Preprints of the
9th International Symposium on Advanced Control of Chemical Processes
The International Federation of Automatic Control
June 7-10, 2015, Whistler, British Columbia, Canada

High-order Differential Dissipativity Analysis of Nonlinear Processes ⋆

Ruigang Wang, Michael James Tippett, Jie Bao

School of Chemical Engineering, University of New South Wales, Sydney, New South Wales 2052, Australia (email: j.bao@unsw.edu.au)

Abstract: Dissipativity theory is an effective tool for system analysis and control design for network systems. Differential dissipativity is an extension of dissipativity achieved by lifting storage functions and supply rates to the tangent bundle. This paper extends differential dissipativity to the high order derivatives of the displacement of external variables. It allows for a more detailed description of nonlinear systems dynamics in terms of dissipativity, which can result in less conservative stability and performance conditions. It can be regarded either as a nonlinear extension of quadratic differential form (QDF) dissipativity for linear systems, or an extension of differential dissipativity to more detailed supply rates. Important features of dissipativity such as the determination of supply rate and storage function, stability conditions for open systems and interconnections for networked dynamics are investigated.

Keywords: Differential dissipativity, nonlinear process, linear differential inclusion.

1. INTRODUCTION

The complexity of plant-wide control for modern chemical plants comes from (1) their large scales (often consisting of a number of process units with hundreds of control variables and states) and the strong interactions between process units (such as heat integration and mass recycling) (Skogestad, 2004). Neither centralized nor decentralized control can deal with the conflicts between computational limits, performance, and interaction handling. Thus, distributed MPC for plantwide control has received much attention recently, as discussed in the recent review papers (Christofides et al., 2013; Scattolini, 2009). A promising approach to plantwide analysis and control is based on the dissipativity theory, where the plantwide stability and performance are determined using the dissipative property of individual process units and the network topology (Xu and Bao, 2009; Tippett and Bao, 2013, 2014).

The dissipativity theory, introduced by Willems (1972), provides a scalable and flexible framework for the analysis of large scale interconnected dynamical systems (e.g., Moylan and Hill (1978)). The main disadvantage of dissipativity based control design is the conservatism of stability and performance conditions obtained from QSR dissipativity. In order to obtain tighter bounds and better control performance, several new types of dissipativity have been introduced. In the linear systems setting, quadratic differential forms (QDFs) have been introduced to define the dissipativity of a dynamical system based on its input/output and their derivatives (Willems and Trentelman, 1998). The QDF supply rates capture more detailed dynamic features of systems, which can result in less conservative stability and performance conditions, as shown in Tippett and Bao (2014).

For nonlinear systems, the notion of incremental dissipativity was introduced by Angeli (2002); Sepulchre (2006), which is useful in analyzing process operability during trajectory tracking (Santoso et al., 2012). However, incremental dissipativity requires the construction of a storage function in the extended space for any two state trajectories, and a priori formulation of the supply rate based on the difference between input/output trajectories, a difficult task in general (Forni and Sepulchre, 2013). A new type of dissipativity theory, differential dissipativity, based on the infinitesimal variations of dynamical systems along their state trajectories was proposed by Forni and Sepulchre (2013). The basic idea goes back to contraction analysis for nonlinear systems developed by Lohmiller and Slotine (1998). It provides a variational approach to incremental dissipativity without explicitly constructing the distance measuring the convergence of solutions.

Motivated by the ideas of QDF dissipativity for linear systems and differential dissipativity for nonlinear systems, this paper fuses these two different types of dissipativity to form a high-order differential dissipativity. It can be regarded either as a nonlinear extension of quadratic differential form (QDF) dissipativity for linear systems, or an extension of differential dissipativity to allow for dynamic supply rates. The latter allows for more flexible and detailed process dynamics features to be captured, which can produce less conservative stability and performance designs.

The remainder of the paper is structured as follows. Some background materials on differential dissipativity are presented in Section 2. Section 3 introduces the concept of high-order differential dissipativity and its important

⋆ This work is supported by ARC Discovery Project DP130103330. The first author acknowledges the financial support of the China Council Scholarship and the UNSW Tuition Fee Scholarship.

features. The notations in this paper follow Forni and Sepulchre (2013, 2014); Tippett and Bao (2014).

2. PRELIMINARIES

2.1 Displacement Dynamical Systems

Differential dissipativity is defined on *displacement dynamical systems*, which can be described as a tangent bundle behavior around the trajectory of the nonlinear system (Lohmiller and Slotine, 1998; Forni and Sepulchre, 2013, 2014). Given the state manifold $\mathcal{M}$ and external variable manifold $\mathcal{W}$, a time-invariant dynamical system Σ and its displacement dynamical system $\delta\Sigma$ is represented by

