

# Potential and Limitations of Multi-stage Nonlinear Model Predictive Control

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**Abstract:** Multi-stage Nonlinear Model Predictive Control (NMPC) is a promising strategy for the design of robust NMPC controllers which is based on describing the evolution of the uncertainty as a scenario tree. The scenario tree makes it possible to consider explicitly that the future control inputs can be adapted to the future information (measurements), thus reducing the conservativeness of the robust approach. This paper reviews the multi-stage approach and illustrates its main advantages using a nonlinear CSTR example. We also provide guidelines for possible multi-stage NMPC users that could help to identify the problems where the use of multi-stage NMPC can result in a significant improvement with respect to standard NMPC or other robust NMPC approaches. Finally, we summarize the different modifications that can be done to the multi-stage approach to enhance its performance. The possible enhancements include: improved performance using parameter estimation, rigorous guarantee of constraint satisfaction, and stability guarantees for the case of discrete-valued uncertainties.

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## 1. INTRODUCTION

Model Predictive Control (MPC) has become a standard technique for process control, mainly because it can deal in a straightforward manner with constraints on the inputs and on the states. Increasingly also its nonlinear version, Nonlinear Model Predictive Control (NMPC), is starting to receive more attention in the industry. One of the main problems of any NMPC approach is that its performance and stability properties are strongly affected by the quality of the model used in the predictions. However, all models are imperfect and therefore it is necessary to introduce some robustification to any NMPC controller used in reality.

The first robust MPC approaches were focused on a min-max approach (Campo and Morari [1987]) using ideas of robust optimization which were presented in Witsenhausen [1968]. These approaches compute a sequence of control inputs that satisfy the constraints for all the cases of the uncertainty and that minimize the worst-case value of the desired objective function. It is well-known (Lee and Yu [1997]) that these approaches are very conservative because they ignore the fact that new measurements will be available in the future and that the future control actions can be adapted. To counteract this problem, closed-loop approaches were proposed e.g. in Scokaert and Mayne [1998] in which a sequence of optimal control *policies* is computed. However, in the general case it is extremely difficult to solve this problem and it has to be simplified by assuming e.g. a fixed structure of the control policies (usually assumed to be affine as in Goulart et al. [2006]). In the last years, different tube-based NMPC methods have been developed (see e.g. Mayne and Kerrigan [2007], Yu et al. [2011]), which solve the nominal NMPC problem and use a second ancillary controller to guarantee that the real (uncertain) system stays close to the nominal trajectory.

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Another possibility to achieve a robust NMPC controller is to formulate the NMPC problem within the framework of stochastic optimization, in a similar way as done in Scokaert and Mayne [1998], Muñoz de la Peña et al. [2005] for the linear case. In this manner, a scenario tree for the different possible realizations of the uncertainty can be formulated, which makes it possible to consider explicitly the fact that new measurements will be available in the future and that the future control inputs can be adjusted accordingly. Such a formulation leads to a multi-stage NMPC method (Lucia et al. [2013]) which has shown very promising results for challenging nonlinear examples both in terms of performance and real-time capabilities.

In this paper, we review the multi-stage NMPC approach and illustrate the main advantages and limitations of its use by means of a nonlinear CSTR example using an economic cost function and we also analyze its performance when using a tracking cost function, which had not been studied before. Additionally, we provide useful guidelines based on previous publications that can help a user of multi-stage NMPC to find the situations where the use of the approach is more beneficial. We also describe how the approach can be extended in different ways in order to enhance the performance of the method.

## 2. MULTI-STAGE NONLINEAR MODEL PREDICTIVE CONTROL

This section reviews the main concepts of the multi-stage NMPC approach presented in Lucia et al. [2013, 2014a]. In multi-stage NMPC, the model uncertainty is represented by a tree of discrete scenarios that branches at each future sampling point for each possible value of the uncertainty as depicted in Fig. 1. The formulation of a scenario tree makes it possible to take explicitly into account that the future decisions can depend on the new information (measurements) that will become available in the future. Thus the future control inputs can be adapted according to the future realizations of the uncertainty and the conservativeness of the approach is reduced compared to other robust methods that search for a single sequence of control

