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Distributed extremum-seeking control over networks of unstable dynamic agents

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Abstract: In this paper, a distributed extremum seeking control technique is proposed to solve a class of real-time optimization problems over a network of dynamic agents with unknown unstable dynamics. Each dynamic agent measures a cost that is shared over a network. A dynamic average consensus approach is used to provide each agent with an estimate of the total network cost. The extremum seeking controller uses the local estimate of the total cost to adjust the value of the local decision variables. The contribution of the proposed technique is the simultaneous stabilization of the network dynamics and the distributed optimization of the total network cost. A dynamic network simulation example is presented to demonstrate the effectiveness of the technique.

Keywords: Extremum seeking control, consensus estimation, distributed optimization

1. INTRODUCTION

Real-Time Optimization (RTO) is a process automation technology whose objective is to predict the economically optimal process operating policy in the near term. The application of RTO requires a considerable expenditure for manufacturers. It involves the development of a process model along with implementation of a suitable optimization routine that can solve the optimization problem in real-time. To circumvent these limitations, a number of alternative techniques have been proposed to solve steady-state optimization problems. One such RTO technique is extremum-seeking control (ESC). ESC has been the subject of considerable research effort over the last decade. This approach, which dates back to the 1920s Leblanc (1922), provides a mechanism by which a system can be driven to the optimum of a measured variable of interest Tan et al. (2010). ESC can be viewed as an empirical RTO implementation in which no exact model description is required, but where the objective function of the RTO problem is available from process measurements. ESC provides an effective control system design technique that can be used to steer an unknown dynamical system to an equilibrium that optimizes a cost function. When dealing with complex dynamical systems, it is generally recognized that overall process objectives are difficult to achieve due to the computational complexity associated with centralized approaches. Thus, a decentralized or a distributed optimization approach is usually favoured in large-scale RTO systems design to achieve global process objectives by solving several local RTO subproblems. The distributed optimization task can be non-cooperative where each local RTO achieves its local optimization objectives. Non-cooperative RTO problems have been tackled using ESC by several researchers (Ghods et al. (2010), Stankovic and Stipanovic (2009), Poveda and Quijano (2013), Frihauf et al. (2012) and Frihauf et al. (2011)). The distributed optimization task can also be cooperative when the local RTOs coordinate actions to optimize the sum of their assigned costs. A particular class of distributed cooperative optimization has been the subject several studies (Bertsekas and J.Tsitsiklis (1989),Nedic and Ozdaglar (2009), Johansson et al. (2009)). For a class of unconstrained optimization problems, it is shown in Nedic and Ozdaglar (2009) that it is possible to achieve overall system objectives by solving local problems and communicating the optimization results via the network. For constrained optimization problems, the Alternating Direct Method of Multipliers (ADMM) can be used to solve distributed and coordinated optimization problems (Boyd et al. (2011), Schizas et al. (2008)). Few ESC techniques have been proposed to solve decentralized and distributed optimization problems have been proposed (Li et al. (2011), Kvaternik and Patel (2012)).

This study proposes the design of distributed optimization of networks of dynamic agents with unknown unstable dynamics. A distributed extremum-seeking controller is proposed to solve the optimization problem. Many researchers have considered various ESC approaches over the last years (see, Krstic (2000), Tan et al. (2006), Krstic and Wang (2000), Adetola and Guay (2007) , Guay et al. (2004), Nesic et al. (2010) ,Moase and Manzie (2012), Ghaffari et al. (2012), Moase et al. (2010), Guay and Dochain (2013) ,Zhang and Ordóñez (2009) ,Fu and Özgüner (2011)). This paper proposes a novel proportional-integral ESC (PIESC) design technique, initially proposed in Guay and Dochain (2014), in a distributed environment to design cooperative systems that solve a distributed optimization problem over networks of unknown unstable dynamic agents. The agent dynamics implement a PIESC controller that can solve the real-time optimization without the explicit need for a two time-scale approach. The main contribution of this paper is to show that the distributed PIESC can be effectively applied to design real-time optimization control systems that can stabilize the network of unstable dynamics to the unknown optimum of the total plant cost.

