Preprints of the
9th International Symposium on Advanced Control of Chemical Processes
The International Federation of Automatic Control
June 7-10, 2015, Whistler, British Columbia, Canada

Proportional-integral extremum-seeking control

M. Guay *

* *Department of Chemical Engineering, Queen's University, Kingston, ON, Canada E-mail address:* `martin.guay@chee.queensu.ca`

Abstract: This paper proposes a proportional-integral extremum-seeking control technique. The technique is a generalization of the standard perturbation based techniques that provides fast transient performance of the closed-loop system to the optimum equilibrium of a measured objective function. The main contribution is that the formal development of this technique does not require the need for a time-scale separation. It is assumed that the equations describing the dynamics of the nonlinear system and the cost function to be minimized are unknown. The cost function and its first time derivative are assumed to be measured. The equilibrium of the unknown dynamics are assumed to be asymptotically stable and the cost function dynamics are assumed to be of relative order one. It is shown that the closed-loop ESC can also stabilize the steady-state optimum for an unknown unstable nonlinear control system. The stabilization result is quite general and provides a new approach to output feedback control of nonlinear systems.The effectiveness of the proposed approach is demonstrated using several simulation examples.

Keywords: Extremum-seeking control, Real-time optimization, Output-feedback control

1. INTRODUCTION

Extremum-seeking control (ESC) has grown to become the leading approach to solve real-time optimization problems **?**. A flurry of activity has resulted from the seminal work of Krstic and coworkers (**?**, **?**, **?**, **?**, **?**, **?**). This strikingly general and practically relevant control approach is equipped with an established and well understood control theoretical framework. Stability conditions for ESC, as highlighted in the proof of Krstic and Wang **?**, rely on two components: an averaging analysis of the persistently perturbed ESC loop and a time-scale separation of ESC closed-loop dynamics between the fast transients of the system dynamics and the slow quasi steady-state extremum-seeking task. Hence, a slower time-scale is required for the optimization dynamics to ensure to minimize its impact on the process dynamics to preserve stability. Invariably, this leads to a slow performance of the closed-loop ESC system. The objective of this study is to develop an ESC technique that does not rely on an explicit time-scale separation in ESC.

Several researchers have argued that the effect of the time-scale separation can only be minimized by incorporating some knowledge of the transient behaviour of the process dynamics. Real-time optimization techniques have been proposed for cases where such information is available. If a process model of the uncertain dynamics is known, the adaptive extremum seeking technique proposed in **?** to stabilize a nonlinear system to the unknown optimum of a known but unmeasured cost function. Nearly identical systems operating in parallel can also be used to develop so-called multi-unit extremum seeking control techniques **?**. In this technique, one proposes the simultaneous perturbation of two similar dynamical systems to generate gradient-like information. Although the concept of multi-unit ESC is interesting, the correctness of the existing convergence results as presented in **?** remains dubious and the need for the existence of an additional similar unit makes the technique rather superfluous in most applications.

This paper attempts to bridge the gap in the application of ESC for fast real-time optimization problems. We adopt a proportional-integral ESC (PIESC) design technique first proposed in **?**. This technique can be interpreted as a generalization of existing approaches where the integral action corresponds to the standard ESC control task used to identify the steady-state optimum. The proportional control action is designed to ensure that the measured cost function is optimized instantaneously. Under suitable assumption on the dynamics of the system and the cost function, this action can be shown to minimize the cost over short times while reaching the optimum steady-state conditions. The approach proposed in the current study is fundamentally different that the method initially proposed in **?**. It addresses the application of the PIESC concept in the context of perturbation based techniques. The resulting technique is quite general and can be applied for the design of fast ESC systems with stable and unstable open-loop dynamics. The perturbation based PIESC approach is shown to locally asymptotically stabilize the unknown optimum with a region of attraction whose size grows with the amplitude of the dither signal.

The paper is organized as follows. A description of the ESC problem along with the key assumptions are given in Section 2. In Section 3, the proposed ESC formulation is presented for a known cost function and process dynamics. The proposed proportional-integral ESC controller is described in Section 4. The application to unstable unknown nonlinear systems is presented in Section 5. Several simulation examples are presented in Section 6 followed by brief conclusions and proposed future work in Section **??**.

2. PROBLEM DESCRIPTION

We consider a class of nonlinear systems of the form:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x) \tag{2}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, u is the vector of input variables taking values in $\mathcal{U} \subset \mathbb{R}^p$ and $y \in \mathbb{R}$ is the variable to be minimized. It is assumed that $f(x)$ and $g(x)$ are smooth vector valued functions of x and that $h(x)$ is a smooth function of x.

