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Observer Design Using Potential Based Realizations

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Abstract: This paper considers observer design for nonlinear dynamical systems which can be approximated by a dissipative Hamiltonian realization. The design approach decomposes the system associated one-form of a given dynamical system over an indeterminate metric using the Homotopy operator to generate exact (potential driven) and anti-exact parts. Then the convexity of the potential system given by the exact part is assessed and we propose a metric equation which yields a Lyapunov function for the potential driven observer system. An application of this method is demonstrated for a two-dimensional van der Pol oscillator.

Keywords: Nonlinear observer, potential-based realization, nonlinear systems

1. INTRODUCTION

Observer design is an important field of research for dynamical systems. While observer theory is well developed for linear system, the problem remains open for nonlinear systems. A recent survey of nonlinear observer design methods was recently published in Kang et al. (2013). One important class of nonlinear observed design techniques is based on the differential geometric control approach first presented in (Krener and Respondek, 1985). The least restrictive nonlinear observer design approach is the so-called Kravaris-Kazantzis observer Kazantzis and Kravaris (1998) with extensions discussed inXiao et al. (2003) , Kravaris et al. (2007) and Andrieu and Praly (2006). Moving horizon nonlinear observers have also been proposed in the literature Kang (2006). While such observers are quite general, they rely heavily on the solution of nonlinear dynamic optimization problems in real-time. As a result, they are computationally demanding.

Design of dissipative observers and separation results were given in (Moreno, 2006, 2008), with an application to chemical reactors observer and controller design in (Schaum et al., 2008). Some results on output feedback (Ortega et al., 1999) and observer design (Lohmiller and Slotine, 1997) can be found in the literature on generalized Hamiltonian systems. Observer design and observer-based control of generalized Hamiltonian systems were studied in (Wang et al., 2005). Of particular importance for the construction given in this paper is the construction of observers for non-integrable systems given in (Lynch and Bortoff, 1997b,a), which is built directly on the homotopy-based approach originally presented in (Banaszuk and Hauser, 1996). The main advantage of the proposed approach is that the approximate dissipative Hamiltonian, even with a constant arbitrary structure, enables one to capture the dynamics of the drift system with a very simple system in the new coordinates, rendering the design of the observer and the observer-based feedback controller straightforward. The design of a potential-based coordinate change coupled with an observer-based feedback could be considered as an alternative to output feedback stabilization problems for dissipative Hamiltonian systems in applications where a Hamiltonian function is not known *a priori*, such as in the study presented in (Ortega et al., 1999, 2000).

An alternative approach has been proposed (Aghannan and Rouchon, 2003) and (Bonnabel, 2010) for the design of position and velocity based observers for Lagrangian systems. The Lagrangian based observer uses a Riemannian manifold endowed with a kinetic energy based metric. These observers work on the class of mechanical systems where position defines the system state. By exploiting the intrinsic symmetry inherent in the Euler-Lagrange equations a contracting observer can be defined on the state space. In Sanfelice and Praly (2012), a metric-based nonlinear observer design technique is proposed. Using a known Riemannian metric, a converging nonlinear observer can be obtained subject to several conditions that include geodesic monotonicity, convexity of the output function, and uniform detectability.

In this paper, we propose an alternative metric-based observer design approach for nonlinear systems of the form:

$$\dot{x} = f(x),\ y = h(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, $y \in \mathbb{R}^m$ are the output variables, $f(x)$ is a vector-valued C^k function of state variables $x \in \mathbb{R}^n$, and $h(x)$ is a C^k function with k≥2. Throughout, the system has an equilibrium at $x = x^*$. Following the approach proposed in (Guay et al., 2013), a Hodge decomposition approach is used to obtain a potential function that allows the definition of a suitable metric that can be used to the asymptotic stability of the set:

$$\varepsilon = \{x, \hat{x} \in \mathbb{R}^n \mid x = \hat{x}\}, \tag{2}$$
$$\dot{\hat{x}} = F(\hat{x}, h(x)), \tag{3}$$
where F is our designated observer function.