This paper is organized as follows. The problem is formulated in Section 2. In Section 3, the control system is presented. A simulation example is presented in Section 4. We conclude in Section 5.

2. PROBLEM DESCRIPTION

We consider a network of nonlinear systems of the form:

$$\dot{x}_i = f_i(x) + g_i(x)u \tag{1}$$

$$y_i = h_i(x) \tag{2}$$

where $x = [x_1^T, \dots x_p^T]^T \in \mathbb{R}^n$ is the vector of state variables and u is the vector of input variables taking values in $\mathcal{U} \subset \mathbb{R}^p$ for the entire network. The dynamic of each agent $i = 1, \dots, p$ is governed by the dynamics (1) with local cost $y_i = h_i(x)$. It is assumed that each agent can only use the local input variable u_i. The overall network cost function is the sum of all the individual costs:

$$J(x) = \sum_{i=1}^{p} h_i(x). \tag{3}$$

It is assumed that the vector fields $f_i(x)$ and $g_i(x)$ are unknown smooth vector valued functions of x and that $h_i(x)$ are unknown smooth functions of x.

The objective is to steer the system to the equilibrium x^* and u^* that achieves the minimum value of $Y = J(x)$. The total network dynamics can be written in the form:

$$\dot{x} = f(x) + G(x)u \tag{4}$$

with total cost

$$Y = J(x) \tag{5}$$

where

$$f(x) = [f_1(x)^T, \dots, f_p(x)^T]^T,$$
$$G(x) = \left[g_1(x)^T, \dots, g_p(x)^T\right]^T.$$

The equilibrium (or steady-state) network map is the vector $x = \pi(u)$ that solves the following equation:

$$f(\pi(u)) + G(\pi(u))u = 0.$$

The corresponding equilibrium cost function is given by:

$$Y = J(\pi(u)) = \ell(u) \tag{6}$$

At equilibrium, the problem is reduced to finding the minimizer u^* of $Y = \ell(u^*)$. In the following, we let $\mathcal{D}(u)$ represent a neighbourhood of the equilibrium $x = \pi(u)$.

Some additional assumptions are required concerning the cost function $J(x)$.

Assumption 1. The total network cost $J(x)$ is such that

(1) $\frac{\partial J(x^*)}{\partial x} = 0$
(2) $\frac{\partial^2 J(x)}{\partial x \partial x^T} > \beta I, \ \forall x \in \mathcal{D}(u)$

where β is a strictly positive constant.

Note that, in contrast to standard ESC, convexity of the cost function $J(x)$ is required. We also require the following properties for the dynamics:

Assumption 2. For the dynamics (1), the following statements are assumed:

(1) there exists $k^*, \alpha > 0$ such that $J(x)$ satisfies the inequality:
$$\frac{\partial J}{\partial x} f(x) + \frac{\partial J}{\partial x} G(x)u - k^* \frac{\partial J}{\partial x} G(x) G(x)^T \frac{\partial J^T}{\partial x} \leq -\alpha \|x - \pi(u)\|^2,$$
$\forall x \in \mathcal{D}(u)$,
(2) the matrix valued function $G(x)$ is full rank $\forall x \in \mathcal{D}(u)$,

$\forall u \in \mathcal{U}$.

Assumption 2 states that h is non-increasing along the vector field of the closed-loop system $f(x) - k^* G(x) G(x)^T \frac{\partial J^T}{\partial x}$ over some neighbourhood of the steady-state manifold $x = \pi(u)$ at a fixed value of the input u. It also states that the cost function is of relative order one in a neighbourhood of the origin. Finally, the following additional assumption concerning the steady-state cost function $\ell(u)$ is required.

Assumption 3. The equilibrium steady-state map $\ell(u)$ is such that

$$\nabla_u \ell(u)(u - u^*) \geq \alpha_u \|u - u^*\|^2$$

for some positive constant α_u $\forall u \in \mathcal{U}$.