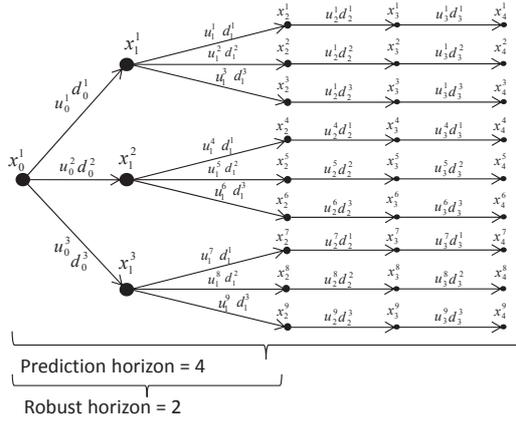


Fig. 1. Scenario tree representation of the evolution of the uncertainty for multi-stage NMPC.

inputs to satisfy the constraints for all the possible values of the uncertainty. Formulating the uncertain decision process as a scenario tree is a well-known approach in the field of multi-stage stochastic programming, which has been extensively used in decision theory and finances (Shapiro [2009]). In the case that the uncertainty is truly discrete-valued, this is the best solution possible for a given prediction horizon. Generally this is not the case, and multi-stage NMPC is an approximation of the best solution.

In the multi-stage NMPC approach, we consider a discrete-time nonlinear system:

$$x_{k+1}^j = f\left(x_k^{p(j)}, u_k^j, d_k^{r(j)}\right), \quad (1a)$$

where each state vector  $x_{k+1}^j \in \mathbb{R}^{n_x}$  at stage  $k+1$  and position  $j$  depends on the parent state  $x_k^{p(j)}$  at stage  $k$ , the vector of control inputs  $u_k^j \in \mathbb{R}^{n_u}$  and the corresponding realization  $r$  of the uncertainty  $d_k^{r(j)} \in \mathbb{R}^{n_d}$  (e.g. in Fig. 1,  $x_2^6 = f(x_1^2, u_1^6, d_1^3)$ ). The uncertainty at the stage  $k$  is defined by  $d_k^{r(j)} \in \{d_k^1, d_k^2, \dots, d_k^s\}$  for  $s$  different possible combinations of values of the uncertainty. We define the set of indices  $(j, k)$  in the scenario tree as  $I$ .  $S_i$  denotes the  $i$ th scenario which is the path from the root node  $x_0$  to one of the leaf nodes and it contains all the states  $x_k^j$  and control inputs  $u_k^j$  that belong to the  $i$ th scenario.

A common way to build a scenario tree is to consider, as possible branches, a combination of values from the assumed extreme values of all the uncertain parameters or disturbances. For the general nonlinear case, it is not guaranteed that this results in robust constraint satisfaction for the values of the uncertainty that are not considered in the tree, but it has been shown to give very good results in practice Lucia et al. [2013, 2014a]. If a rigorous guarantee for robust constraint satisfaction of all the possible values of the uncertainty (including those that are not in the tree) is required, the multi-stage approach can be combined with reachability analysis as shown in Lucia et al. [2014b].

Generating the full scenario tree including all the extrema of the uncertainty space makes the size of the resulting optimization problem grow rapidly with increasing length of the prediction horizon  $N_p$  and with increasing number of uncertainties. A possible strategy to avoid the exponential growth of the scenario tree over the prediction horizon is to consider that the

uncertainty remains constant after a certain stage (called robust horizon  $N_r$ ), until the end of prediction horizon (Fig. 1).

The optimization problem that has to be solved at each sampling instant can be written as:

$$\min_{x_{k+1}^j, u_k^j, \forall (j,k) \in I} \sum_{i=1}^N \omega_i J_i(X_i, U_i) \quad (2a)$$

subject to:

$$x_{k+1}^j = f\left(x_k^{p(j)}, u_k^j, d_k^{r(j)}\right), \quad \forall (j, k+1) \in I, \quad (2b)$$

$$0 \geq g\left(x_{k+1}^j, u_k^j, d_k^{r(j)}\right), \quad \forall (j, k) \in I, \quad (2c)$$

$$u_k^j = u_k^l \text{ if } x_k^{p(j)} = x_k^{p(l)}, \quad \forall (j, k), (l, k) \in I, \quad (2d)$$

where  $X_i, U_i$  are the set of states and control inputs that belong to the scenario  $S_i$  with the probability of occurrence  $\omega_i$ . The constraints on inputs and states are denoted by  $g(\cdot)$ . The cost of each scenario is denoted by  $J_i(\cdot)$  and can be written as:

$$J_i(X_i, U_i) := \sum_{k=0}^{N_p-1} L\left(x_{k+1}^j, u_k^j\right), \quad \forall x_{k+1}^j, u_k^j \in S_i. \quad (3)$$

The constraints (2d) are called non-anticipativity constraints which imply that the control inputs cannot anticipate the realization of the uncertainty, i.e. the control inputs  $u_k^j$  that branch at the same parent node  $x_k^{p(j)}$  must be the same and the constraints in (2c) are general nonlinear constraints.