The objective is to steer the system to the equilibrium x^* and u^* that achieves the minimum value of $y(= h(x^*))$. The equilibrium (or steady-state) map is the n dimensional vector $x = \pi(u)$ that solves the following equation:

$$f(\pi(u)) + g(\pi(u))u = 0.$$

The corresponding equilibrium cost function is given by:

$$y = h(\pi(u)) = \ell(u) \tag{3}$$

At equilibrium, the problem is reduced to finding the minimizer u^* of $y = \ell(u^*)$. In the following, we let $\mathcal{D}(u)$ represent a neighbourhood of the equilibrium $x = \pi(u)$.

Some additional assumptions are required concerning the cost function $h(x)$.

Assumption 1. The cost $h(x)$ is such that

(1) $\frac{\partial h(x^*)}{\partial x} = 0$
(2) $\frac{\partial^2 h(x)}{\partial x \partial x^T} > \beta I, \ \forall x \in \mathbb{R}^n$

where β is a strictly positive constant.

Note that, in contrast to standard ESC, convexity of the cost function $h(x)$ is required. We also require the following properties for the dynamics:

Assumption 2. The dynamics (1) are such that:

(1) there exists a positive constant $\alpha_e > 0$, the cost function $h(x)$ is such that:
$$\frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x)u \leq -\alpha_e \|x - \pi(u)\|^2, \ \forall x \in \mathcal{D}(u),$$
(2) the matrix valued function $g(x)$ is full rank $\forall x \in \mathcal{D}(u)$,

$\forall u \in \mathcal{U}$.

Assumption 2 states that h is non-increasing along the vector field $f(x) + g(x)u$ over some neighbourhood of the steady-state manifold $x = \pi(u)$ at a fixed value of the input u. It also states that the cost function is of relative order one in a neighbourhood of the origin. Finally, the following additional assumption concerning the steady-state cost function $\ell(u)$ is required.

Assumption 3. The equilibrium steady-state map $\ell(u)$ is such that

$$\nabla_u \ell(u)(u - u^*) \geq \alpha_u \|u - u^*\|^2$$

for some positive constant α_u $\forall u \in \mathcal{U}$.

3. EXTREMUM SEEKING CONTROLLER WITH FULL INFORMATION

In this section, we propose the extremum-seeking control approach that will form the basis of the development in later sections. Let us first consider the cost function $y = h(x)$ and compute its time derivative:

$$\dot{y} = L_f h + L_g h u \tag{4}$$

where $L_f h$ and $L_g h$ are the Lie derivatives of $h(x)$ with respect to $f(x)$ and $g(x)$, respectively. The Lie derivative is the directional derivative of the function $h(x)$ given by:

$$L_f h = \frac{\partial h}{\partial x} f, \ L_g h = \frac{\partial h}{\partial x} g.$$

By the relative order assumption, it follows that $L_g h \neq 0$ in a neighbourhood of the unknown optimum x^*.

The proposed controller is given by:

$$u = -kL_g h + \hat{u} \tag{5}$$

$$\dot{\hat{u}} = -\frac{1}{\tau_I} L_g h \tag{6}$$

where τ_I and k are positive constants. Let the optimal steady-state input be given by u^*. The error in the deviation bias is denoted by $\tilde{u} = u^* - \hat{u}$.

We first establish the convergence of the closed-loop ESC system (5), (6) and (1) with full information.

Theorem 1. Consider the nonlinear (1) with cost function (2). Let Assumptions 1, 2 and 3 hold. Then there exists a τ_I^* such that, for all $\tau_I > \tau_I^*$, the nonlinear system in closed-loop with the proportional-integral ESC controller (5), (6) converges to the equilibrium $x^* = \pi(u^*)$ that minimizes the cost function $h(x)$.

Proof: We pose the candidate Lyapunov function:

$$V = y + \frac{1}{2}\tilde{u}^T \tilde{u}.$$

Its time derivative is given by:

$$\dot{V} = L_f h - k\|L_f g\|^2 + L_g h \hat{u} - \tilde{u}\dot{\hat{u}}.$$

Let $\dot{\hat{u}} = -\frac{1}{\tau_I} L_g h$. Upon substitution of $\tilde{u} = u^* - \hat{u}$, one obtains:

$$\dot{V} = L_f h + L_g h \hat{u} - k\|L_f g\|^2 + \frac{1}{\tau_I} L_g h \tilde{u}$$

