a robust model predictive control scheme tailored to our problem setup. This scheme combined with the state estimator and the bounding approach presented above enables the robust output feedback control of the full order system (1) using only the reduced order model (4) for the required online computations. We first outline the proposed robust model predictive control scheme and then discuss implementation issues.

In the previous sections we showed that the estimation error and the dynamics of the estimates $\hat{x}$ are given by

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + d_k, \tag{17a}$$
$$z_k = D\hat{x}_{k|k} + Eu_k + \underbrace{z_k - \hat{z}_{k|k}}_{\Delta z_k}, \tag{17b}$$

where $d_k \in \mathbb{D}$ and $\Delta z_k \in \Delta\mathcal{Z}$. In addition, as outlined in the previous section it is possible to compute offline boxes $\overline{\mathbb{D}} \supseteq \mathbb{D}$ and $\overline{\Delta\mathcal{Z}} \supseteq \Delta\mathcal{Z}$ as outer-approximations of these two sets. This result is the basis for the proposed robust control approach.

We consider a tube-based, robust MPC approach similarly as in Mayne et al. (2006), see also Rawlings and Mayne (2009) for more details. Note that also other robust MPC approaches such as Chisci et al. (2001) might fit into the considered framework, but are not discussed here due to space limitations. Loosely speaking, in tube-based, robust MPC the controller is based on a combination of a predictive control law for the disturbance free case and a linear feedback to take future disturbances into account. This combination allows to reduce the conservatism, while the required optimal control problem to be solved is of similar complexity as in standard MPC.

We need the following definition (Rawlings and Mayne (2009)).

Definition 6. (Robust invariant set $\overline{S}_\infty(K)$)
The set $\overline{S}_\infty(K)$ is a robust invariant set for the dynamics $\hat{x}_{k+1|k+1} = (A+BK)\hat{x}_{k|k} + d_k$, $d_k \in \overline{\mathbb{D}}$, if for every $s \in \overline{S}_\infty(K)$ and every $d \in \overline{\mathbb{D}}$ it holds that $(A+BK)s + d \in \overline{S}_\infty(K)$.

Note that $\overline{S}_\infty(K)$ exists only if K is such that $A+BK$ is asymptotic stable (unless $\overline{\mathbb{D}} = \{0\}$).

The robust MPC optimizes a convex quadratic cost function using a nominal state trajectory $\overline{\mathbf{x}} = (\overline{x}_k^T, \dots, \overline{x}_{k+N}^T)^T$ and input sequence $\overline{\mathbf{u}} = (\overline{u}_k^T, \dots, \overline{u}_{k+N-1}^T)^T$, where $N > 1$ is the horizon length. As in Mayne et al. (2006); Rawlings and Mayne (2009) this nominal state trajectory need to satisfy the dynamics[2]

$$\overline{x}_{i+1} = A\overline{x}_i + B\overline{u}_i, \; i = k, \dots, k+N-1, \tag{18a}$$
$$\overline{x}_k = \{\hat{x}_{k|k}\} \oplus \overline{S}_\infty(K), \tag{18b}$$

and the constraints

$$\overline{u}_i \in \underline{\mathbb{U}}, \qquad D\overline{x}_i + E\overline{u}_i \in \underline{\mathbb{Z}}, \qquad \overline{x}_{k+N} \in \mathbb{X}^f, \tag{18c}$$

where $i = k, \dots, k+N-1$ and the tightened constraints $\underline{\mathbb{U}}$ and $\underline{\mathbb{Z}}$ are given by

$$\underline{\mathbb{U}} = \mathbb{U} \ominus K\overline{S}_\infty(K),$$
$$\underline{\mathbb{Z}} = \mathbb{Z} \ominus (D+EK)\overline{S}_\infty(K) \ominus \Delta\mathbb{Z}.$$

The set $\mathbb{X}^f$ is the so-called terminal set.

The cost function is given by

$$V(\overline{\mathbf{x}}, \overline{\mathbf{u}}) = \overline{x}_{k+N}^T Q^f \overline{x}_{k+N} + \sum_{i=k}^{k+N-1} \overline{x}_i^T Q \overline{x}_i + \overline{u}_i^T R \overline{u}_i, \tag{19}$$

where the state penalty matrix $Q = Q^T$ and input penalty matrix $R = R^T$ are design parameters, which need to be chosen such that R is positive definite, Q positive semi-definite and $(A, Q^{\frac{1}{2}})$ observable and Q^f is given by $Q^f = (A+BK)^T Q^f (A+BK) + K^T RK + Q$. Under these assumptions it is possible to establish the following results.