3. DISTRIBUTED ESC CONTROLLER

3.1 Consensus algorithm

In this study, we consider a consensus estimation approach that provides each agent with an estimate of the total cost J using only its local cost measurement and communication protocol with neighbouring agents. This problem is handled by using a dynamic average approach where the objective of each agent is to estimate the mean of the "inputs" to a system. Here, the use of "inputs" refers to h_i, the measurement of the local cost. By estimating the mean over all agents, the consensus provides an average cost estimate of the total cost given by $\frac{1}{p}\sum h_i$ which provides an estimate of the average cost $\frac{1}{p}J$. Since the number of agents is assumed to be fixed, this provides an estimate of the total cost J. Therefore the consensus algorithm takes local cost measurements as inputs and produces total cost estimates as outputs.

In the consensus estimation approach, agent i updates their estimate using two sources of information: 1) their measurement of h_i and 2) the estimates of $\frac{1}{p}J$ from other agents which are acquired from the network. Graph theory supplies the machinery necessary to model a network. A graph is a collection of vertices whereby each vertex is formed through the meeting of two or more edges. Here, a vertex represents one agent and an edge represents a pathway of communication. Three matrices are used to define a network:

(1) *Degree matrix:* Let $D \in \mathbb{R}^{p \times p}$ be a diagonal matrix where d_{ii} is the number of agents which agent i can communicate with for $i = 1, \dots, p$.

(2) *Adjacency matrix:* Let $A \in \mathbb{R}^{p\times p}$ where $a_{ij} = 1$ (0) if information can (cannot) flow from agent i to agent j and $a_{ii} = 0$ for $i, j = 1, \ldots, p$. A graph is *undirected* when $A = A^T$ meaning that all agents can both transmit and receive information.
(3) *Laplacian matrix:* Let $L = D - A$. By definition, an undirected graph has rank$(L) = p - 1$. If rank$(L) < p - 1$, then a graph is *directed* and some agents can only transmit or receive information.

The proportional-integral estimator designed by Freeman et al. (2006) is the preferred technique to solve the consensus problem in the distributed ESC design approach proposed in this manuscript. It is important to note that only specific properties of the dynamic consensus are required for the successful application of the proposed distributed PIESC. Let $\widehat{J}_i$ denote agent i's estimate of J and $\widehat{J} = [\widehat{J}_1, \ldots, \widehat{J}_p]^T$. The consensus estimator is given by

$$\begin{bmatrix} \dot{\widehat{J}}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} \gamma I - \kappa_P L & \kappa_I L \\ -\kappa_I L & 0 \end{bmatrix} \begin{bmatrix} \widehat{J}(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} \gamma I \\ 0 \end{bmatrix} h(x) \tag{7}$$

where $[\widehat{J}\ w]^T \in \mathbb{R}^{2p}$ is the internal estimator state and $\kappa_P,\ \kappa_I,\ \gamma > 0$ are user-specified constants. If the γ-term is large relative to the κ-terms, then more weight is given to local costs thus the actions of agents are more selfish in nature. By contrast, larger κ-terms result in more coordinated actions. The terms, κ_P and κ_I, are analogous to the proportional and integral terms of a PI controller. A large κ_P (κ_I) gives weight to the current (past) estimates of J which are received from the network.

3.2 Dynamics of agent i

The dynamics of the total cost J are given by:

$$\dot{J}(t) = L_f J + L_G J u \tag{8}$$

where $L_f J$ and $L_G J$ are the Lie derivatives of J with respect to f and G. The dynamics of J as seen by each agent, i, is given by:

$$\frac{\dot{J}}{p} = \frac{1}{p}\left(L_f J + \frac{\partial J}{\partial x} G_i u_i + \frac{\partial J}{\partial x} G_j u_j\right) \tag{9}$$

where $j \neq i$.

The local cost dynamics (9) are parameterized as follows:

$$\frac{\dot{J}}{p} = \theta_{0i}(t) + \theta_{1i}(t) u_i \tag{10}$$

where

$$\theta_{1i}(t) = \frac{1}{p}\frac{\partial J}{\partial x} G_i$$

and

$$\theta_{0i}(t) = \frac{1}{p}\left(L_f J + \frac{\partial J}{\partial x} G_j u_j\right).$$

The parameter $\theta_{0i}(t)$ measures the local contribution of the network drift dynamics and the effect of the inputs of the other local agents. The correct estimation of $\theta_{0i}(t)$ allows one to estimate the term $\theta_{1i}(t)$ that measures the effect of the local input on the global cost.