By assumption 2, it follows that:

$$\begin{aligned}\dot{V} \leq & -\alpha\|x - \pi(\hat{u})\|^2 - k\|L_g h\|^2 \\ & + \frac{1}{\tau_I}(L_g h - \frac{\partial h(\pi(\hat{u}))}{\partial x}\frac{\partial \pi(\hat{u})}{\partial u})\tilde{u} + \frac{1}{\tau_I}\frac{\partial h(\pi(\hat{u}))}{\partial x}\frac{\partial \pi(\hat{u})}{\partial u}\tilde{u}.\end{aligned}$$

At equilibrium, it is easy to deduce that $\frac{\partial \pi(\hat{u})}{\partial u} = g(\pi(\hat{u}))$. Therefore, one can write the third term of the last equation as:

$$\frac{1}{\tau_I}(L_g h - \frac{\partial h(\pi(\hat{u}}{\partial x}\frac{\partial \pi(\hat{u})}{\partial u})\tilde{u} = \frac{1}{\tau_I}(L_g h(x) - L_g h(\pi(\hat{u})))\tilde{u}$$

By smoothness of g and h, it follows that $L_g h$ is Lipschitz with constant L_G. As a result,

$$\frac{1}{\tau_I}(L_g h - \frac{\partial h(\pi(\hat{u}))}{\partial x}\frac{\partial \pi(\hat{u})}{\partial u})\tilde{u} \leq \frac{1}{\tau_I} L_G\|x - \pi(\hat{u})\|\|\tilde{u}\|.$$

This yields:

$$\dot{V} \leq -\alpha\|x-\pi(\hat{u})\|^2 - k\|L_gh\|^2 \frac{1}{\tau_I} L_G\|x-\pi(\hat{u})\|\|\tilde{u}\| + \frac{1}{\tau_I}\frac{\partial h(\pi(\hat{u}))}{\partial x}\frac{\partial \pi(\hat{u})}{\partial u}\tilde{u}.$$

By assumption 3, the last term of the last inequality can be upper bounded to yield:

$$\dot{V} \leq -\alpha\|x-\pi(\hat{u})\|^2 - k\|L_gh\|^2 + \frac{1}{\tau_I} L_G\|x-\pi(\hat{u})\|\|\tilde{u}\| - \frac{\alpha_u}{\tau_I}\|\tilde{u}\|^2$$

which can be written in matrix form as:

$$\dot{V} \leq -k\|L_gh\|^2 - [\|x-\pi(\hat{u})\|,\ \|\tilde{u}\|]\begin{bmatrix} \alpha & -\frac{L_G}{2\tau_I} \\ -\frac{L_G}{2\tau_I} & \frac{\alpha_u}{\tau_I}\end{bmatrix}\begin{bmatrix}\|x-\pi(\hat{u})\| \\ \|\tilde{u}\|\end{bmatrix}$$

The minimum eigenvalue of the matrix

$$\Lambda = \begin{bmatrix} \alpha & -\frac{L_G}{2\tau_I} \\ -\frac{L_G}{2\tau_I} & \frac{\alpha_u}{\tau_I}\end{bmatrix}$$

is positive if

$$L_G^2 - 4\tau_I\alpha_u\alpha < 0.$$

This means that:

$$\tau_I > \frac{L_G^2}{4\alpha_u\alpha}.$$

Since $g(x)$ is everywhere full rank and x^* is the unique point where $\nabla_x h(x^*) = 0$. Thus, for all $\tau_I > \frac{L_G^2}{4\tau_I\alpha}$, the system reaches the unique point where $L_gh = 0$ which occurs at the point x^* with corresponding input u^*.

This completes the proof. ■

4. PROPORTIONAL-INTEGRAL PERTURBATION BASED ESC

The PI principle is in fact very general and applicable to improve many ESC techniques. It was first applied in **?** using a time-varying parameter estimation approach. In this section, we consider the application of the PI ESC approach to the standard perturbation ESC. To do so, we consider the same assumptions as stated above. The treatment below is restricted to the single input case. The proposed PIESC closed-loop dynamical system is described by:

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ \dot{\hat{u}} &= -\frac{1}{\tau_I}\dot{y}\sin(\omega t) \\ u &= -\frac{k}{a}\dot{y}\sin(\omega t) + \hat{u} + a\sin(\omega t).\end{aligned} \tag{7}$$

For the purpose of the proof of the main result of the paper, it is assumed that $\dot{y}$ is available for measurement. Some relaxations of this assumption are discussed below.

Next, we state the main result of this paper.