Corollary 7. (Robust, reduced order output MPC)
Let Assumption 2 hold and the terminal set $\mathbb{X}^f$ be given such that $\forall x \in \mathbb{X}^f$ we have $Kx \in \underline{\mathbb{U}}$ and $(A+BK) \in \mathbb{X}^f$ and $(D+EK)x \in \underline{\mathbb{Z}}$.

If the optimization problem $O(\hat{x}_{0|0})$ given by

$$\min_{\overline{\mathbf{x}}, \overline{\mathbf{u}}} V(\overline{\mathbf{x}}, \overline{\mathbf{u}}) \text{ subject to (18),} \tag{20}$$

is feasible at $k = 0$ and $u_k = \overline{u}_k^\star + K(\hat{x}_{k|k} - \overline{x}_k^\star)$ is used as feedback, where $\overline{x}_k^\star$ and $\overline{u}_k^\star$ are obtained from the optimal solution of (20), then for any $k \geq 0$

- *$u_k \in \mathbb{U}$, $z_k \in \mathbb{Z}$ (robust constraint satisfaction)*
- *for any $w_k^f \in \mathbb{W}$ and $v_k^f \in \mathbb{V}$ the optimization problem $O(\hat{x}_{k+1|k+1})$ is feasible (robust recursive feasibility)*

[2] $\oplus$, $\ominus$ denote the Minkowski sum/difference, respectively.

- *x_k converges to $\overline{S}_\infty(K)$ (robust stability).*

This corollary is a rather straightforward combination of the results of Mayne et al. (2006) and the results derived in the previous section. Therefore a detailed proof is avoided here. In summary, the corollary shows that under mild conditions the closed loop system will satisfy the constraints and the system state will be robustly stabilized.

Note that $\mathbb{X}^f$ can be chosen either as an ellipsoid or a polytope, compare Rawlings and Mayne (2009).

Remark 8. (Computation of sets $\overline{S}_\infty(K)$)
The sets $\overline{S}_\infty(K)$ can be computed using tailored algorithms, see Rawlings and Mayne (2009) and the references therein.

Remark 9. (Optimal control problem)
The optimal control problem (20) *is a convex quadratic program (QP), if the terminal constraint $\mathbb{X}^f$ is a convex polytope and convex quadratically constrained quadratic program (QCQP), if $\mathbb{X}^f$ is an ellipsoid. Various tailored solution methods have been proposed in the context of MPC to solve the arising optimal control problem such as e.g. Rao et al. (1998); Ferreau et al. (2008); Wang and Boyd (2010); Kögel and Findeisen (2011, 2013); Patrinos and Bemporad (2014). Often the multi-stage structure of the optimization problem* (20) *is exploited, e.g. by considering the states as optimization variables (so-called sparse formulation) to accelerate the solution speed.*

Remark 10. (Reduced order / full order MPC)
To illustrate the advantage of the proposed reduced order MPC approach, let us consider interior point methods (see e.g. Rao et al. (1998); Wang and Boyd (2010); Kögel and Findeisen (2013)) that utilize the multi-stage structure. In this case the computational effort for an iteration is in general $O(N(m+p+q)^3)$, i.e. the effort approximately grows linearly with the horizon length N and cubically with the problem dimensions (including the state dimension m). Loosely, speaking, if we assume that $m+p+q$ is half/one tenth of $n+p+q$, then the reduced order MPC can speed up the solution speed by a factor of (approximately) 8 *or* 1000*, respectively.*

Note that if in addition to the multi-stage structure of (20) *also structure of the dynamics and/or the cost function is used, then methods as Rao et al. (1998); Wang and Boyd (2010); Kögel and Findeisen (2013) have a lower computational effort. As an example one might consider systems with finite impulse response for which tailored interior point methods, e.g. Kögel and Findeisen (2013), have a computational effort quadratically in the system dimension, if suitably adapted.*

5. NUMERICAL EXAMPLES

We illustrate the approach using two examples. First we consider a SISO toy example. The second is the control of a high-purity distillation column. `MATLAB` and `YALMIP` (Löfberg (2004)) were used for these studies.

5.1 Toy example

We consider a SISO system with the transfer function

$$G(s) = \frac{270}{(s+1)(s+3)} + \frac{50}{(s+2)(s+5)(s+1)} + \frac{0.001}{s+0.01}$$

where the performance and measurement output are the same except of the measurement noise (i.e. $C^f = D^f$). The system has $n = 5$ states. We consider as a reduced model an approximation by a system with one state, which we obtained by balanced reduction (`balred` command in `MATLAB`, without matched DC gains). For the chosen sampling time of 0.1 Fig. 2 shows the Bode plot of $G(s)$ and its reduced order approximation.