The organization of the paper is as follows. Section 2 presents the necessary mathematical background used for the method of decomposition, notably the homotopy operator and its properties. The design of a potential based observer for the control system is then presented. The homotopy operator is used to obtain the observer error system's characteristic potential and a metric equation is proposed. After which, the Lyapunov function is evaluated for stability given the metric equation. An example for a two dimensional van der Pol oscillator is presented in Section 4 which illustrates the approach. Concluding remarks are then given and the method is compared to another approach for a problem.

2. MATHEMATICAL BACKGROUND

In this section a cursory overview of the necessary mathematical background is given. Of primary concern to us will be the Hodge decomposition and the homotopy operator. Additionally, a shallow examination of exterior calculus and differential forms will be given. A full account of exterior calculus is given in (Edelen, 1985). Although it won't be touched upon in this section a background that the dynamical systems satisfy the conditions to be represented as a dissipative Hamiltonian system. Background can be found in: (Wang et al., 2003) and (Hudon et al., 2008).

2.1 EXTERIOR CALCULUS

Let X be a smooth vector space, $X \in \Gamma^\infty(\mathbb{R}^n)$, then we define the smooth mapping:
$$X : \mathbb{R}^n \to T\mathbb{R}^n, \tag{4}$$
where $T\mathbb{R}^n$ is the tangent space. This mapping is defined by assigning to every point x $\in \mathbb{R}^n$ a tangent vector $X|_x \in T\mathbb{R}^n$. Then the dual space is the set of all linear functionals $T^*\mathbb{R}^n$ that map the tangent space back into $\mathbb{R}^n$:
$$\omega \in T^*\mathbb{R}^n : T\mathbb{R}^n \to \mathbb{R}^n, \tag{5}$$
thus, the dual space is a vector space over the space of functions on $\mathbb{R}^n$ (referred to as a 0-form later). We then define the dual space $T^*(\mathbb{R}^n)$ as the space of differential forms of degree one, $\Lambda^1(\mathbb{R}^n)$, with the representation:
$$\omega = \sum \omega_i(x) dx_i, \tag{6}$$
where $\{dx_i\}$ is the natural basis of the dual space. Then let the space of differential forms of degree zero be denoted $\Lambda^0(\mathbb{R}^n)$, and as above it is the function space: f:$\mathbb{R}^n \to \mathbb{R}^n$.