Theorem 2. Consider the nonlinear closed-loop system (3) with cost function (2). Let Assumptions 1, 2 and 3 hold. Then there exists a τ_I^* such that for all $\tau_I > \tau_I^*$ the trajectories of the nonlinear system (3) converge to an $\mathcal{O}(1/\omega)$ neighbourhood of the unknown optimum equilibrium, $x^* = \pi(u^*)$. Moreover, there exists $\omega^* > 0$ such that, for any $\omega > \omega^*$, the unknown optimum is a locally asymptotically stable equilibrium of the system with a region of attraction whose size grows with the ratio $\frac{a}{k}$. Furthermore, $\|x - x^*\|$ enters an $\mathcal{O}(\frac{1}{\omega}) + \mathcal{O}(\frac{k}{\omega a}) + \mathcal{O}(\frac{a}{\omega})$ neighbourhood of the origin and $\|\hat{u}-u^*\|$ enters an $\mathcal{O}(\frac{1}{\omega}) + \mathcal{O}(\frac{1}{\omega a\tau_I}) + \mathcal{O}(\frac{a}{\tau_I\omega})$ of the origin.

Proof: (sketch)

The first step of the proof is to evaluate the derivative $\dot{y}$. It is relative straightforward to show that:

$$\begin{aligned}\dot{y} =& (1 - \left(\frac{k}{a}\right)L_gh\sin(\omega t) + \left(\frac{k}{a}\right)^2(L_gh)^2\sin(\omega t)^2 \\ &- \left(\frac{k}{a}\right)_3(L_gh)^3\sin(\omega t)^3 \\ &+ \left(\frac{k}{a}\right)^4(L_gh)^4\sin(\omega t)^4 + \ldots)(L_fh + L_gh\hat{u}) \\ &- a(-1 + \left(\frac{k}{a}\right)L_gh\sin(\omega t) - \left(\frac{k}{a}\right)^2(L_gh)^2\sin(\omega t)^2 \\ &+ \left(\frac{k}{a}\right)^3(L_gh)^3\sin(\omega t)^3 \\ &- \left(\frac{k}{a}\right)^4(L_gh)^4\sin(\omega t)^4 + \ldots)(L_gh)\sin(\omega t)\end{aligned}$$

Correspondingly, the integral action term $\dot{\hat{u}}$ is given by:

$$\begin{aligned}\tau_I\dot{\hat{u}} =& \frac{1}{a}((-\sin(\omega t) + \left(\frac{k}{a}\right)L_gh\sin(\omega t)^2 \\ &- \left(\frac{k}{a}\right)^2(L_gh)^2\sin(\omega t)^3 + \left(\frac{k}{a}\right)^3(L_gh)^3\sin(\omega t)^4 \\ &- \left(\frac{k}{a}\right)^4(L_gh)^4\sin(\omega t)^5 + \ldots)(L_fh + L_gh\hat{u}) \\ &- a(\sin(\omega t) - \left(\frac{k}{a}\right)L_gh\sin(\omega t)^2 \\ &+ \left(\frac{k}{a}\right)^2(L_gh)^2\sin(\omega t)^3 - \left(\frac{k}{a}\right)^3(L_gh)^3\sin(\omega t)^4 \\ &+ \left(\frac{k}{a}\right)^4(L_gh)^4\sin(\omega t)^5 + \ldots)L_gh\sin(\omega t)) \\ =& F(x, \hat{u}, t).\end{aligned}$$

We then compute the average system for $\hat{u}$ as

$$\dot{\hat{u}}_{av} = \frac{\omega}{2\pi}\int_0^{2\pi} F(x, \hat{u}, \tau)d\tau = F_{av}(x_{av}, \hat{u}_{av}).$$

Evaluating the integral, the average system is given by:

$$\begin{aligned}\tau_I\dot{\hat{u}}_{av} =& \frac{1}{a}((M_2\left(\frac{k}{a}\right)L_gh + M_4\left(\frac{k}{a}\right)^3(L_gh)^3 \\ &+ \ldots)(L_fh + L_gh\hat{u}_{av}) - a(M_2 + M_4\left(\frac{k}{a}\right)^2(L_gh)^2 \\ &+ M_6\left(\frac{k}{a}\right)^4(L_gh)^4 + \ldots)L_gh)\end{aligned}$$

where the positive constants $M_i = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sin^i(\omega\sigma)d\sigma$ for $i = 2, 4, \ldots$. The sequence of the numbers M_i can be evaluated as: $\frac{1}{2}$, $\frac{3}{8}$, $\frac{5}{16}$, $\frac{35}{128}$...