$$F(x, \dot{x}, w) = 0 \tag{1a}$$

$$DF(x, \dot{x}, w)[\delta x, \dot{\delta x}, \delta w] = 0, \tag{1b}$$

where $F : T\mathcal{M} \times \mathcal{W} \to \mathbb{R}^p$ is the nonlinear map represents Σ with $(x, w)(\cdot) : \mathbb{R} \to \mathcal{M} \times \mathcal{W}$, $DF(\cdot) : T_{(x,\dot{x},w)}\mathcal{M} \times \mathcal{W} \to T_{F(x,\dot{x},w)}\mathbb{R}^p$ represents the tangent map of F, $(\delta x, \delta w) \in T_x\mathcal{M} \times T_w\mathcal{W}$ represents an infinitesimal displacement on (x, w) and $w = (y, u)$ with output y and input u. The displacement system $\delta\Sigma$ characterizes the infinitesimal difference between two neighborhood solutions. A graphical representation of a displacement is shown in Figure 1. Consider a parameterized $\mathcal{C}^2$ solution family $(x, w)(\cdot, s) : \mathbb{R} \to \mathcal{M} \times \mathcal{W} \in \Sigma$ with $s \in [0, 1]$. For any s, the parameterized curve must satisfy the system (1a). By taking the derivative with respect to parameter s, we can obtain

$$0 = \frac{\partial}{\partial s} F(x(t, s), \dot{x}(t, s), w(t, s)) = DF(\cdots) \left[\frac{\partial x}{\partial s}, \frac{\partial^2 x}{\partial t \partial s}, \frac{\partial w}{\partial s}\right]. \tag{2}$$

By comparing it with the displacement system (1b), the infinitesimal variation on $(x, w)(\cdot, s)$ can be defined as $(\delta x, \delta w) := (\frac{\partial x}{\partial s}, \frac{\partial w}{\partial s}) \in T_x\mathcal{M} \times T_w\mathcal{W}$.

Forni and Sepulchre (2013) proposed a differential approach by checking a pointwise geometric condition in the tangent bundle. This contraction property is measured by the Riemannian-Finsler metric which is considered more convenient and intuitive than the distance metric since the base manifold $\mathcal{M}$ might contain more complex structure than Euclidean space.

2.2 Differential Dissipativity

The differential dissipativity extends the incremental dissipativity properties in the sense that it depends only on the tangent vector field along the solution. Also, it leads to less conservative results since it does not need to check the dissipativity condition between any two operating trajectories. In differential dissipativity, the *differential storage function* $S : T\mathcal{M} \to \mathbb{R}_{\geqslant 0}$ is seen as the infinitesimal energy associated to the infinitesimal variation $\delta x(\cdot)$ on a given solution $x(\cdot)$. The energy can be either increased or decreased through the *differential supply rate* Q from external variable w.

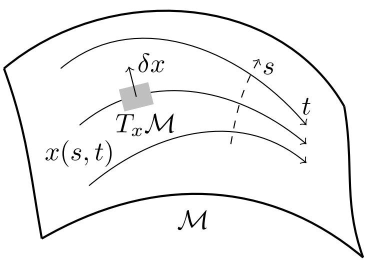

Fig. 1. Displacement dynamical system $\delta\Sigma$

Definition 1. (Forni and Sepulchre (2013)). A function $S : T\mathcal{M} \to \mathbb{R}_{\geqslant 0}$ is a *differential storage function* for the dynamical system Σ in (1a) if there exist $c_1, c_2 \in \mathbb{R}_{\geqslant 0}$, $p \in \mathbb{R}_{\geqslant 1}$ and $K : T\mathcal{M} \to \mathbb{R}_{\geqslant 0}$ such that

$$c_1 K(x, \delta x)^p \leqslant S(x, \delta x) \leqslant c_2 K(x, \delta x)^p, \tag{3}$$

for all $(x, \delta x) \in T\mathcal{M}$, where S and K satisfies the following conditions:

(i) S and K are $\mathcal{C}^1$ functions for each $(x, \delta x) \in T\mathcal{M}$;
(ii) $K(x, 0) = 0$ and $K(x, \delta x) > 0$ for each $x \in \mathcal{M} \setminus \Omega$ and $\delta x \neq 0$, where Ω is a set of isolated points of $\mathcal{M}$;
(iii) $K(x, \lambda\delta x) = \lambda K(x, \delta x)$ for each $\lambda > 0$, $x \in \mathcal{M} \setminus \Omega$ and $(x, \delta x) \in T\mathcal{M}$;
(v) $K(x, \delta x_1 + \delta x_2) < K(x, \delta x_1) + K(x, \delta x_2)$ for each $x \in \mathcal{M} \setminus \Omega$ and $\delta x_1, \delta x_2$ such that $\delta x_1 \neq \lambda\delta x_2$ and $\lambda \in \mathbb{R}$ (strict convexity).

The function K provides a Finsler metric on the state manifold $\mathcal{M}$. Since a Finsler metric is a non-symmetric norm on each tangent space $T_x\mathcal{M}$, it is suitable for nonlinear system due as it is non-homogeneous with respect to different directions. And it is much more convenient than the general distance metric due as it is a local property. The differential storage S can be connected to the energy of the displacement δx using (3). So the storage function acts like a Lyapunov function candidate which is upper and lower bounded by a special type of $\mathcal{K}_\infty$ function – $K(x, \delta x)^p$ where p is the degree of homogeneity of S. The definition of differential dissipativity is given by lifting the dissipativity to the tangent bundle.

Definition 2. (Forni and Sepulchre (2013)). A function $Q : \mathcal{M} \times T\mathcal{W} \to \mathbb{R}$ is a *differential supply rate* for the dynamical system Σ in (1a) if

$$\int_0^t |Q(x, w, \delta w)(\tau)| \, \mathrm{d}\tau < \infty, \tag{4}$$

for each $t \geqslant 0$ and each $(x, \delta x, w, \delta w)(\cdot) \in \delta\Sigma$.