Let the space of infinitely differential vectors be denoted by $\Gamma^\infty(\mathbb{R}^n)$ and one-forms $\Lambda^1(\mathbb{R}^n)$. Then, with $\partial_i = \frac{\partial}{\partial x_i}$ and dx_i as the standard bases for vectors and one-forms respectively, we defined the smooth vector field:
$$X(x) = \sum_{i=1}^{n} v^i(x)\partial_i \,\epsilon\, \Gamma(\mathbb{R}^n), \tag{7}$$
v^i(x) are the vector functions. And, define the smooth differentiable one-form:
$$\omega(x) = \sum_{i=1}^{n} \omega_i(x) dx_i \in \Lambda^1(\mathbb{R}^n), \tag{8}$$
where the coeffecient functions $\omega_i(\mathrm{x}_1, \mathrm{x}_2, \ldots, x_n)$, $i = 1, 2, ..., n$ are differentiable. The wedge (exterior) product is then defined as an algebra on $\Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n)$ and satisfies the properties:
$$\wedge : \Lambda^1 \times \Lambda^1 \to \Lambda^2, \tag{9}$$
$$dx_i \wedge dx_j = -dx_j \wedge dx_i,$$
$$dx_i \wedge f(x)dx_j = f(x)dx_i \wedge dx_j,$$
for all $f(x) \in \Lambda^0(\mathbb{R}^n)$. Thus, the vector space of two-forms over $\mathbb{R}^n$ is defined with a basis of:
$$\{dx_i \wedge dx_j, i < j\}. \tag{10}$$
We then generalize the exterior product to:
$$\wedge : \Lambda^k \times \Lambda^l \to \Lambda^{k+l}, \tag{11}$$
with the similar properties as above for the wedge product on space of one-forms. Which has a distributive property, as well as an anti-commutative property like the one-forms:
$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma,$$
$$\alpha \wedge \beta = (-1)^{deg(\alpha)deg(\beta)} \beta \wedge \alpha,$$
where α, β, and $\gamma \in \Lambda(\mathbb{R}^n)$. Consequently, we express the expanded differential form of degree $k \leq n$ as:
$$\omega(x) = \sum_{i_1 < i_2 < \cdots < i_k} \omega_{i_1 i_2 \cdots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}, \tag{12}$$
where, ω(x) $\in \Lambda^k(\mathbb{R}^n)$, and $\omega_{i_1 i_2 \cdots i_k}$ defines $\binom{n}{k}$ functions in the domain of $\Lambda^k(\mathbb{R}^n)$. The operators, which will be useful, can now be defined over the differential space. The exterior derivative is a unique operator on $\Lambda(\mathbb{R}^n)$ with the following properties:
$$d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n),$$
$$d(\alpha + \beta) = d\alpha + d\beta,$$
$$d(\alpha \wedge \beta) = (d\alpha) \wedge (d\beta),$$
$$df = \sum_{i}^{n} (\partial f / \partial x_i) dx_i \,, \quad f \in \Lambda^0(\mathbb{R}^n),$$
$$d \circ d\alpha = 0.$$
The Hodge star operator, $\star$, is a linear operator that maps a form of the degree $p \leq n$, where n is the degree of the manifold to a degree (n - p) form:
$$\star : \Lambda^p \to \Lambda^{n-p}, \tag{13}$$
such that for $\omega \in \Lambda^p(\mathbb{R}^n)$:
$$\omega \wedge *\omega = \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n,$$
where $\omega_i \in \Lambda^1(\mathbb{R}^n)$. For example on the 2-dimensional manifold, with metric tensor $G = dx_1 \otimes dx_1 + dx_2 \otimes dx_2$, the Hodge star of dx_1 yields dx_2 (ie. $\star dx_1 = dx_2$ and $\star dx_2 = -dx_1$). The interior product is defined in the usual way:
$$\lrcorner : \Gamma^\infty(\mathbb{R}^n) \times \Lambda^k(\mathbb{R}^n) \to \Lambda^{k-1}(\mathbb{R}^n), \tag{14}$$
with the following properties:
$$D \lrcorner f = 0,$$
$$D \lrcorner \omega = \omega(D),$$
$$D \lrcorner (\alpha + \beta) = D \lrcorner \alpha + D \lrcorner \beta,$$
$$D \lrcorner (\alpha \wedge \beta) = (D \lrcorner \alpha) \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge (D \lrcorner \beta),$$
where $\omega \in \Lambda^1(\mathbb{R}^n)$, $\forall\, D \,\epsilon\, \Gamma^\infty(\mathbb{R}^n)$, $\forall\, \alpha, \beta \in \Lambda^k(\mathbb{R}^n)$ with $k = 1, 2, ..., n$, and $\forall\, f \in \Lambda^0(\mathbb{R}^n)$.

2.2 HOMOTOPY OPERATOR

We now show how to construct the homotopy operator $\mathbb{H}$ as well as define its properties. Then explore its uses. First, we define the star-shaped region, S, which is an open region in the n-dimensional space of $\mathbb{R}^n$. S is called star-shaped with respect to one of its points, P_0, if S is contained in some neighbourhood, U, for P_0. In this neighbourhood,U, coordinate functions assign coordinates $({x^0}_1, {x^0}_2, ..., {x^0}_n)$ to the point P_0. Then for any point, P in S the coordinate functions of U assign coordinates to P, so the set of points with coordinates $({x^0}_i + \lambda(x_i - {x^0}_i))$ for all $\lambda \in [0,1]$ belongs to S. Basically, this mean that all points in S are connected to a P_0 linearly. Note in this paper we assume without loss of generality that $P_0 = 0$. The vector field is defined as:

$$\mathfrak{X}(x_i) = (x_i - x_i^0)\partial_i = (\frac{d}{d\lambda}(x_i^0 + \lambda(x_i - x_i^0)))\partial_i. \quad (15)$$