This sequence has the following interesting property. The ratio of the each subsequent terms yields the following sequence:

$$\frac{M_2}{1} = \frac{1}{2}, \frac{M_4}{M_2} = \frac{3}{4}, \frac{M_6}{M_4} = \frac{5}{6}, \frac{M_8}{M_6} = \frac{7}{8}, \ldots$$

and thus $\frac{M_{2(i+1)}}{M_{2i}} \leq 1$ for all $i \in \mathbb{N}$. Finally, it can be shown that the sequence is the expansion of the following expression:

$$\frac{1}{\sqrt{1-x^2}} = M_2 + M_4x^2 + M_6x^4 + M_8x^6 + \ldots$$

which exists for all $|x| < 1$.

Now since $u = k\dot{\hat{u}}\tau_I + \hat{u} + a\sin(\omega t)$, it follows that we can write the average value:

$$\begin{aligned}u_{av} = &\left(\frac{k}{a}\right)^2(M_2 + M_4\left(\frac{k}{a}\right)^2(L_gh)^2\\ &+ \ldots)(L_fh + L_gh\hat{u}_{av})L_gh - k(M_2 + M_4\left(\frac{k}{a}\right)^2(L_gh)^2\\ &+ M_6\left(\frac{k}{a}\right)^4(L_gh)^4 + \ldots)L_gh + \hat{u}_{av}\end{aligned}$$

The overall average system takes the form:

$$\begin{aligned}\dot{x}_{av} =& f(x_{av}) + g(x_{av})u_{av}\\ \tau_I\dot{\hat{u}}_{av} =& \frac{1}{a}\alpha(L_fh + L_gh\hat{u}_{av})kL_gh - \alpha L_gh\end{aligned}$$

where

$$\alpha = (M_2 + M_4\left(\frac{k}{a}\right)^2(L_gh)^2 + M_6\left(\frac{k}{a}\right)^4(L_gh)^4 + \ldots).$$

Next, we pose the following Lyapunov function candidate for the averaged system:

$$V = h(x_{av}) + \frac{1}{2}\tilde{u}_{av}^T\tilde{u}_{av}. \quad (8)$$

where $\tilde{u}_{av} = u^* - \hat{u}_{av}$.

Proceeding as the proof of Theorem 1, we write:

$$\begin{aligned}\dot{V} =& (L_fh + L_gh\hat{u}_{av}) - \alpha k(L_gh)^2\\ &+ \alpha(\left(\frac{k}{a}\right)^2(L_gh)^2 - \frac{1}{a\tau_I}\tilde{u}_{av}^TL_gh)(L_fh + L_gh\hat{u}_{av})\\ &+ \tilde{u}_{av}^T\frac{\alpha}{\tau_I}(\frac{\partial h(\pi(\hat{u}_{av}))}{\partial x}\frac{\partial\pi(\hat{u}_{av})}{\partial u})\\ &+ \tilde{u}_{av}^T\frac{\alpha}{\tau_I}(L_gh - \frac{\partial h(\pi(\hat{u}_{av}))}{\partial x}\frac{\partial\pi(\hat{u}_{av})}{\partial u})\end{aligned}$$

By smoothness of the f, g and h, there exists a constant L_F such that: $\|L_fh + L_gh\hat{u}_{av}\| \leq L_F\|x_{av} - \pi(\hat{u}_{av})\|$. By Assumptions 2 and 3, one obtains:

$$\begin{aligned}\dot{V} \leq& -\alpha_e\|x_{av} - \pi(\hat{u}_{av})\|^2\\ &- \alpha_e\alpha\left(\frac{k}{a}\right)^2(L_gh)^2\|x_{av} - \pi(\hat{u}_{av})\|^2\\ &+ \frac{\alpha L_F}{a\tau_I}\|\tilde{u}_{av}\|\|L_gh\|\|x_{av} - \pi(\hat{u}_{av})\| - \alpha k(L_gh)^2\\ &- \frac{\alpha_u\alpha}{\tau_I}\|\tilde{u}_{av}\|^2 + \tilde{u}_{av}^T\frac{\alpha}{\tau_I}(L_gh - \frac{\partial h(\pi(\hat{u}_{av}))}{\partial x}\frac{\partial\pi(\hat{u}_{av})}{\partial u})\end{aligned}$$