Definition 3. (Forni and Sepulchre (2013)). A dynamical system Σ in (1a) is *differentially dissipative* with respect to the differential supply rate Q if there exists a differential storage function S such that

$$S(x, \delta x)(t) - S(x, \delta x)(0) \leqslant \int_0^t Q(x, w, \delta w)(\tau) \mathrm{d}\tau, \tag{5}$$

for all $t \geqslant 0$ and all $(x, \delta x, w, \delta w)(\cdot) \in \delta\Sigma$ in (1b). When Q is independent on x, that is, $Q : T\mathcal{W} \to \mathbb{R}$, we say that Σ is *uniformly differentially dissipative.*

If $S \in \mathcal{C}^1$, the differential dissipativity condition is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} S(x(t), \delta x(t)) \leqslant Q(x(t), w(t), \delta w(t)). \tag{6}$$

Following theorem reveals the connection between incremental stability and differential dissipativity.

Theorem 4. (Forni and Sepulchre (2013)). Suppose the dynamical system Σ is differentially dissipative with differential storage function S and differential supply rate Q. Suppose also that for external variable $w = (y, u)$, output y, input u, it holds that $Q(x, y, u, \delta y, 0) = 0$ for each $x \in \mathcal{M}$, and each $(y, u, \delta y, 0) \in T\mathcal{W}$. Then, there exists a class $\mathcal{K}$ function α such that

$$d(x_1(t), x_2(t)) \leqslant \alpha(d(x_1(0), x_2(0))) \tag{7}$$

for each $t \geqslant 0$ and each $(x_1, y_1, u_1)(\cdot), (x_2, y_2, u_2)(\cdot) \in \Sigma$, such that $u_1(\cdot) = u_2(\cdot)$, where d is the pseudo-distance induced by $S^{\frac{1}{p}}$ (see Definition 1).

So if the system is differential dissipative, the distance of two solutions from different initial states with the same control input is bounded.

3. MAIN RESULTS

3.1 Higher-order Differential Dissipativity

The basic idea of high-order differential dissipativity is to investigate the dissipativity properties not only on the displacement system but also its high order derivatives. As such, the storage function would relay on the norm defined on $\hat{\delta x} = [\delta x, \delta x^{(1)}, \ldots, \delta x^{(n)}]^T$. Therefore, high-order differential dissipativity measures the contraction property of the neighborhood trajectories with certain smoothness which can bring less conservative results.

Following the steps of Willems and Trentelman (1998), we develop the concept of nonuniform quadratic differential forms (NQDFs) and high-order differential dissipativity based on NQDFs. Let ϕ_{kl} be symmetric matrices whose coefficients are functions of $\overline{x} = (x, u, u^{(1)}, \ldots, u^{(n)}) \in \overline{X}$ and $\mathbb{R}^{n\times n}[\xi, \eta]$ denote the set of symmetric polynomial matrices in the noncommutative indeterminates ξ and η. Explicitly, an element $\Phi \in \mathbb{R}^{n\times n}[\xi, \eta]$ is given by

$$\Phi(\xi, \eta) = \sum_{k,l}^{n} \xi^k \phi_{kl}(\overline{x})\eta^l. \tag{8}$$

Definition 5. (Nonuniform quadratic differential form). Consider the indeterminates ξ, η as representation of differential the operators $\frac{\mathrm{d}}{\mathrm{d}t}^T$ and $\frac{\mathrm{d}}{\mathrm{d}t}$, then the Φ induced NQDF with respect to infinitesimal displacement of external variable $w = (y, u)$ can be defined by

$$Q_\Phi(\delta w) := \sum_{k,l}^{n} \left(\frac{\mathrm{d}^k \delta w}{\mathrm{d}t^k}(t)\right)^T \phi_{kl}(\overline{x}) \left(\frac{\mathrm{d}^l \delta w}{\mathrm{d}t^k}(t)\right). \tag{9}$$

Remark 6. Although the form is the same as Willems and Trentelman (1998), there are two main differences:

(1) ϕ_{kl} is not a constant matrix but a matrix function of $\overline{x}$. From local viewpoint, $\overline{x}$ is the minimum set of independent variables which can approximate the local behavior up to certain order Conte et al. (2007).
(2) the indeterminates do not commute due to the nonuniform nature of $\phi_{kl}(\overline{x})$. So some nice results based on QDFs need to be carefully treated such as differentiation formula of a QDF Willems and Trentelman (1998).