The homotopy operator is now introduced by the property that $\mathbb{H}$ is a linear operator on the elements of $\Lambda(\mathbb{R}^n)$ that satisfies the identity:

$$\omega = d(\mathbb{H}\omega) + \mathbb{H}d\omega, \quad (16)$$

where ω is a differential form such that $\omega \in \Lambda^k(\mathbb{R}^n)$. To now define the operator, again letting ω be a differential form with degree k on the star-shaped region, S centred at the equilibrium point x^*. Then the operator is defined, on these coordinates, as:

$$(\mathbb{H}\omega)(x) = \int_0^1 \mathfrak{X} \lrcorner \omega(x_i^* + \lambda(x_i - x_i^*))\lambda^{k-1}d\lambda, \quad (17)$$

where $\mathfrak{X}$ is the associated vector field on S, $\lambda \in [0,1]$, and $k = \deg(\omega)$. $\mathbb{H}$ is well defined on S, since both $\mathfrak{X}$ and $\omega(x,\lambda)$ are also well defined on S (Edelen, 1985). The properties of $\mathbb{H}$ are as follows:

$$\mathbb{H} : \Lambda^k(S) \to \Lambda^{k-1}(S)\ \forall k \geq 1\ and\ \Lambda^0(S) \to 0,$$

$$d\mathbb{H}+\mathbb{H}d = \mathbb{I}\ \forall k \geq 1, and\ \mathbb{H}df(x) = f(x)-f(x^0)\ for\ k = 0,$$

$$\mathbb{H}\mathbb{H}\omega(x) = 0,\ \mathbb{H}\omega(x^0) = 0,$$

$$\mathfrak{X}\lrcorner\mathbb{H} = 0,\ \mathbb{H}\mathfrak{X}\lrcorner = 0.$$

By the second property for $\omega \in \Lambda^k(\mathbb{R}^n)$ we get:

$$\omega = d(\mathbb{H}\omega) + \mathbb{H}d\omega, \quad (18)$$

since $d\circ d(\mathbb{H}\omega) = 0$, $d(\mathbb{H}\omega)$ is a closed form. By the Poincaré lemma every closed form on S is exact, so $d(\mathbb{H}\omega)$ is also exact Farber (2004). The exact part of ω is denoted $\omega_e = d(\mathbb{H}\omega)$ and the anti-exact part, $\omega_a = \mathbb{H}d\omega$. Thus the homotopy operator can be used to decompose some differential form into an exact and anti-exact part. The anti-exact part is the non-dissipative part (Edelen, 1985). The dissipation inequality is given by:

$$j(x,\omega) = \frac{\partial}{\partial x}\psi(x,\omega) + U(x,\omega) \quad (19)$$

where U is the anti-exact part or the one-form, and we define:

$$\psi(x,\omega) = \int_0^1 p(\lambda x,\omega)\frac{d\lambda}{\lambda} \quad (20)$$

where $p \geq 0$ is the dissipation potential. We will go on to show how to use this with the homotopy operator.

3. OBSERVER DESIGN

In this section we show how the homotopy operator can be used to construct an observer system given in (37). The homotopy operator approach is used to define a Lyapunov function for the estimation error $e = x - \hat{x}$.

3.1 Existence of observability functions

Observability functions are central to the design of observers in a number of approaches. For example, the design of minimum variance and minimax estimators require the existence of function that solve forward dynamic programming partial differential equations. Observability functions can be interpreted as solutions of similar nonlinear partial differential equations in the absence of uncertainties and disturbances. In this section, we demonstrate the application of the potential based realization to develop conditions for the existence of observability functions.

To apply a potential-based formulation of this problem, we pose the vector field:

$$F(x) = f(x) - \frac{1}{2}\Gamma(x) \quad (21)$$

where $\Gamma(x)$ is such that $x^T\Gamma(x) = h(x)^T h(x)$. Note that under the assumption that $h(0) = 0$, Γ can always be constructed as follows:

$$\Gamma = \int_0^1 \frac{\partial\|h(\lambda x)\|^2}{\partial x}d\lambda.$$

Using the approach proposed above, we obtain the one-form:

$$\omega = (f(x) - \frac{1}{2}\Gamma(x))dx$$

The homotopy based decomposition yields the potential $W_o(x)$ such that

$$\omega = -dW_o(x) + U_o(x).$$

Consider the storage function $V(x) = \frac{1}{2}\|x\|^2$. Its time derivative is given by:

$$\dot{V} = x^T f(x) = -x^T\nabla W_o(x)^T + \frac{1}{2}x^T\Gamma(x)$$
$$= -x^T\nabla W_o(x)^T + \frac{1}{2}\|h(x)\|^2.$$

As a result, it follows that if $W_o(x)$ is convex then:

$$\dot{V} \leq +\frac{1}{2}\|h(x)\|^2$$

or

$$V(t) \leq V(0) + \int_0^t \|h(x(\tau))\|^2 d\tau.$$

Thus, the analysis demonstrates that if the potential W_o is convex then the function V is an observability function. One can therefore use this function to synthesize an observer. We can summarize the result as follows.