### 3. CASE STUDY

We illustrate some of the advantages of multi-stage NMPC using a nonlinear CSTR benchmark problem adapted from Klatt and Engell [1998]. The dynamics of the CSTR are described by the following set of differential equations:

$$\dot{c}_A = F(c_{A0} - c_A) - k_1 c_A - k_3 c_A^2, \quad (4a)$$

$$\dot{c}_B = -F c_B + k_1 c_A - k_2 c_B, \quad (4b)$$

$$\begin{aligned} \dot{T}_R = F(T_{in} - T_R) + \frac{k_W A}{\rho c_p V_R} (T_K - T_R) \\ - \frac{k_1 c_A \Delta H_{AB} + k_2 c_B \Delta H_{BC} + k_3 c_A^2 \Delta H_{AD}}{\rho c_p}, \end{aligned} \quad (4c)$$

$$\dot{T}_K = \frac{1}{m_K c_{pK}} (\dot{Q}_K + k_W A (T_R - T_K)), \quad (4d)$$

where the reaction rates  $k_i$  follow the Arrhenius law:

$$k_i = k_{0,i} e^{\frac{-E_{A,i}}{R(T_R + 273.15)}}. \quad (5)$$

The ODEs are derived from component balances for the concentration of component A ( $c_A$ ) and for the concentration of component B ( $c_B$ ). The energy balances for the temperature of the reactor  $T_R$  and for the coolant temperature ( $T_K$ ) form the last two differential equations. The control inputs are the inflow ( $F = \dot{V}_{in}/V_R$ ) normalized by the volume of the reactor and the heat removed by the coolant ( $\dot{Q}_K$ ). The parameters that appear in the model equations are the same as those used in Klatt and Engell [1998]. The initial conditions of the states together with the constraints on the states are described in Table 1. The constraints for the control inputs are shown in Table 2.

### 4. ECONOMIC OPERATION UNDER UNCERTAINTY

In this section we consider as the control task for the case study the maximization of the production of component B

Table 1. Initial conditions and state constraints.

State	Init. cond.	Min.	Max.	Unit
$c_A$	0.8	0.1	2.0	$\text{mol l}^{-1}$
$c_B$	0.5	0.1	2.0	$\text{mol l}^{-1}$
$T_R$	134.14	50.0	$T_{\max}$	$^{\circ}\text{C}$
$T_J$	134.0	50.0	180.0	$^{\circ}\text{C}$

Table 2. Bounds on the manipulated variables.

Control	Min.	Max.	Unit
$F$	5	100	$\text{h}^{-1}$
$\dot{Q}_K$	0	-8500	$\text{kJ h}^{-1}$

( $\dot{n}_B = \dot{V}_{\text{in}} c_B$ ). It is considered that the activation energy  $E_{A,3}$  is uncertain and it is assumed that it varies by  $\pm 10\%$  with respect to its nominal value. The stage cost minimized at each time stage for each scenario is chosen as:

$$L = -\dot{n}_B + r_1 \Delta F^2 + r_2 \Delta \dot{Q}_K^2, \quad (6)$$

where the penalty terms for the control movements are chosen as  $r_1 = 10^{-5}$  and  $r_2 = 10^{-7}$ . The prediction horizon is  $N_p = 40$  steps and the sampling time of the controller is  $t_{\text{step}} = 0.005$  h. We consider the upper constraint for the reactor temperature to be  $T_{\max} = 140^{\circ}\text{C}$ . For the multi-stage case, a scenario tree is generated using the maximum, minimum and nominal value of the uncertainty and a robust horizon  $N_r = 1$ . Fig. 2 shows the results of applying standard NMPC with an economic cost function to the CSTR under consideration for different values of the uncertain parameter. It can be seen that when there is no plant model mismatch, all the constraints are satisfied because the predictions of the controller are perfect. However, if the model is not perfect the constraints on the temperature  $T_R$  or on the concentration  $c_A$  are violated. Moreover, it can be observed that the control loop performs a limit cycle for the higher activation energy. In contrast, if multi-stage NMPC is used, none of the scenarios violates the constraints, as shown in Fig. 3. It can be seen that multi-stage NMPC stays sufficiently far away from the constraints such that they are not violated for any of the cases, that is, multi-stage NMPC computes an *automatic* backoff from the constraints. The results of multi-stage NMPC also show that in this case the constraint on  $c_A$  is not important for the robust operation of the reactor and the constraint on  $T_R$  is the limiting one. This robustness is achieved at the cost of a reduced production of  $\dot{n}_B$ . The amount of material obtained with multi-stage NMPC is significantly smaller in this case because of the required robustness. The product yield under standard NMPC control is misleading for the case of a lower activation energy because the temperature constraint is violated largely (note that in that case the constraint is violated by almost  $20^{\circ}\text{C}$ ). If it is possible to estimate the uncertain parameter during the operation, a new scenario tree can be generated with a reduced uncertainty range in order to enhance the performance, as shown in Lucia and Paulen [2014].

## 5. SETPOINT TRACKING UNDER UNCERTAINTY

In this section we discuss the tracking of a predefined setpoint for the concentration of component B ( $c_B$ ). It is considered that the activation energy  $E_{A,3}$  is uncertain and it is assumed that it varies by  $\pm 10\%$  with respect to its nominal value. The stage cost is in this case:

$$L = (c_B - c_B^{\text{ref}})^2 + r_1 \Delta F^2 + r_2 \Delta \dot{Q}_K^2, \quad (7)$$

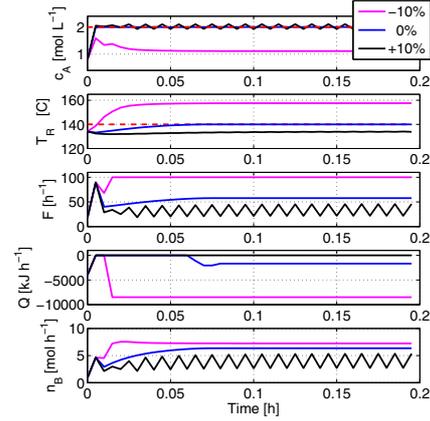


Fig. 2. Concentration of component A, reactor temperature, control inputs and product obtained by standard NMPC with an economic cost function for different values of the uncertain parameter ( $\pm 10\%$  w.r.t. the nominal value).

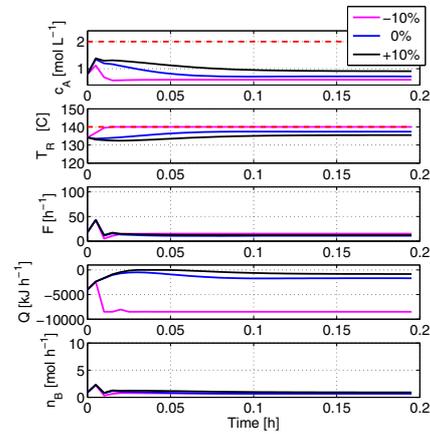


Fig. 3. Concentration of component A, reactor temperature, control inputs and product obtained by multi-stage NMPC with an economic cost function for different values of the uncertain parameter ( $\pm 10\%$  w.r.t. the nominal value).

with the same penalty terms  $r_1$  and  $r_2$  and NMPC parameters as in the previous section. The setpoint is chosen to be  $c_B^{\text{ref}} = 0.5$  for  $t \leq 0.3$  h and  $c_B^{\text{ref}} = 0.7$  for  $t > 0.3$  h.

The results for the tracking problem for standard NMPC are shown in Fig. 4. The state constraints are chosen such that they remain inactive to analyze only the tracking performance ( $T_{\max} = 180^{\circ}\text{C}$ ). Standard NMPC results in a steady state error (Fig. 4) for all the cases of the uncertainty except when a perfect model is used (0% variation with respect to the nominal value of the parameter). If multi-stage NMPC is used, the steady state error cannot be completely avoided, but the reason for this offset is different than for standard NMPC. In standard NMPC the steady-state error occurs because the controller minimizes the tracking cost function using a wrong model for the predictions. According to this model the calculated input would drive the system to the setpoint but once the control is applied to the system, it remains in the same position. For multi-stage NMPC the controller calculates sequences of control inputs (one for each scenario) that minimize the distance to the setpoint on the average. Since the first control input is common