Definition 7. (Higher-order differential dissipativity). From the input-output sense, the system Σ in (1a) is high-order differentially dissipative with a storage function Q_Ψ, and a supply rate Q_Φ, both of which are NQDFs, if the following dissipation inequality is satisfied

$$\int_0^\infty Q_\Phi(\delta w)\mathrm{d}t \geqslant Q_\Psi(\delta w), \tag{10}$$

for all allowable trajectories of the external variable in displacement dynamical system (1b) with compact support. When the Φ matrix is constant, we can say that the system Σ is *uniformly high-order differentially dissipative.*

Remark 8. The nonuniform QDF on displacement dynamics is a natural extension of QDF for the linear system since the displacement dynamics of a linear system has the same form as global linear system. Therefore, NQDF based differential dissipativity can be seen as the nonlinear version of QDF dissipativity. Incremental dissipativity measures the difference of any pair of solutions while differential dissipativity gauges the local infinitesimal variation of continuous varying solutions. For example, for nonlinear input-output system $\ddot{y}+y\dot{y}+y = u+\dot{u}$, differential dissipativity captures the dynamics on $\delta w \in \mathcal{C}^1$. While the high-order dissipativity considers the dissipativity properties of smoother continuous varying solutions since the NQDF norm is defined on high-order derivatives. The dissipativity property for the example would be more specific on $\delta w \in \mathcal{C}^2$. Therefore, it is much less conservative than the former two types of dissipativity.

Example 9. Consider a simple nonlinear system

$$\dot{x} = -x^3 + u,\ y = x$$
$$\dot{\delta x} = -3x^2\delta x + \delta u,\ \delta y = \delta x.$$

Its extended input-output relation with respect to Equation (17),(18) can be expressed as

$$\begin{pmatrix}\delta y\\ \dot{\delta y}\\ \ddot{\delta y}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ -3x^2 & 1 & 0\\ 15x^4 - 6xu & -3x^2 & 1\end{pmatrix}\begin{pmatrix}\delta x\\ \delta u\\ \dot{\delta u}\end{pmatrix}.$$

It holds the higher order differential dissipativity with storage function and supply rate as bellow

$$Q_\Psi(\delta y, \delta u) = \begin{pmatrix}\delta y\\ \dot{\delta y}\\ \delta u\\ \dot{\delta u}\end{pmatrix}^T \begin{pmatrix}1/2 & 0 & 0 & 0\\ 0 & 1/2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\delta y\\ \dot{\delta y}\\ \delta u\\ \dot{\delta u}\end{pmatrix},$$

$$Q_\Phi(\delta y, \delta u) = \begin{pmatrix}\delta y\\ \dot{\delta y}\\ \delta u\\ \dot{\delta u}\end{pmatrix}^T \begin{pmatrix}\phi_{11} & 0 & \phi_{13} & \phi_{14}\\ 0 & \phi_{22} & 0 & 0\\ \phi_{13} & 0 & \phi_{33} & 0\\ \phi_{14} & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\delta y\\ \dot{\delta y}\\ \delta u\\ \dot{\delta u}\end{pmatrix},$$

$$\phi_{11} = -3x^2(1 + 12x^4 - 6xu),\ \phi_{22} = -1,\ \phi_{33} = -3x^2 - 1,$$
$$\phi_{13} = 1/2{+}12x^4 - 3x^2 - 3xu,\ \phi_{14} = -\frac{3}{2}x^2.$$

3.2 Dissipativity Determination of a Nonlinear Process

Although the storage function is defined on the external variables, it can be represented by states and inputs. Besides, this formulation is intuitive and useful for interconnection analysis, which is important application of dissipativity theory. However, this inequality is difficult to verify since $\delta w = (\delta y, \delta u)$ is not independent. Tippett and Bao (2014) gives a state space formulation of dissipativity based on $\overline{x}$. Similarly, we develop a state space formulation based on $\delta\overline{x} = (\delta x, \delta u, \delta u^{(1)}, \ldots, \delta u^{(n)}$. Define

$$\begin{aligned} \hat{\delta u}^T(t) &= \left(\delta u^T \; \delta u^{(1)^T} \; \cdots \; \delta u^{(n)^T}\right), \\ \hat{\delta y}^T(t) &= \left(\delta y^T \; \delta y^{(1)^T} \; \cdots \; \delta y^{(n)^T}\right), \end{aligned} \tag{11}$$

as the extended inputs and outputs, then the supply rate can be written as

$$Q_\Phi(\delta y, \delta u) = \begin{pmatrix} \hat{\delta y}(t) \\ \hat{\delta u}(t) \end{pmatrix}^T \begin{pmatrix} Q(\overline{x}) & S(\overline{x}) \\ S^T(\overline{x}) & R(\overline{x}) \end{pmatrix} \begin{pmatrix} \hat{\delta y}(t) \\ \hat{\delta u}(t) \end{pmatrix}, \tag{12}$$

where the coefficients of matrices Q, S, R are functions of $\overline{x}$. This shows that a NQDF is a quadratic form similar to the conventional (Q, S, R) type supply rate. The following result provides a condition for determining the dissipativity of a nonlinear system with NQDF supply rate and storage functions.