Theorem 1. Consider the nonlinear system (1) and the one-form $\omega = \star j$ constructed from the corresponding vector field (21). If the potential W_o is locally convex then the nonlinear systems admits an observability function given by $V = \frac{1}{2}\|x\|^2$.

Proof:The proof follows from the above discussion. ■

The existence of the observability function can be used to design observers for this class of systems.

3.2 Design of metric observers

To accomplish this we can use the homotopy operator to re-express the vector field of a nonlinear system of type (1) as a potential driven system of the form:

$$f(x) = \sum_{i=1}^{k} Q_i(x) \frac{\partial P_i(x)}{\partial x}, \quad (22)$$

where P_i(x) are the potentials, $Q_i(x)$ are the structure matrices, and k is from f $\in C^k$, where k is finite (Hudon et al., 2008). First, for the state space, $\mathbb{R}^n$, we define the error vector field:

$$F_e = (f(x) - f(\hat{x}))\frac{\partial}{\partial x} - (f(x) - f(\hat{x})\frac{\partial}{\partial \hat{x}}, \quad (23)$$

where $\hat{x}$ is the observer state, and $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial \hat{x}}$ denotes the vector field coordinates. Note that $\frac{\partial}{\partial x_i} \cdot dx_j = 1$ if $i = j$, zero otherwise. Next, define an arbitrary spanning metric tensor, G, and the n-dimensional volume form for the error space:

$$G = \sum_{i=1}^{n} \sum_{j=i}^{n} (a_{ij} dx_i \otimes dx_j + a_{ij} d\hat{x}_i \otimes d\hat{x}_j), \quad (24)$$

$$\mu = dx_1 \wedge ... \wedge dx_n \wedge d\hat{x}_1 \wedge ... \wedge d\hat{x}_n, \quad (25)$$

where a_{ij} can take any value in $\mathbb{R}$. As an example, the canonical metric tensor would be of the form:
$G_{can} = dx_1 \otimes dx_1 + ... + dx_n \otimes dx_n + d\hat{x}_1 \otimes d\hat{x}_1 + ... + d\hat{x}_n \otimes d\hat{x}_n$.
Taking the interior product of the volume form, μ, along the vector yields the ($2n$ - 1) form:

$$j = F_e \lrcorner \mu = (-1)^{i-1} \sum_{i=1}^{n} ((f_i(x) - f_i(\hat{x}))\Lambda_i - (f_i(x) - f_i(\hat{x}))\hat{\Lambda}_i), \quad (26)$$

$$\Lambda_i = dx_1 \wedge ... \wedge dx_{i-1} \wedge dx_{i+1} \wedge ... \wedge dx_n,$$
$$\hat{\Lambda}_i = d\hat{x}_1 \wedge ... \wedge d\hat{x}_{i-1} \wedge d\hat{x}_{i+1} \wedge ... \wedge d\hat{x}_n,$$

j represents the expression of the vector field F_e as an ($2n$ -1) form in the metric induced space. We then compute the vector associated one-form, ω, using the Hodge star operator:

$$\omega = \star j. \quad (27)$$

Now that we have the metric induced one-form the Hodge decomposition can be used to decompose it into:

$$\omega = \omega_e + \omega_a + \gamma, \quad (28)$$

γ is the harmonic part of the one-form and only appears if both $d\omega = 0$, and $(-1)^{(n(k+1)+1)} \star d \star (\omega) = 0$, the harmonic part will not be investigated here. Once the decomposition is attained with the homotopy operator, as shown in (18):

$$\omega = d\mathbb{H}\omega + \mathbb{H}d\omega,$$

then we note that the exact part of the one-form is in gradient-like form:

$$\omega_e = d\mathbb{H}\omega = dP = \nabla_e P de, \quad (29)$$

where the function $P(x,\hat{x}) = P(\mathrm{e}+\hat{x},\ \hat{x})$ is denoted the potential of the system on the domain S. Additionally, the anti-exact part on the field $\mathfrak{X}$ of S:

$$\mathfrak{X} \lrcorner \mathbb{H} d\omega = 0, \quad (30)$$

by the fourth property of $\mathbb{H}$ from above. Thus, in error coordinates ($e = x$ - $\hat{x}$), the homotopy operator yields the potential associated with the metric:

$$P(e, \hat{x}) = \mathbb{H}\omega = \int_0^1 e\frac{\partial}{\partial e} \lrcorner (f(\hat{x} + \lambda e) - f(\hat{x})) d\lambda. \quad (31)$$

It is easy to see that, by construction, $P(0, \hat{x}) = 0$ and that $\nabla_e P(0, \hat{x}) = 0$. We then obtain the result:

$$e\frac{\partial}{\partial e} \lrcorner \omega = e^T \frac{\partial P(\hat{x} + e, \hat{x})^T}{\partial e} = e^T \Theta(\hat{x} + e, \hat{x}) e, \quad (32)$$

where $\Theta(x)$, the Hessian matrix, is defined by:

$$\Theta(e, \hat{x}) = \int_0^1 \frac{\partial^2 P(\lambda e)}{\partial x \partial x^T} d\lambda. \quad (33)$$

Lemma 1. If $f \in \mathcal{C}^2$ on $S \subset \mathbb{R}^n$, then $\Theta(x)$ is Lipschitz on S.

Proof: Since we have $f(\mathrm{x}) \in \mathcal{C}^2 \iff f(\mathrm{e} + \hat{x}) \in \mathcal{C}^2$, then the following argument works equally for x, although will be shown for e. Additionally,

$$\frac{\partial^2}{\partial e^2}(f(x) - f(\hat{x})) \in \mathcal{C}^2,$$

due to the above properties. Note that when we say a differential form is $\mathcal{C}^2$ we mean that for a given k-form, the all coefficients $\omega_i(x)$ are $\mathcal{C}^2$, since by the definition of exterior derivative the continuity on the geometry is equivalent to the continuity of the coefficients. We then show that the interior product preserves these properties. We then explicitly define the interior product, a linear map (Atkinson and Han, 2005), using Einstein summation notation, as (Edelen, 1985):

$$\mathcal{V} \lrcorner \omega = \frac{1}{(k-1)!} v^j \omega_{j i_2 ... i_k} dx^{i_2} \wedge ... \wedge dx^{i_k}, \quad (34)$$

where $\mathcal{V} \in \Gamma^2(\mathbb{R}^n)$ and $\omega \in \Lambda^k$. Then so long as all $v^j(e)$ and $\omega_{j i_1 ... i_k}(e)$ are $\mathcal{C}^2$ then $\mathcal{V} \lrcorner \omega \in \mathcal{C}^2$ since:

$$\frac{\partial}{\partial e}[v^j \omega_{j i_2 ... i_k}] = \frac{\partial v^j}{\partial e}[\omega_{j i_2 ... i_k}] + v^j \frac{\partial \omega_{j i_2 ... i_k}}{\partial e}.$$

For our problem, $F_e \lrcorner \mu$, $v^j = \pm(f_j(e + \hat{x})$ - $f_j(e$ - $x)) \in C^2$ and $\omega_{j i_2 ... i_k} = 1$ is smooth. A similar evaluation is used for the Hodge star operator, which is also linear (Warner, 1983), and is explicitly defined as:

$$\star\omega = \frac{\omega_{i_1 i_2 ... i_k}}{det(G)}(\pm dx_{k+1} \wedge ... \wedge dx_n), \quad (35)$$

where $\omega = \omega_{i_1 ... i_k} dx_1 \wedge ... \wedge dx_n$. It is assumed $det(G) \neq 0$ by the given metric assumptions. Then if $(\omega)_j \in \mathcal{C}^2 \iff (\star\omega)_j \in \mathcal{C}^2$. Then since $(\star\omega)_i \in \mathcal{C}^2$ and $\mathcal{C}^2$ is a vector space, by the second property of the homotopy operator (18) $(\star\omega)_i \in \mathcal{C}^2 \iff (d\mathbb{H}\omega)_i \in \mathcal{C}^2$. Then, by property (32), we obtain:

$$|e^T \nabla_e P(x) - e^T \nabla_e P(\hat{x})| \leq L||e||^2$$
$$\implies |e^T(\Theta(x) - \Theta(\hat{x}))e| \leq L||e||^2.$$