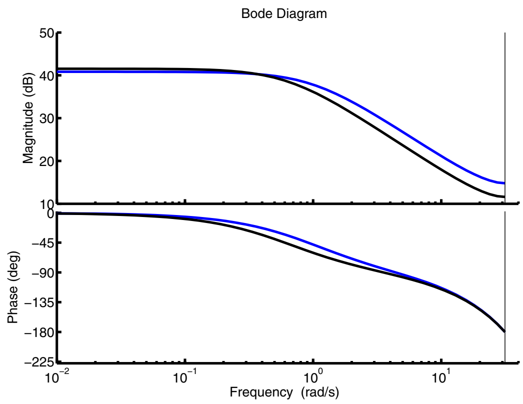

Fig. 2. Bode plot of $G(s)$ (blue) and reduced order approximation (black).

We assume that the noises are bounded by $\mathbb{V} = \{v | \, |v| \leq 0.01\}$, $\mathbb{W} = \{w | \, |w| \leq 0.01\}$ and that the constraints are given by $|u_i| \leq 2$, $|z_i| \leq 100$ and $|u_i - u_{i-1}| \leq 1$. Note that the input rate constraints can be incorporated by an extension of the system state.

For the state estimation we use a fixed interval least square estimator as discussed in Section 3.3 using $W = V = 1$ and $M = 5$. For $P = 100$ we can obtain $\Delta\mathbb{Z} = \{\Delta z | \, |\Delta z| \leq 6.418\}$ and $\mathbb{D} = \{d | \, |d| \leq 2.618\}$. For the robust MPC setup we choose $N = 20$, $R = 1$ and $Q = 0.01$. Fig. 3 and 4 illustrate the closed loop behavior. The behavior of the system is as expected: the system is robustly stabilized to the steady state ($z_k = 0$, $u_k = 0$) and the constraints are satisfied.

5.2 Fractional distillation column

We considers the control of a fractional distillation column, the so-called Column A presented in Skogestad and Morari (1988). The distillation column consists of 40 stages and separates a binary mixture into products with a purity of 99%. The feed is at the middle of the column (stage 21) and has a purity of 50%. The control inputs (manipulated variables) are the distillate and bottom flow rates, the reflux flow and boil-up flow. The performance output (controlled variables) are the purity of the products as well as the liquid holdup in the condenser and reboiler.

We only use a linearized model available from `http://www.nt.ntnu.no/users/skoge/book/matlab_m/cola/cola.html`, which we scaled for this simulation study. The system has $n = 82$ states and is not asymptotic stable: it has two integrating states. We choose a sampling time of 3 minutes. We assume that measurements of the

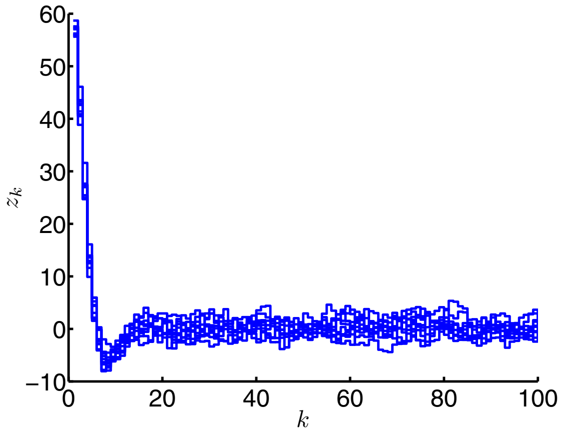

Fig. 3. Performance output z_k for SISO system. Same initial conditions. Different noise realizations.

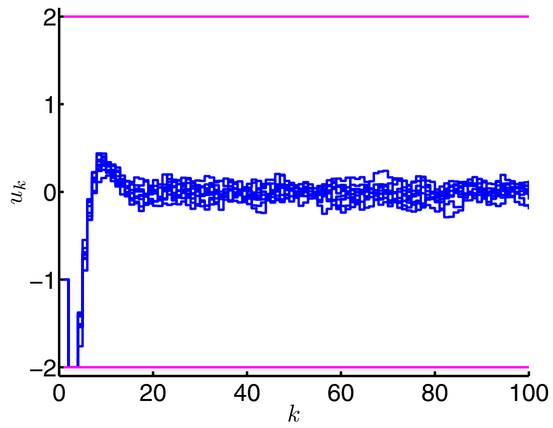

Fig. 4. Plot of controlled input u_k (blue) for SISO system. Same initial conditions. Different noise realizations.

composition and liquid holdup of stages 1, 11, 21, 31 and the condenser are available.