Proposition 10. Let the differential operator $\mathbf{s} = \frac{\mathrm{d}}{\mathrm{d}t}$, consider the state formulation of displacement dynamical system in (1b)

$$\begin{pmatrix} \mathbf{s}\delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} A(\overline{x}) & B(\overline{x}) \\ C(\overline{x}) & D(\overline{x}) \end{pmatrix} \begin{pmatrix} \delta x \\ \delta u \end{pmatrix} \tag{13}$$

where A, B, C, D are the linearization matrix at every operation point. The system in (13) is dissipative with respect to the supply rate and storage function pair induced by $\Phi \in \mathbb{R}^{n\times n}[\xi, \eta]$ and $\Psi \in \mathbb{R}^{(n-1)\times(n-1)}[\xi, \eta]$, respectively, with their coefficient matrices ϕ and ψ partitioned as $\phi = \begin{pmatrix} \phi_Q(\overline{x}) & \phi_S(\overline{x}) \\ \phi_S^T(\overline{x}) & \phi_R(\overline{x}) \end{pmatrix}$ and $\psi = \begin{pmatrix} \psi_Q(\overline{x}) & \psi_S(\overline{x}) \\ \psi_S^T(\overline{x}) & \psi_R(\overline{x}) \end{pmatrix}$, if and only if the following nonuniform LMI is satisfied in the desired region $\mathcal{D} \subset \overline{X}$:

$$\begin{pmatrix} \mathbb{T}_{11} & \mathbb{T}_{12} \\ \mathbb{T}_{12}^T & \mathbb{T}_{22} \end{pmatrix} \geqslant 0 \tag{14}$$

where

$$\begin{aligned} \mathbb{T}_{11} &= \hat{C}^T \gamma_Q \hat{C}, \\ \mathbb{T}_{12} &= \hat{C}^T \gamma_Q \hat{D} + \hat{C}^T \gamma_S, \\ \mathbb{T}_{22} &= \hat{D}^T \gamma_Q \hat{D} + \hat{D}^T \gamma_S + \gamma_s^T \hat{D} + \gamma_R, \end{aligned} \tag{15}$$

$$\begin{pmatrix} \gamma_Q & \gamma_S \\ \gamma_S^T & \gamma_R \end{pmatrix} = \phi - \begin{pmatrix} 0 & 0 \\ \psi & 0 \end{pmatrix} - \begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{s}\psi \end{pmatrix}, \tag{16}$$

and

$$\hat{C} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \hat{D} = \begin{pmatrix} d_{(0,0)} & 0 & \cdots & 0 & 0 \\ d_{(1,0)} & d_{(1,1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ d_{(n,0)} & d_{(n,1)} & \cdots & d(n, n-1) & d_{(n,n)} \end{pmatrix}, \tag{17}$$

with following iterative formulas $(1 \leqslant i \leqslant n, 1 \leqslant j < i)$

$$\begin{aligned} c_0 &= C, \quad c_i = \mathbf{s}c_{i-1} + c_{i-1}A \\ d_{(0,0)} &= d_{(i,i)} = D \\ d_{(i,0)} &= c_{i-1}B + \mathbf{s}d_{(i-1,0)} \\ d_{(i,j)} &= d_{(i-1,j-1)} + \mathbf{s}d_{(i-1,j)} \end{aligned} \tag{18}$$

Usually, it is difficult to determine the supply rate and storage function, especially when these functions are state dependent. Although the uniform dissipativity might be more conservative than NQDF, it is more practicable to be estimated when the states and inputs are constrained. Santoso et al. (2012) introduced a method to estimate the incremental dissipativity based on LDIs. Similarly, an LDI method can be applied to estimate the uniform high-order differential dissipativity.

Consider the high-order displacement system

$$\begin{pmatrix} \mathbf{s}\delta x \\ \hat{\delta y} \end{pmatrix} = G(\overline{x}) \begin{pmatrix} \delta x \\ \hat{\delta u} \end{pmatrix}, \tag{19}$$

where matrix $G(x, \overline{u} \in \Delta$. Then every trajectory of the displacement system is also a trajectory of the LDI defined by:

$$\begin{pmatrix} \mathbf{s}\delta x \\ \hat{\delta y} \end{pmatrix} = \Delta \begin{pmatrix} \delta x \\ \hat{\delta u} \end{pmatrix}. \tag{20}$$

If every trajectory of the LDI defined by Δ has a certain property (e.g., stability) then the trajectories of the original displacement system will also have that same property (Boyd et al., 1994). LDIs can be expressed in various ways, we use one of the most common ways – a polytope. In this case, Δ can be represented by the polytope's vertices given as follows:

$$\Delta = \mathrm{Co}\left\{ \begin{pmatrix} \hat{A}_1 & \hat{B}_1 \\ \hat{C}_1 & \hat{D}_1 \end{pmatrix}, \begin{pmatrix} \hat{A}_2 & \hat{B}_2 \\ \hat{C}_2 & \hat{D}_2 \end{pmatrix}, \ldots, \begin{pmatrix} \hat{A}_m & \hat{B}_m \\ \hat{C}_m & \hat{D}_m \end{pmatrix} \right\}. \tag{21}$$

where $G_i = (\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i)$,$1 \leqslant i \leqslant m$ are the vertices of the set Δ. By the definition of the convex hull, for any $G \in \Delta$, it is possible to write $G = \sum_{i=1}^m \mu_i G_i$ with $\mu_i \geqslant 0$ for all i and $\sum_{i=1}^m \mu_i = 1$.

Then, the dissipativity condition (14) can be numerically solved subject to every G_i to determine the uniform supply rate and storage function. Usually, the coefficient matrices ϕ and ψ of a nonlinear process determined by condition (14) and (21) are not unique. When it is desired to perform an open-loop analysis on a system, a set of feasibility problems can be solved with constraints on the bounds of ϕ_Q or ϕ_R, e.g., to use $\phi_Q < 0$ if the equilibrium operator trajectory of the nonlinear process is known to be stable.