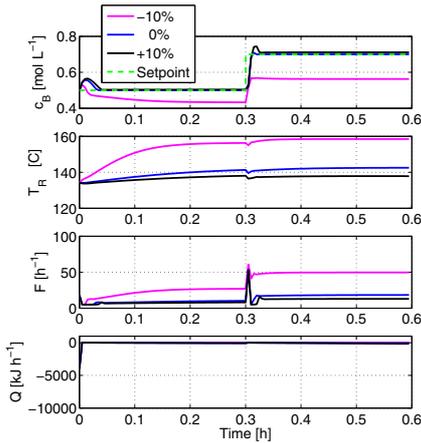


Fig. 4. Concentration of component B, reactor temperature, and control inputs obtained by standard NMPC tracking a pre-defined setpoint for different values of the uncertain parameter ( $\pm 10\%$  w.r.t. the nominal value).

Table 3. Performance comparison between standard NMPC and multi-stage NMPC.

Unc. in $E_{A,3}$	Accumulated cost	
	Standard NMPC	Multi-stage NMPC
+10%	0.0602	0.0827
+0%	0.0389	0.0467
-10%	1.3041	1.1272
Average	1.403	1.256

due to the non-anticipativity constraints, it is not possible to drive the system to the setpoint for all the scenarios in the first stage and this results in the steady-state error. Table 3 shows that multi-stage NMPC achieves a better performance on the average (11%) comparing the average accumulated cost over the three scenarios. The accumulated cost was calculated by integrating the tracking error over the whole time period. Standard NMPC has better performance when the model is perfect, but it has no control about the loss of performance for the rest of the cases. Multi-stage NMPC calculates the inputs that result in the best average performance. If the performance of standard NMPC (with the nominal model) is very similar to the average performance over different values of the uncertainty (e.g. in the unconstrained linear case), standard NMPC and multi-stage NMPC will provide almost identical solutions.

The strategies that are commonly used to achieve offset-free behavior in standard NMPC can be also applied to multi-stage NMPC. For example, a bias term can be used for each scenario based on the difference of the measured and predicted output. Another very simple strategy to achieve steady state accuracy is to adapt the reference based on the integrated tracking error. This update is given by:

$$\tilde{c}_B^{\text{ref}} \leftarrow \tilde{c}_B^{\text{ref}} + k_{\text{bias}}(c_B^{\text{ref}} - c_B^{\text{meas}}). \quad (8)$$

where  $\tilde{c}_B^{\text{ref}}$  is the actual reference used in the cost function and  $c_B^{\text{meas}}$  is the measured concentration. Using  $k_{\text{bias}} = 0.2$  steady-state accuracy (see Fig. 5) is achieved for all scenarios for both standard NMPC and for multi-stage NMPC (the results for standard NMPC are omitted here for brevity). There is no significant difference between the performance of standard and multi-stage NMPC. The reason for this is that the bias term

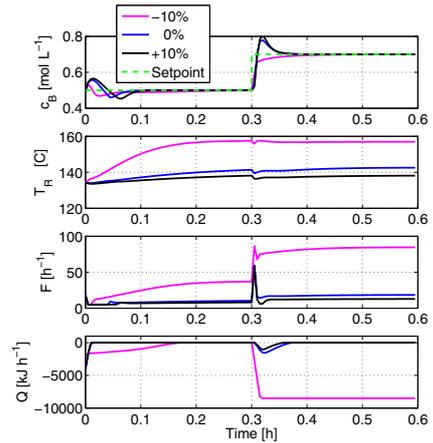


Fig. 5. Concentration of component B, reactor temperature, and control inputs obtained by multi-stage NMPC with bias term tracking a pre-defined setpoint for different values of the uncertain parameter ( $\pm 10\%$  w.r.t. the nominal value).