We consider box constraints on the measurement noise $\mathbb{V} = \{v| \|v\|_\infty \leq 0.001\}$ and constraints of the form $\|u_k|_\infty \leq 6$, $\|z_k\|_\infty \leq 10$ and $\|u_k - uk - 1\|_\infty \leq 1.5$. With respect to the process noise we assume that the main disturbance is in the feed: the feed flow and composition can each vary between -0.05 and 0.05 from the nominal value and additionally on every state acts a small disturbance with values between -10^{-4} and 10^{-4}.

We consider a reduced order model with $m = 10$ states obtained via balanced reduction and use a fixed interval estimator based on least square estimation (compare Section 3.3) using $W = I$ and $V = 10I$ and $M = 30$. We obtained the outer bounds on the sets $\mathbb{D}$ and $\Delta\mathbb{Z}$ assuming that Assumption 2 holds for $P = 90$. For the robust MPC setup we choose $N = 120$, $R = 0.01I$ and $Q = D^T D$ and $\mathbb{X}^f$ as a polytopic set.

Figure 5 and 6 show simulations of the closed loop system using the linearized system. In the first period the steady state is $z_k = 0$ and $u_k = 0$ and jumps in the second and third period to different set points. We observe that the constraints are satisfied and that the system is robustly regulated to the set points.

The average computations necessary at each time instance take about 240 ms using a 3.4 GHz `Intel i7-4770` CPU and the quadratic programming solver `quadprog` from `MATLAB`. In comparison using the full order model the average time is about 3.6s, i.e. using the reduced order model allows to reduce the computational effort by a factor of 15 in this example.

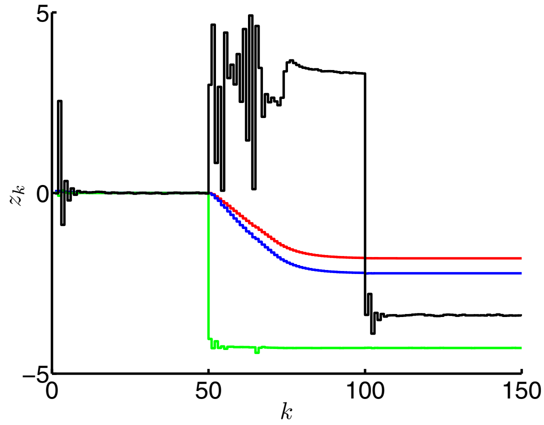

Fig. 5. Plot of performance output z_k for linearized distillation column. Red: purity of distillate product. Blue: purity of bottom product. Green: liquid holdup in condenser. Black: liquid holdup in reboiler.

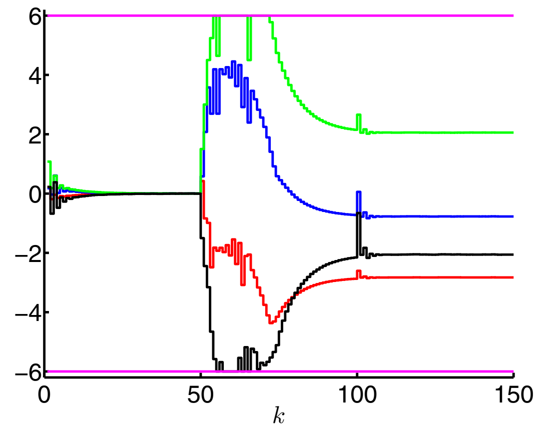

Fig. 6. Plot of controlled input u_k for linearized distillation column. Red: reflux flow. Blue: boil-up flow. Green: distillate product flow rate. Black: bottom product flow rate. Magenta: input constraints.

6. SUMMARY AND FUTURE WORKING DIRECTIONS

In this work we considered the control of a linear, discrete-time, constrained process subject to additive process and measurements noise using an output feedback based on model predictive control and, especially, a reduced order model. We outlined a robust model predictive control approach for this problem setup using a fixed interval, linear state estimator and a robust MPC. In detail, we proposed an offline analysis based on linear programming to bound the maximum possible estimation errors. This allows to design the robust MPC such that it can take the effects of the process and measurement noise as well as the reduction error into account. The results are illustrated using simple examples.

In future works we will analyze and evaluate the proposed framework in more details. Future works consider an extension to set-point tracking, similar as Alvarado et al. (2007). Also methods to improve the design of the estimator, reduced model and robust MPC are of interest as well as co-design methods, i.e. methods to combine these design processes into a single process. Another possible extensions of the proposed approach is to use nonlinear estimators. Finally, we aim to apply the proposed approach also to applications arising in the optimal charging of Lithium ion batteries and to other applications.

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