Thus Θ is Lipschitz on the star-shaped domain in some area around $e = 0$. This proves the lemma. ■

3.3 Design Equation

In this section, we consider the special case in which $y = h(x) = Cx$. The proposed design equations are given. The following metric equation is proposed:

$$\dot{M} = -2\Theta(\hat{x}) - \beta M - MQM + 2C^T RC \ , \ M(0) = \alpha, \quad (36)$$

where M is a symmetric matrix of the time-varying metric coefficients, a_{ij}, α is a matrix of the initial values of the metric coefficients, Q and R are arbitrarily chosen positive matrices of scalars. The metric dependent observer system is given by:

$$\dot{\hat{x}} = f(\hat{x}) + M(t)^{-1}C^T R(h(x) - h(\hat{x})) = F(e, \hat{x}), \quad (37)$$

which reduces the error based on the metric difference in measured output.

3.4 Stability Analysis

Theorem 2. Assume that the equation (36) admits a unique positive definite solution $M(t)$, then the observer (37) is the such that the origin is a asymptotically stable equilibrium of the error dynamics

$$\dot{e} = f(\hat{x} + e) - F(e, \hat{x}).$$

Proof: Pose the Lyapunov function for the error dynamics, $V = \frac{1}{2}e^T Me$.

We then get,

$$\dot{V} = e^T M\dot{e} + \frac{1}{2}e^T \dot{M}e. \quad (38)$$

The error dynamics are given by:

$$\dot{e} = f(x) - f(\hat{x}) - M^{-1}C^T RCe$$

Differentiating V, we obtain:

$$\begin{aligned}\dot{V} =& e^T M(f(x) - f(\hat{x})) - e^T C^T RCe \\ &+ \frac{1}{2}e^T(-2\Theta(\hat{x}) - \beta M - MQM + 2C^T RC)e.\end{aligned}$$

By the property of the homotopy operator, $e^T M(f(x) - f(\hat{x})) = e^T\Theta(e, \hat{x} + e)e$. Therefore, $\dot{V}$ becomes:

$$\dot{V} = e^T(\Theta(e, \hat{x}+e) - \Theta(\hat{x}))e - \frac{\beta}{2}e^T Me - \frac{1}{2}e^T MQMe. \quad (39)$$

Using the previous lemma,

$$\dot{V} \leq L||e||^2 - \frac{\beta}{2}e^T Me - \frac{1}{2}e^T MQMe, \quad (40)$$

Since M is positive definite, it follows that $\frac{\beta}{2}e^T Me \geq \frac{\beta\lambda}{2}\|e\|^2$ where λ is the minimum eigenvalue of M. As a result, one can write:

$$\dot{V} \leq L||e||^2 - \left(\frac{\beta\lambda}{2} + \frac{\lambda^2\lambda_Q}{2}\right)\|e\|^2, \quad (41)$$

where λ_Q is the minimum eigenvalue of Q. As a result, there always exists a β and Q such that $\dot{V} < -\sigma\|e\|^2$ on S for some positive constant $\sigma > 0$. This proves the asymptotic stability of the error dynamics and the convergence of the metric based observer system. ∎

4. EXAMPLE

In this section, we consider a nonlinear dynamical system, the van der Pol oscillator, to illustrate the use of our method of observer design. The system is given by:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = 2x_2(1 - x_1^2) - x_1,\ y = x_1 = h(x).$$

We start with a metric tensor of the form:

$$\begin{aligned}G = a_{11}dx_1 \otimes dx_1 + a_{12}dx_1 \otimes dx_2 + a_{22}dx_2 \otimes dx_2 + \\ a_{11}d\hat{x}_1 \otimes d\hat{x}_1 + a_{12}d\hat{x}_1 \otimes d\hat{x}_2 + a_{22}d\hat{x}_2 \otimes d\hat{x}_2. \quad (42)\end{aligned}$$