Proceeding as in the proof of Theorem 1, the last term can be upper bounded to obtain:

$$\begin{aligned}\dot{V} \leq& -\alpha_e\|x_{av} - \pi(\hat{u}_{av})\|^2\\ &- \alpha_e\alpha\left(\frac{k}{a}\right)^2(L_gh)^2\|x_{av} - \pi(\hat{u}_{av})\|^2\\ &+ \frac{\alpha L_F}{a\tau_I}\|\tilde{u}\|\|L_gh\|\|x_{av} - \pi(\hat{u}_{av})\| - \alpha k(L_gh)^2 \quad (9)\\ &- \frac{\alpha_u\alpha}{\tau_I}\|\tilde{u}_{av}\|^2 + \frac{\alpha L_G}{\tau_I}\|\tilde{u}_{av}\|\|x_{av} - \pi(\hat{u}_{av})\|.\end{aligned}$$

It can then be shown that one must pick τ_I to be such that:

$$\tau_I > \max\left[\frac{L_G^2}{4\alpha_u\alpha_e}, \frac{L_G^2}{k^2\alpha_u\alpha_e}, \frac{L_F^2}{\alpha_e\alpha_uk^2}\right].$$

The next step is to establish the stability of the unknown optimum equilibrium $x^* = \pi(u^*)$. We must confirm the boundedness of α. It is relatively easy to show that:

$$\alpha = \frac{1}{\sqrt{1 - \left(\frac{k}{a}\right)^2 L_gh^2}}.$$

As a result, one must choose k and a such that $\left(\frac{k}{a}\right)^2 L_gh^2 < 1$. By smoothness, there exists a positive constant L_γ such that:

$$\|L_gh\| \leq L_\gamma\|x - \pi(u^*)\|.$$

Let us consider the set

$$\mathcal{M} = \left\{x, \hat{u} \,|\, \|x - \pi(u^*)\| \leq \frac{a}{k}\frac{1}{L_\gamma}, \|\hat{u} - u^*\| \leq \beta_u\right\}$$

for some constant β_u. If follows that α is well defined for every x in the interior of $\mathcal{M}$ since $\left(\frac{k}{a}\right)\|L_gh\| < 1$.

Let $r = \min_{(x,\hat{u})\in\partial\mathcal{M}}(V - h(x^*))$ where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$ and define

$$\Omega_r(a/k) = \{x \in \mathbb{R}^n\ \hat{u} \in \mathbb{R}^p \mid (V - h^*) < r\}.$$

Since α is well defined on the set $\Omega_r(a/k)$, it follows that $\dot{V}$ is bounded and negative definite for any x_{av} and $\hat{u}_{av}$ in Ω_r. As a result, every initial conditions $x_{av}(0) = x(0)$ and $\hat{u}_{av}(0) = \hat{u}$ starting in $\Omega_r(a/k)$ converges to the unknown optimum equilibrium which asymptotically stable over $\Omega_r(a/k)$. Finally, note that the size of the $\mathcal{M}$ is directly proportional to the ratio $\frac{a}{k}$. As a result, we achieve local asymptotic stability of the unknown optimum equilibrium over $\Omega_r(a/k)$.

Next we apply averaging results. Once can write the averaging system as:

$$\begin{aligned}\dot{x}_{av} =& f(x_{av}) + g(x_{av})u_{av}\\ \tau_I\dot{\hat{u}}_{av} =& \frac{1}{a}\alpha(L_fh + L_gh\hat{u}_{av})kL_gh - \alpha L_gh\end{aligned}$$

with control

$$u_{av} = \left(\frac{k}{a}\right)^2\alpha(L_fh + L_gh\hat{u}_{av})L_gh - k\alpha L_gh + \hat{u}_{av}$$

or,

$$\dot{\xi}_{av} = F_{1av}(\xi_{av})$$

where $\xi_{av} = [x_{av}^T, \hat{u}_{av}^T]^T$. By the development above, we established that the unknown optimum equilibrium is an asymptotically stable equilibrium of the average system. The nominal system can be written in the following perturbation form:

$$\dot{\xi} = F_1(t, \xi) + F_2(t, \xi, \epsilon)$$

where $\epsilon > 0$ is a perturbation parameter and $\xi = [x^T, \hat{u}^T]^T$. Using standard averaging results , it can be shown that:

$$\|\xi(\tau) - \xi_{av}(\tau)\| \leq \|\zeta(\tau) - \xi_{av}(\tau)\| + \frac{M}{\omega L_{F_1}} e^{L_{F_1}\tau} - \frac{M}{\omega L_{F_1}}.$$

Since the approximation holds for all τ, we get that the nominal system enters an $\mathcal{O}(M)$ neighbourhood of the unknown equilibrium.

Using the results of **?**, **?** and **?**, it follows, from the asymptotically stability of the averaged system, that there exists $\omega^* > 0$ such that, for any $\omega > \omega^*$, the unknown optimum is locally asymptotically stable equilibrium of the nominal system for initial conditions in $\bar{\Omega}_{a/k}$.