3.3 Dissipativity and stability

Dissipativity theory has a close connection with input-output stability. For the QDF dissipativity in Tippett and Bao (2014), it implies that for certain vanishing input, the output and its derivatives asymptotically converge to zero. Also, it implies the asymptotic Lyapunov stability of the state space of the *minimal realization* (or any realization with the *zero detectability condition* of the input-output system. Analogous results for NQDFs are developed below:

Theorem 11. A system which is dissipative with positive semidefinite storage function Q_Ψ with respect to supply rate Q_Φ in (12) is asymptotically stable with finite $\mathcal{L}_2$ gain from $\hat{\delta u}$ to $\hat{\delta y}$

$$\left\|\hat{\delta y}\right\|_2 \leqslant \rho \left\|\hat{\delta u}\right\|_2 \tag{22}$$

where positive constant α and positive defined function $\beta(\overline{x})$ and $\rho(\overline{x})$ satisfy

$$\begin{aligned} \rho(\overline{x}) &= \left\|(-Q)^{-\frac{1}{2}}\right\|_2 \left(\left\|(-Q)^{-\frac{1}{2}}S\right\|_2 + \beta(\overline{x})\right) \\ \beta^2(\overline{x})I &\leqslant R - S^T Q^{-1} S \end{aligned} \tag{23}$$

if $Q < -\alpha^2 I$ for $\forall \overline{x} \in \overline{X}$.

Proof. Note that dissipativity condition with respect to (12) implies

$$\int_{-\infty}^{\infty} \hat{\delta y}^T Q \hat{\delta y} + 2\hat{\delta y}^T S \hat{\delta u} + \hat{\delta u}^T R \hat{\delta u} \mathrm{d}t \geqslant 0 \tag{24}$$

Let $\hat{Q} = -Q$. If $Q < -\alpha^2 I$, we can then complete the square to obtain

$$\begin{aligned}&\int_{-\infty}^{\infty} \left[\hat{Q}^{\frac{1}{2}}\widehat{\delta y} - \hat{Q}^{-\frac{1}{2}}S\widehat{\delta u}\right]^2 \mathrm{d}t \\ &\leqslant \int_{-\infty}^{\infty} \widehat{\delta u}^T \left(R + S^T\hat{Q}^{-1}S\right)\widehat{\delta u}\mathrm{d}t\end{aligned} \tag{25}$$

By applying the reverse triangle inequality and defining $\beta(\overline{x})$ as a positive defined function such that $\beta^2(\overline{x})I \leqslant R - S^T Q^{-1} S$ we then obtain

$$\left\|\hat{Q}^{\frac{1}{2}}\widehat{\delta y}\right\|_2 - \left\|\hat{Q}^{-\frac{1}{2}}S\widehat{\delta u}\right\|_2 \leqslant \beta(\overline{x})\left\|\widehat{\delta u}\right\|_2 \tag{26}$$

which reduces to $\left\|\widehat{\delta y}\right\|_2 \leqslant \rho(\overline{x})\left\|\widehat{\delta u}\right\|_2$ with (23).

To show the stability, for vanishing input the dissipation inequality becomes

$$\mathbf{s}Q_\Psi(\widehat{\delta y}, 0) \leqslant \widehat{\delta y}^T Q\widehat{\delta y} \tag{27}$$

As $Q < -\alpha^2 I$, this implies $\mathbf{s}Q_\Psi(\widehat{\delta y}, 0) < 0$, which implies asymptotic stability.

3.4 Interconnection of dissipative systems

Interconnections play an essential role in the application of dissipativity theory. Now we show that the negative feedback system of two high-order differentially dissipative systems is also high-order differentially dissipative.

Theorem 12. Consider a negative feedback configuration shown of a controller $\Sigma_1 : e \to u$ and a process system $\Sigma_2 : u \to y$, which are high-order differentially dissipative with respect to the following supply rates:

$$Q_{\Phi_1}(\delta u, \delta e) = \widehat{\delta u}^T Q_1\widehat{\delta u} + 2\widehat{\delta u}^T S_1\widehat{\delta e} + \widehat{\delta e}^T R_1\widehat{\delta e} \tag{28}$$

$$Q_{\Phi_2}(\delta y, \delta u) = \widehat{\delta y}^T Q_2\widehat{\delta y} + 2\widehat{\delta y}^T S_2\widehat{\delta u} + \widehat{\delta u}^T R_2\widehat{\delta u} \tag{29}$$

where the coefficients of $Q_i, S_i, R_i, i = 1, 2$ are functions of $\overline{x}$. Then, the closed-loop system is also high-order differentially dissipative and satisfies the following dissipation condition:

$$\begin{aligned}\mathbf{s}Q_{\Psi_1+\Psi_2}(\delta y, \delta u) \leqslant &\begin{pmatrix}\widehat{\delta u}\\ \widehat{\delta y}\end{pmatrix}^T \begin{pmatrix}Q_1 + R_2 & -S_1 + S_2^T\\ -S_1^T + S_2 & R_1 + Q_2\end{pmatrix}\begin{pmatrix}\widehat{\delta u}\\ \widehat{\delta y}\end{pmatrix} \\ &+ 2\begin{pmatrix}\widehat{\delta u}\\ \widehat{\delta y}\end{pmatrix}^T \begin{pmatrix}S_1\\ -R_1\end{pmatrix}\widehat{\delta r} + \widehat{\delta r}^T R_1\widehat{\delta r}\end{aligned} \tag{30}$$

where Ψ_1, Ψ_2 are the storage function matrix of system Σ_1, Σ_2. Furthermore, the closed loop system is high-order differential stable if

$$Q_{cl} := \begin{pmatrix}Q_1 + R_2 & -S_1 + S_2^T\\ -S_1^T + S_2 & R_1 + Q_2\end{pmatrix} < -\alpha^2 I, \tag{31}$$

where α is positive constant.