The simulation results can be see for the change of the metric coefficients with time in Figure **??**. Then the

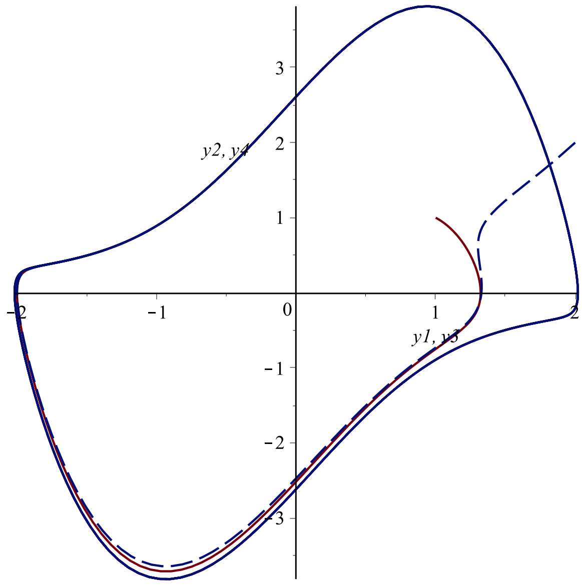

Fig. 1. A plot of the Van der Pol oscillator (solid) and metric based observer (dashed).

associated vector field with respect to the dynamical system (23) is given by:

$$\begin{aligned}F_e(x, \hat{x}) =& (f_1(x) - f_1(x))(\partial_{x_1} - \partial_{\hat{x}_1}) \\ &+ (f_2(x) - f_2(\hat{x}))(\partial_{x_2} - \partial_{\hat{x}_2}).\end{aligned}$$

We note that the equilibrium point x^* is at $\{x_1 = 0, x_2 = 0\}$. Then calculating the associated one-form, by (27) we get:

$$\begin{aligned}\omega =& a_{11}(x_2 - \hat{x}_2)dx_1 \\ &- (2a_{22}x_2 - 2a_{22}x_2x_1^2 - a_{22}x_1 - 2a_{22}\hat{x}_2 \\ &+ 2a_{22}\hat{x}_2\hat{x}_1^2 + a_{22}\hat{x}_1 + a_{12}x_2 - a_{12}\hat{x}_2)dx_2 \\ &+ a_{11}(x_2 - \hat{x}_2)d\hat{x}_1 \\ &(2a_{22}x_2 - 2a_{22}x_2x_1^2 - a_{22}\hat{x}_2 + 2a_{22}\hat{x}_2\hat{x}_1^2 \\ &+ a_{22}\hat{x}_1 + a_{12}x_2 - a_{12}\hat{x}_2)d\hat{x}_2.\end{aligned}$$

The homotopy operator of ω is taken, $\mathbb{H}\omega$, then the Hessian, Θ, is calculated with respect to x_1 and x_2 to calculate $\omega_e|_x = d(\mathbb{H}\omega)|_x$. Then we define the values of $R = 1$ and $Q = 100$. Figure 1 shows the phase portrait of the state of the system along with the state estimates. The observer is shown to recover effectively the unknown states of the system (i.e., x_2).

5. CONCLUSION

The problem of observer design without a metric known a priori on a system which allows a dissipative Hamiltonian realization was considered. Using the Homotopy operator an associated potential is generated which either is convex, or if not can be made convex by the manipulation of the metric equation. Thus, the Lyapunov function for the given observer system is shown to be locally asymptotically stable around the origin. By allowing for non-convex potentials the observer design algorithm was expanded to the class of energy-like driven dynamical systems. Finally, an example is shown to demonstrate the method of observer design. It should be noted that if a suitable metric can be inferred a priori the design of an observer can be greatly simplified. For example, the motivating problem in Sanfelice and Praly (2012), let:

$$G = \frac{1}{1+x_1^2} dx_1 \otimes dx_1 + \frac{x_1 x_2}{1+x_1^2} dx_1 \otimes dx_2 + dx_2 \otimes dx_2. \quad (43)$$

Then we get the corresponding one-form:

$$\omega = -x_2 dx_1. \quad (44)$$

Since the attained one-form is anti-symmetric the system has a convex potential and the observer system is asymptotically convergent.

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