The distributed ESC approach requires the estimation of the local parameters $\theta_{0i}(t)$ and $\theta_{1i}(t)$ for each agent. This task is performed using a time-varying parameter estimation approach such as the one proposed in Guay and Dochain (2014). Since the local agent do not have access to the total cost, the estimation must rely on the estimated consensus cost $\widehat{J}$.

We first define the regressor vector $\phi_i = \left[1,\ u_i^T\right]^T$. The vector of parameter estimates is given by $\hat{\theta}_i = \left[\hat{\theta}_{0i},\ \hat{\theta}_{1i}^T\right]^T$. Let the estimation error be given by $e_i = \widehat{J}_i - \hat{z}_i$. The estimator model of (10) for agent i is given by

$$\dot{\hat{z}}_i = \phi^T \hat{\theta}_i + K e_i(t) + c_i(t)^T \dot{\hat{\theta}}_i(t), \quad \hat{z}(0) = \hat{z}_0, \tag{11}$$

where $K > 0$ is a user-specified constant. The dynamics of $c_i(t)$ are described by the system:

$$\dot{c}_i(t) = -K c_i(t) + \phi_i, \quad c(0) = 0. \tag{12}$$

Let $\eta_i = e_i - c_i(t)^T \hat{\theta}_i$ be an auxiliary variable. The dynamics of this variable is approximated by

$$\dot{\hat{\eta}}_i(t) = -K \hat{\eta}_i(t) \tag{13}$$

The parameter estimation update law is given as follows. We define a scaling matrix $\Sigma_i \in \mathbb{R}^{2\times 2}$, with dynamics given by

$$\dot{\Sigma}_i(t) = c_i(t) c_i^T(t) - k_T \Sigma_i(t) + \delta I, \quad \Sigma_i(0) = \alpha_1 I, \tag{14}$$

where $k_T, \delta, \alpha_1 > 0$ are user-defined constants.

Equations (11)-(14) form the framework of the parameter update law presented in Adetola and Guay (2008), namely:

$$\dot{\hat{\theta}}_i = \text{proj}\left\{\Sigma_i^{-1} c_i (e_i - \hat{\eta}_i - \sigma \hat{\theta}_i), \hat{\theta}_i\right\}, \quad \hat{\theta}_i(0) = \hat{\theta}_{i_0} \tag{15}$$

where $\text{proj}\left\{\tau, \widehat{\theta}\right\}$ denotes a Lipschitz projection operator Krstic et al. (1995).

The projection operator is designed such that

$$\text{proj}\left\{\tau, \hat{\theta}_i\right\}^T \beta(t) \geq \tau^T \beta(t) \tag{16}$$

where $\hat{\theta}_i, \beta \in \Theta$ and Θ is a ball centered on $\hat{\theta}_i(t)$ with a finite radius. This ball forms the uncertainty set of the parameter estimation routine. As a result of (16),

$$\hat{\theta}_{i_0} \in \Theta \implies \hat{\theta}_i(t) \in \Theta, \quad \forall t \geq 0 \tag{17}$$

We must assume that the cost function undergoes sufficient excitation for $\hat{\theta}_i$ to converge to its true values. This idea has been formalized in the following assumption.

Assumption 4. For agent, i, there exist constants, $\alpha_2 > 0$ and $T > 0$, such that

$$\int_t^{t+T} c_i(\tau) c_i(\tau)^T d\tau \geq \alpha_2 I, \quad t > 0. \tag{18}$$

The machinery for the real-time parameter estimation is now in place. The local extremum seeking control algorithm can be design. In this study, we propose the application of a proportional-integral ESC algorithm given by

$$\begin{aligned} \dot{u}_i &= -k_g \hat{\theta}_{1i} + \hat{u}_i + d_i(t) \\ \dot{\hat{u}}_i &= -\frac{1}{\tau_I} \hat{\theta}_{1i} \end{aligned} \tag{19}$$

where $k_g > 0$ is a user-defined constant called the proportional ESC gain constant and $\tau_I >$ is a positive constant called the integral ESC gain constant. The dither signal, $d_i(t)$, is designed such that $||d_i(t)|| \leq D\ \forall t \geq 0$, where $D > 0$ is a constant.