Remark 13. It is difficult to solve the closed-loop dissipativity formulation of (31) since the coefficients of Q_{cl} are functions of $\overline{x}$. However, by choosing certain type of base function for the coefficients, such as polynomials, the closed-loop dissipativity condition can be determined through sum-of-squares optimization (Prajna et al., 2004). The solution will be much less conservative than the open-loop case.

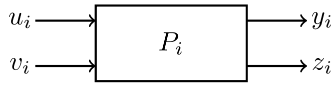

Fig. 2. Partitioning of subsystem inputs and outputs

For system analysis and control design of plant-wide systems, the relationship between the global dissipativity properties and that of individual subsystems and network topology is briefly introduced. The input to each process unit P_i is partitioned into two components: the manipulated input u_i and the input from interconnected units v_i. Meanwhile, the output of each unit is divided into two parts: the measured output y_i and the output z_i for interconnection as shown in Figure 2. For example, consider a continuous stirred-tank reactor (CSTR), the measured output is usually the temperature of reactor while the interconnection output is the the temperature and concentration of products. Meanwhile the manipulated input is usually the flow rate of jacket while the interconnected input is the temperature and concentration of input reactants.

The entire plant-wide system of n process units is depicted in Figure 3. Each process unit is stacked diagonally to form the plant-wide system *without interconnections*, $\tilde{P}$. The inputs and outputs of this system are the vectors consisting of the inputs and outputs of each system concatenated with one another. That is, the measured output of this diagonal system is $\mathbf{y} = \mathrm{col}(y_1, \ldots, y_n)$, where y_i is the outputs of the ith process unit. The signals $\mathbf{u}, \mathbf{v}$ and $\mathbf{z}$ are defined in a similar way as the concatenation of the appropriate input signals to each process unit. The topology of the interconnections between processes units is captured by the matrix H. Assume that the network topology is time invariant. So matrix H is a constant matrix with elements of either 0 or 1. In this way the interconnection topology of the process network is captured by the relation

$$\mathrm{v} = H\mathrm{z}. \tag{32}$$

For example, consider two systems $\Sigma_1 : v_1 \to z_1$ and $\Sigma_2 : v_2 \to z_2$ with feedback interconnection $v_1 = z_2$, $v_2 = z_1$, then the interconnection matrix is $H = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. With this representation of the plant-wide system, the global dissipativity properties can be formulated as a linear combination of those of the individual process units. Assume that the supply rate of the ith process unit is Q_{ϕ_i},

$$Q_{\phi_i}(\delta w_i) = \begin{pmatrix}\widehat{\delta y_i}\\ \widehat{\delta z_i}\\ \widehat{\delta u_i}\\ \widehat{\delta v_i}\end{pmatrix}^T \begin{pmatrix}Q_{yy}^i & Q_{yz}^i & S_{yu}^i & S_{yv}^i\\ Q_{zy}^i & Q_{zz}^i & S_{zu}^i & S_{zv}^i\\ S_{uy}^i & S_{uz}^i & R_{uu}^i & R_{uv}^i\\ S_{vy}^i & S_{vz}^i & R_{vu}^i & R_{vv}^i\end{pmatrix}\begin{pmatrix}\widehat{\delta y_i}\\ \widehat{\delta z_i}\\ \widehat{\delta u_i}\\ \widehat{\delta v_i}\end{pmatrix} \tag{33}$$

where $\delta w_i = (\delta y_i, \delta z_i, \delta u_i, \delta v_i)$ is the external variables, the subscripts y, z, u and v denote terms associated with the measured output, interconnecting output, control input and interconnecting input, respectively. The supply rate of subsystem $\tilde{P}$ can be written as

$$Q_{\tilde{\phi}}(\delta w) = \widehat{\delta w}^T \tilde{\phi}\widehat{\delta w} \tag{34}$$

where $w = \mathrm{col}(w_1, \ldots, w_n)$ and $\tilde{\phi} = \mathrm{diag}(\phi_1, \ldots, \phi_n)$. Let $w' = (y, z, u, v)^T$ and G be a permutation matrix such that $\widehat{\delta w} = G\widehat{\delta w}'$. From the interconnection topology (32), we eliminate variable v and obtain