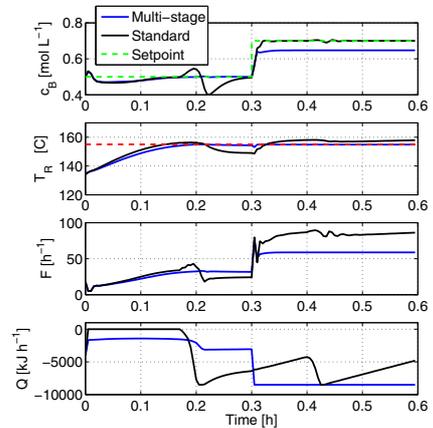


Fig. 6. Concentration of component B, reactor temperature (with constraint), and control inputs obtained by standard NMPC and by multi-stage NMPC with bias term tracking a pre-defined setpoint for a value of the uncertain parameter 10% smaller than the nominal value).

adapts the output model based on the measurement information. In particular when the plant reaches the steady state the input-output behavior is perfectly corrected for all the cases of the uncertainty. During the dynamic part of the trajectory the bias update is an approximation of the exact correction, but this does not have to be worse than optimizing an average performance for several scenarios (where only one scenario is the real one) as it is done in multi-stage NMPC.

The importance of using a robust approach becomes apparent in this case if constraints are active. Now it is considered that the upper bound on the temperature of the reactor is  $T_{\text{max}} \leq 155$ . If standard NMPC with the bias term is used, tracking is achieved but the constraint is violated (for the case when the uncertainty is 10% smaller than the nominal value). In contrast, multi-stage NMPC with the bias term realizes that the defined setpoint is unreachable and stays as close as possible without violating the temperature constraint as can be seen in Fig. 6.

The rigorous analysis of the offset-free behavior is out of the scope of this paper and the reader is referred to Morari and

Maeder [2012] for an overview on the theoretical assumptions required to guarantee offset-free tracking in NMPC.

## 6. WHEN AND HOW TO USE MULTI-STAGE NMPC?

This section presents some guidelines that provide insight on the kind of problems that can be tackled by multi-stage NMPC and on the problems where its use is more advantageous than other approaches. Once it has been decided that multi-stage NMPC is a suitable approach for the problem under consideration the controller has to be designed. We present some guidelines for the design of a multi-stage NMPC controller and for the use of the different extensions and modifications that have been presented in other papers.

### 6.1 When to Use Multi-stage NMPC

Multi-stage NMPC is a robust NMPC approach and as such, it is in general suboptimal when compared to a standard NMPC that uses the perfect model of the system, because the multi-stage approach accounts for possible uncertainties. Therefore, the best option to achieve a robust NMPC scheme is to estimate the uncertainties if it is possible, removing the uncertainty from the problem. However, if the uncertainties vary over time, the application of standard NMPC (even in the case of exact and instantaneous estimation of varying parameters) may result in constraint violations. On the other hand, the use of multi-stage NMPC can prevent the constraint violations because it takes into account in advance that the uncertainty might change as illustrated in Lucia et al. [2013].

For the cases when the uncertainty cannot be removed by parameter estimation and disturbance feedforward, the use of multi-stage NMPC is beneficial in comparison with standard NMPC or with other robust approaches that do not take feedback into account. The benefits can be seen in the form of increased average performance over the different possible scenarios of the uncertainty and in the form of robust constraint satisfaction. A typical way of introducing feedback in MPC is the use of affine control policies (Goulart et al. [2006]). However, an affine policy is suboptimal even in the constrained linear case. This suboptimality can be significant for nonlinear examples resulting in a considerable worse performance than multi-stage NMPC as shown in Lucia et al. [2014a]. Furthermore, the flexibility of the approach makes it possible to integrate it with estimation techniques (see Lucia et al. [2013]) or with optimal experiment design (see Lucia and Paulen [2014]) to enhance the performance based on measurement information.

For the case of an economic cost function and tight constraints, an important improvement in the performance is expected by using multi-stage NMPC as it has been shown in several results (see Lucia et al. [2013, 2014a]). If the control task is the unconstrained tracking of a pre-defined setpoint the benefits of multi-stage NMPC might be small compared to the use of standard NMPC with a bias term to achieve steady-state accuracy as illustrated above. For each problem it should be analyzed by simulation studies whether the possible improvements justify the increase in the complexity of the controller.

Stochastic information about the uncertainty can be incorporated in the multi-stage formulation by choosing the weights for each scenario. In multi-stage NMPC, the constraints are satisfied with certainty for all scenarios. If chance constraints are sufficient, other approaches can be considered such as the

scenario approach in the convex case (Calafiore and Campi [2006]) or the use of polynomial chaos expansions (Mesbah et al. [2014]).