Theorem 1. Let Assumptions 1 to 4 hold. Consider the distributed extremum-seeking controller (19), the parameter estimation algorithm (14) and (15) and the dynamic average consensus algorithm (7). Then there exists controller gains k_g, k_T, K, σ and τ_I^* and dynamic consensus gains γ, κ_P and κ_I such that, for all $\tau_I > \tau_I^*$, the system converges exponentially to an $\mathcal{O}(D/k_g)$, $\mathcal{O}(k_g/k_T)$, $\mathcal{O}(\sigma z_\theta)$ neighbourhood of the minimizer x^* of the measured cost function y.

Proof: The proof is omitted due space restriction. It follows the same lines as the results of (Guay and Dochain (2014)) and using the properties of the proposed consensus algorithm stated in (Freeman et al. (2006)). ■

4. SIMULATION EXAMPLE

Consider a system with 5 states, 5 inputs and linear dynamics given by:

$$\dot{x} = \begin{bmatrix} 0.1108 & 1 & 0 & 0 & 0 \\ 0 & 0.1367 & 1 & 0 & 0 \\ 0 & 0 & 0.0548 & 1 & 0 \\ 0 & 0 & 0 & 0.1172 & 1 \\ 0 & 0 & 0 & 0 & 0.1000 \end{bmatrix} x + u$$

This system has poles at 0.1108, 0.1367, 0.0548, 0.1172 and 0.1000 and which are all slow, unstable poles. This system will be controlled by 5 agents. The i^{th} agent measures a local cost y_i and manipulates the input u_i. The local costs are quadratic functions of the states:

$$y_1 = (x_1 - 1)^2 + 2(x_2 - 2)^2, \quad y_2 = (x_2 - 2)^2 + 2(x_3 - 4)^2$$
$$y_3 = (x_3 - 3)^2 + 2(x_4 - 6)^2, \quad y_4 = (x_4 - 4)^2 + 2(x_5 - 8)^2$$
$$y_5 = (x_5 - 5)^2 + 2x_1^2$$

The total cost is the sum of the local costs: $J(x) = \sum_{i=1}^{5} y_i$. Each agent estimates the total cost using its local cost and information received from neighbouring agents. The Laplacian matrix used in this simulation is:

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

The tuning parameters used in this simulation were: $\gamma = 1 \times 10^4$, $\sigma = 1 \times 10^{-6}$, $\kappa_P = 1 \times 10^4$, $k_T = 50$, $\kappa_I = 1 \times 10^4$, $k_g = 0.5$, $K = 50$, $\tau_i = 1$, $\delta = 5 \times 10^{-7}$. The dither signals used were: $d(t) = 10[\sin(123t), \sin(155t), \sin(187t), \sin(219t), \sin(251t)]^T$. The optimal cost, state and input can be computed to be: $J^* = 9.833$, $x^* = [$ 0.5000, 2.0000, 3.6667, 5.3333, 7.0000 $]^T$ and $u^* = [$ -2.0554, -3.9400, -5.5344, -7.6253, -0.7000 $]^T$. The control system was simulated starting at the following initial conditions: $x_i(t_0) = 0$, $\hat{J}_i(t_0) = y_i(t_0)$, $w_i(t_0) = 0$, $\hat{z}_i(t_0) = y_i(t_0)$, $c_i(t_0) = [0, 0]^T$, $\Sigma_i(t_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\hat{\theta}_i(t_0) = [1e - 4, 0]^T$, $\hat{u}_i(t_0) = 0$.

The simulation results (Figures 1, 2, and 3) show that the distributed extremum-seeking controller is able to stabilize the unstable system and find the minimum of the total cost function. Initially, some oscillatory transient behaviour is observed as the parameter estimates, $\hat{\theta}_i$ are still far from the true values of the parameters, $\hat{\theta}_i$. However, once the algorithm has been running for a short period of time, the controllers have accurate estimates of the gradient and can then move in the appropriate direction. All 5 input biases converge, $\hat{u}_i$, converge to the optimal inputs, u_i^*. Despite the instability of the open-loop system, the closed loop system is stable and so the states, x_i converge to their optimal values, x_i^*. At these conditions, the convex cost function is also minimized.