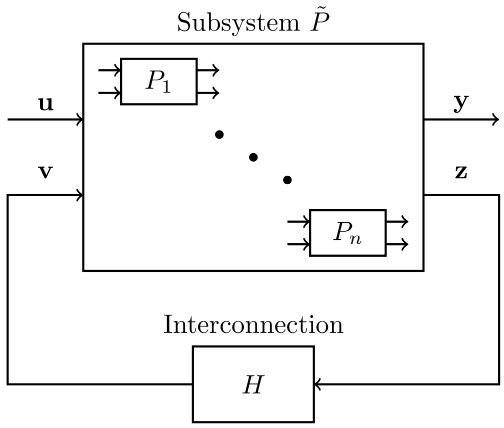

Fig. 3. Network view of plant-wide process

$$\hat{\delta w}' = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & H_p & 0 \end{pmatrix} \begin{pmatrix} \hat{\delta y} \\ \hat{\delta z} \\ \hat{\delta u} \end{pmatrix} = F\hat{\delta w}''. \tag{35}$$

where $H_p = \mathrm{diag}(H, H, \ldots, H)$ with the length of differential order. The supply rate of subsystem $\tilde{P}$ can be written as a NQDF on w''

$$Q_{\tilde{\phi}}(\delta w) = \hat{\delta w}'^T G^T \tilde{\phi} G \hat{\delta w}' = \hat{\delta w}''^T (GF)^T \tilde{\phi} GF \hat{\delta w}''. \tag{36}$$

Proposition 14. Consider the subsystems and interconnection in Figure 3. Assume the supply rate and storage function of subsystem $\tilde{P}$ is induced by the coefficient matrix $\tilde{\phi}$ and $\tilde{\psi}$, then the global supply rate and storage function of plant-wide process are $(GF)^T\tilde{\phi}GF$ and $(GF)^T\tilde{\psi}GF$, respectively, by considering $w'' = (y, z, u)^T$ as the external variables.

4. CONCLUSION

In this paper, a novel type of dissipativity for nonlinear processes has been developed by adopting the concepts of quadratic differential forms and differential dissipativity. The nonuniform quadratic differential forms (NQDF) based supply rates and storage functions can produce less conservative results for stability and performance than that resulting from other types of dissipativity such as conventional QSR, incremental and differential dissipativity. A condition for determining the high-order differential dissipativity of nonlinear processes is proposed. The local uniform supply rate and storage function can be obtained using an LDI based method. The dissipativity-based stability conditions for interconnected systems are investigated in this paper.

Future work may include a systematic approach for determining NQDF for the high-order dissipativity which is essential for system analysis and control design. A possible approach based on the skew field and noncommutative algebra (Halás, 2008) might be the natural generalization of operator approach for linear systems (Willems and Trentelman, 1998). Another interesting direction is to investigate the plant-wide control of nonlinear processes, especially for distributed MPC.

REFERENCES

Angeli, D. (2002). A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3), 410–421.

Boyd, S.P., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory*, volume 15. SIAM.

Christofides, P.D., Scattolini, R., de la Pena, D.M., and Liu, J. (2013). Distributed model predictive control: A tutorial review and future research directions. *Computers & Chemical Engineering*, 51, 21–41.

Conte, G., Moog, C.H., and Perdon, A.M. (2007). *Algebraic methods for nonlinear control systems.* Springer.

Forni, F. and Sepulchre, R. (2014). A differential Lyapunov framework for contraction analysis. *IEEE Transactions on Automatic Control*, 59(3), 614–628.

Forni, F. and Sepulchre, R. (2013). On differentially dissipative dynamical systems. In *Proc. 9th IFAC Symposium on Nonlinear Control Systems, Toulouse; France.*

Halás, M. (2008). An algebraic framework generalizing the concept of transfer functions to nonlinear systems. *Automatica*, 44(5), 1181–1190.

Lohmiller, W. and Slotine, J.J.E. (1998). On contraction analysis for non-linear systems. *Automatica*, 34(6), 683–696.

Moylan, P. and Hill, D. (1978). Stability criteria for large-scale systems. *Automatic Control, IEEE Transactions on*, 23(2), 143–149.

Prajna, S., Papachristodoulou, A., and Parrilo, P.A. (2004). Sostools: Sum of squares optimization toolbox for MATLAB–User's Guide.

Santoso, H., Hioe, D., Bao, J., and Lee, P.L. (2012). Operability analysis of nonlinear processes based on incremental dissipativity. *Journal of Process Control*, 22(1), 156–166.

Scattolini, R. (2009). Architectures for distributed and hierarchical model predictive control–a review. *Journal of Process Control*, 19(5), 723–731.

Sepulchre, R. (2006). Oscillators as systems and synchrony as a design principle. In *Current trends in nonlinear systems and control*, 123–141. Springer.

Skogestad, S. (2004). Control structure design for complete chemical plants. *Computers & Chemical Engineering*, 28(1), 219–234.

Tippett, M.J. and Bao, J. (2013). Dissipativity based distributed control synthesis. *Journal of Process Control*, 23(5), 755–766.

Tippett, M.J. and Bao, J. (2014). Control of plant-wide systems using dynamic supply rates. *Automatica*, 50(1), 44–52.

Willems, J.C. and Trentelman, H.L. (1998). On quadratic differential forms. *SIAM Journal on Control and Optimization*, 36(5), 1703–1749.

Willems, J.C. (1972). Dissipative dynamical systems part i: General theory. *Archive for rational mechanics and analysis*, 45(5), 321–351.

Xu, S. and Bao, J. (2009). Distributed control of plantwide chemical processes. *Journal of Process Control*, 19(10), 1671–1687.