From the bounds above, we can easily evaluate $\mathcal{O}(M)$ in terms of the parameters of the controller. First, we note that the term $\frac{1}{\sqrt{1-\gamma^2}}$ is order $\mathcal{O}(1)$. From the bound on F_{21}, one obtains that $\|x - x^*\|$ enters an $\mathcal{O}(\frac{1}{\omega}) + \mathcal{O}(\frac{k}{\omega a}) + \mathcal{O}(\frac{a}{\omega})$ neighbourhood of the origin. While, from F_{22}, it follows that $\|\tilde{\hat{u}}\|$ enters an $\mathcal{O}(\frac{1}{\omega}) + \mathcal{O}(\frac{1}{\omega a \tau_I}) + \mathcal{O}(\frac{a}{\tau_I \omega})$. This completes the proof. ■

Remark 1. It is assumed that the derivative of the cost function $\dot{y}$. In general, this may prove somewhat difficult. As a result, one can considers the system

$$\dot{v} = -\omega_h v + y, \; w = -\omega_h^2 v + \omega_h y$$

where w is an estimate of $\dot{y}$ and $\omega_h >> \omega$ is a positive tuning parameter taken to be larger than the dither frequency of the proportional-integral extremum seeking controller which takes the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ \dot{v} &= -\omega_h v + y \\ \dot{\hat{u}} &= -\frac{1}{\tau_I}(-\omega_h^2 v + \omega_h y)\sin(\omega t) \\ u &= -\frac{k}{a}(-\omega_h^2 v + \omega_h y)\sin(\omega t) + \hat{u} + a\sin(\omega t). \end{aligned} \tag{10}$$

The properties of the ESC system (10) will recover the properties of the original ESC system analyzed in Theorem 2 as ω_h increases. We also remark that if this filter is used then the initial condition for the dynamics of v should be taken as $v(0) = y/\omega_h$ to avoid a sudden jump arising for the direct feedthrough term in w.

5. APPLICATION TO UNSTABLE SYSTEMS

In this section, we consider the following relaxation of Assumption 2.

Assumption 4. The dynamics (1) are such that:

(1) there exists a positive constant $\alpha_e > 0$ and a gain k^* such that the cost function $h(x)$ fulfills the following inequality:
$$L_f h + L_g h u - k^* \|L_g h\|^2 \leq -\alpha_e \|x - \pi(u)\|^2,$$
$\forall x \in \mathcal{D}(u)$,
(2) the matrix valued function $g(x)$ is full rank $\forall x \in \mathcal{D}(u)$,

$\forall u \in \mathcal{U}$.

Assumption 4 indicates the class of control systems such that the closed-loop system is stabilized by the Jurdjevic-Quinn damping feedback $u = -kL_g h$. The result state next is a slight generalization of Theorem 2.

Theorem 3. Consider the nonlinear closed-loop system (3) with cost function (2). Let Assumptions 1, 3 and 4 hold. Then there exists a τ_I^* such that for all $\tau_I > \tau_I^*$ the trajectories of the nonlinear system (3) converge to an $\mathcal{O}(1/\omega)$ neighbourhood of the unknown optimum equilibrium, $x^* = \pi(u^*)$. Moreover, there exist $\omega^* > 0$ and $k^* > 0$ (from Assumption 4) such that, for any $\omega > \omega^*$ and $k > 2k^*$, the unknown optimum is a locally asymptotically stable equilibrium of the system with a region of attraction whose size grows with the ratio $\frac{a}{k}$. Furthermore, $\|x - x^*\|$ enters an $\mathcal{O}(\frac{1}{\omega}) + \mathcal{O}(\frac{k}{\omega a}) + \mathcal{O}(\frac{a}{\omega})$ neighbourhood of the origin and $\|\hat{u} - u^*\|$ enters an $\mathcal{O}(\frac{1}{\omega}) + \mathcal{O}(\frac{1}{\omega a \tau_I}) + \mathcal{O}(\frac{a}{\tau_I \omega})$ of the origin.

6. SIMULATION EXAMPLE

6.1 Example 1

We consider the following unknown dynamical system

$$\dot{x}_1 = -x_1 + u$$

with the following cost function: $y = 1 + 4(x_1 - 1.2)^2$. By linearity of the stable dynamics and the convexity of the cost function, it follows that all assumptions are met. For the controller, we consider the following tuning parameters: $a = 5$, $\omega = 100$, $k = 0.5$, $\tau_I = 1$. The initial conditions are $x_1(0) = \hat{u}(0) = 0$. The simulation results are shown in Figures **??**. The figure shows the cost function value y, the integrator state $\hat{u}$ and the state variable x for the closed-loop system. The ESC performs very remarkably well and reaches the unknown optimum faster than the open-loop time scale of the system. In the second simulation, we consider a frequency $\omega = 1000$. Results are shown in Figure **??**. The impact of the increase in dither frequency is clearly shown in a reduction of the amplitude of the oscillation in the state variable, x_1.