For simple cases where it is possible to find the invariant sets that are necessary for the design of tube-based controllers, this approach might be preferred over the multi-stage NMPC controller because it can be implemented with the same computational complexity as standard NMPC (see Yu et al. [2011]) and provide set-theoretic guarantees about the possible trajectories of the controlled system. However, for general nonlinear systems it is very difficult to find the necessary elements for the design of tube-based NMPC. Furthermore, the issue of optimal performance under uncertainty is not addressed by tube-based NMPC.

For cases in which the most important uncertainties can be summarized in only a few parameters (or disturbances) with known bounds, multi-stage NMPC is a very promising strategy that provides excellent performance while satisfying tight constraints for all scenarios if enough computation power is available. Industrially relevant case studies can be solved in real-time if an efficient implementation of multi-stage NMPC is used.

### 6.2 How to Use Multi-stage NMPC

Once multi-stage NMPC has been chosen as the control approach for a given system, the following steps must be taken to design the controller.

The first step consists in designing the scenario tree. An easy rule to generate a suitable scenario tree is to consider the combination of the maximum, minimum, and optionally also the nominal values of the different uncertain parameters as scenarios. If there are many uncertain parameters the resulting scenario tree might be intractable, and it is therefore necessary to lump the effect of several uncertainties into a few critical uncertainties or to perform sampling.

Then the robust horizon has to be chosen. Several simulation studies (e.g. Lucia et al. [2013]) show that branching the tree only in the first stage (robust horizon  $N_R = 1$ ) results in very good results with low computational effort. Nevertheless, it has to be kept in mind that this assumes that at the next sampling time different control inputs can be taken depending on the uncertainty. This is not true because at the next sampling time a new scenario tree (shifted in time) will be solved which imposes that all the control inputs have to be the same in the first stage. This can potentially lead to recursive infeasibility of the controller, although we have not encountered this when dealing with quite a number of examples.

Then it is possible to make use of the different enhancements and extensions to improve the performance of multi-stage NMPC. If some information about the uncertainty is available, it can be introduced on-line into the scenario tree. This can be done either by adjusting the probabilities of the different scenarios of the tree (see Lucia et al. [2013]), or by *narrowing* the tree based on the new bounds of the uncertainty that are provided by confidence ellipsoids or other estimation techniques (see Lucia and Paulen [2014]).

If theoretical guarantees are required it is possible to have an a priori guarantee of stability and recursive feasibility under the usual assumptions (terminal set and terminal penalty term)

for standard and min-max NMPC for the case of discrete-valued uncertainties. If only a guarantee of robust constraint satisfaction is needed, the reachable sets of each scenario can be computed as shown in Lucia et al. [2014b].

An efficient implementation is necessary for the successful application of multi-stage NMPC. Based on a number of simulation studies, the use of interior point methods has shown a superior performance compared to sequential quadratic programming methods for the solution of the large scale optimization problems that result from the formulation of multi-stage NMPC, as compared in Lucia and Engell [2013]. In particular the combination of a full discretization of the nonlinear dynamics based on orthogonal collocation, the efficient generation of exact first- and second-order derivatives (which can be achieved very easily using e.g. CasADi (Andersson et al. [2012])) and the solution of the NLP with the solver IPOPT (Wächter and Biegler [2006]) has shown excellent performance. This method was used in this paper to solve the example problems. We describe the possibility to generate such an efficient implementation with a very low effort in Lucia et al. [2014c]. Scenario generation techniques that can achieve a very good approximation of the multi-stage cost with a reduced number of scenarios were presented in Leidereiter et al. [2014].

## 7. CONCLUSIONS

This paper shows the advantages and limitations of the multi-stage NMPC approach. We illustrated by means of a nonlinear CSTR example that generating a simple scenario tree and using multi-stage NMPC results in robust constraint satisfaction for both an economic and a tracking cost function. We show that the constraint satisfaction is achieved by an automatically calculated backoff. In addition, the approach can be used to analyze which constraints are limiting the robust operation of the system. We also provide guidelines that help identifying the kind of problems for which the multi-stage approach can be more beneficial: Solving highly nonlinear problems with tight constraints and an economic objective will very often result in a significant improvement compared to standard NMPC or other robust NMPC approaches.

Finally, we reviewed different modifications that can be applied to the approach to enhance its performance, e.g. including parameter estimation for a better performance or reachability analysis for a rigorous guarantee of robustness.

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