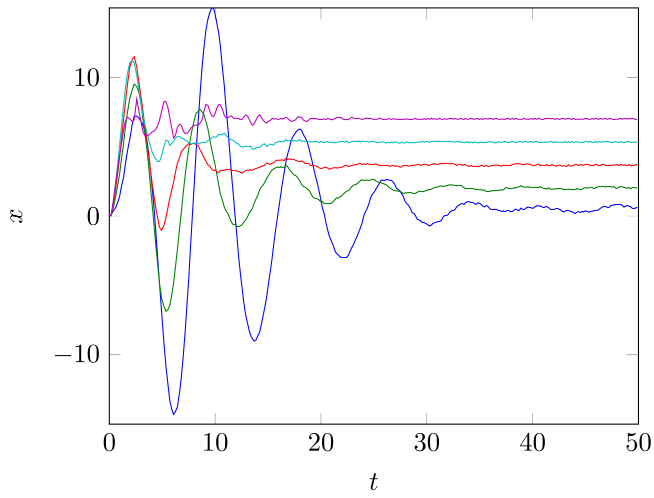

Fig. 1. State trajectories for the system controlled and stabilized by PIESC with consensus

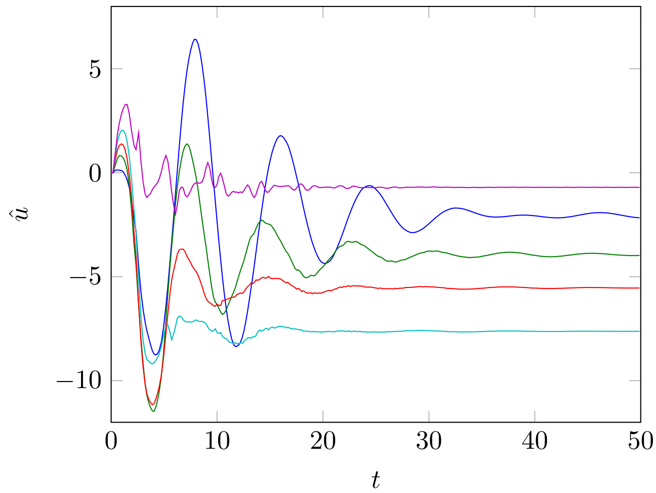

Fig. 2. Input bias trajectories for the system controlled and stabilized by PIESC with consensus

5. CONCLUSIONS

In this paper, a novel distributed extremum seeking control technique is proposed to solve a class of real-time optimization problems over a network of dynamic agents with unknown dynamics. Using the local estimates of the total costs, the proposed distributed extremum seeking controller adjusts the local decision variables for each agent independently to optimize the total network cost.

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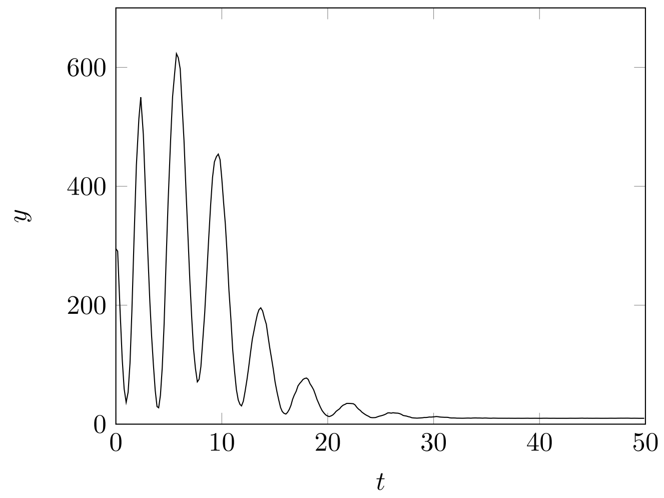

Fig. 3. Total cost trajectory for the system controlled and stabilized by PIESC with consensus

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