6.2 Example 2

In this section, we consider the following dynamical system taken from **?**:

$$\dot{x}_1 = x_1^2 + x_2 + u, \; \dot{x}_2 = -x_2 + x_1^2$$

The cost function to be minimized is given by: $y = 1 - x_1 + x_1^2$. Following the analysis in **?**, it follows that the system fulfils the assumptions of the proposed approach.

The optimum occurs at $u^* = -0.5$, $x_1^* = 0.5$, $x_2^* = 0.255$ where $y^* = -1.25$. The tuning parameters are chosen as: $k = 10$, $\tau_I = 0.1$, $a = 10$ and $\omega = 100$. The simulation results are shown in Figure **??**. The proposed PIESC stabilizes the unstable nonlinear system to the unknown optimum equilibrium of the cost function y. The transient performance is surprisingly good given that the equilibrium of this nonlinear system is highly unstable.

7. CONCLUSION

A perturbation-based proportional-integral ESC technique is proposed. The proposed technique yields a closed-loop system with a local asymptotic stable equilibrium at the unknown optimum equilibrium conditions. The unknown optimum has a region of attraction whose size is proportional to the amplitude of the dither signal for a fixed optimization gain. It is also shown that the closed-loop ESC can stabilize the equilibrium optimum for an unknown unstable nonlinear control system.

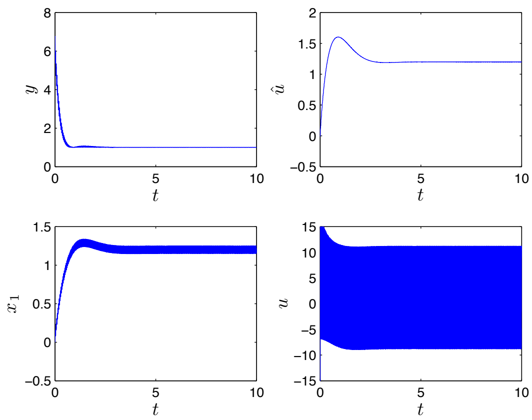

Fig. 1. Performance of the PI-ESC for Example 1 with $\omega = 100$. The upper row shows the cost function and $\hat{u}$. The state variable x and the input variables u are shown in the bottom row.

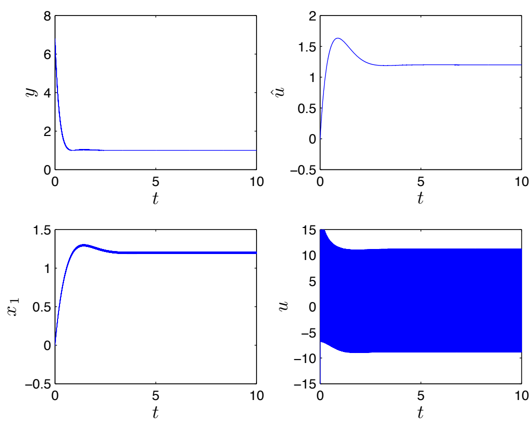

Fig. 2. Performance of the PI-ESC for Example 1 with $\omega = 1000$. The upper row shows the cost function and $\hat{u}$. The state variable x and the input variables u are shown in the bottom row.

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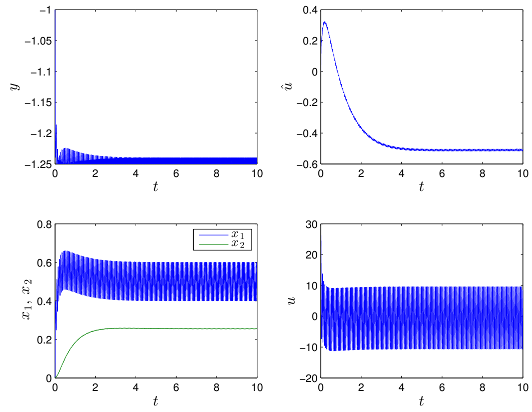

Fig. 3. Performance of the PI-ESC for Example 2. The upper row shows the cost function and $\hat{u}$. The state variables x_1, x_2 and the input variables u are shown in the bottom row.

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