# Fractional-order process simulator based on exact step response discretization $^{\star}$

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**Abstract:** In the last decades, there is a boom of fractional calculus (FC) applications in many technical areas including process control. The multiple fractional-order pole models (MFPM) were proven as suitable for modeling processes that are essentially monotone from their physical nature (holds true for a big percentage of chemical processes). Due to implementation aspects, the time-domain real-time simulation of MFPM is a challenging problem. For this purpose, often a high order zero/pole transfer function is employed. This paper describes a novel multiple fractional pole simulator based on exact step response discretization. The proposed technique decreases computational burden and the resulting simulator can be deployed even to real-time embedded devices. The advantages of SW tools developed are demonstrated on various examples including predictive control feedback loop simulation.

Keywords: Fractional calculus, multiple fractional-order pole model, step response, discretization, Simulink function block

## 1. INTRODUCTION

Fractional Calculus is an interdisciplinary and emerging research area (Ortigueira (2008); Elwakil (2010)). In the last years, a boom of fractional calculus (FC) applications started in many technical areas including process control. It was studied from both controller synthesis (Podlubny (1999); Monje et al. (2008); Cech and Schlegel (2013)) and system identification side (Schlegel and Cech (2005); Charef et al. (1992)). The scientific effort resulted into number of practical outputs documented e.g. in Pommiere et al. (2002); Sabatier et al. (2007). The generalization of integrals and derivatives to arbitrary real order (FO - Fractional Order) leads to more flexible transfer functions P(s) with non-integer power of complex variable s. The multiple-fractional pole model (MFPM) proposed in Charef et al. (1992); Schlegel and Čech (2005); Schlegel et al. (2014) was proven suitable for modeling essentially monotone industrial processes with distributed parameters. Such model provides frequency response flexibility superior to traditional integer-order models. However, time domain simulation or even real-time implementation is a challenging problem. Commonly, FO element is approximated by a continuous high-order filter as in Charef et al. (1992); Cech and Schlegel (2011). It can be shown, that the 'ideal' realization always leads to infinite order filter,

see Oldham and J.Spanier (2002); Monje et al. (2010). Hence, several limitations must be taken into account and we will speak further only about approximation. Traditionally, the quality of approximation is measured by the integral quadratic distance between frequency response of ideal FO element and its corresponding integer-order (IO) equivalent. Charef's and Oustaloup's methods together with their modifications are typical representatives of such approach (Charef et al. (1992); Oustaloup et al. (1996); Monje et al. (2010)). They approximate the fractional elements by classical transfer function with zeros and poles equidistantly spread in the logarithmic space. Unfortunately, the quality obtained is not sufficient namely for approximating filters with low order. Often also the methods based on continued fraction expansion (CFE) are used (Vinagre et al. (2000)). All approximations exhibit good performance only on certain frequency band. Other imperfections appear when the filter is discretized. For general fractional differential equation the methods can be divided into two groups called direct and indirect discretization (see e.g. Djouambi et al. (2013) for more references).

An alternative approach to MFPM implementation is presented in this paper. It is based on an exact step response discretization. The main idea is the numerical integration of an exact analytic expression of the multiple FO pole impulse response. The resulting step response is then sampled with a given period T and a corresponding discrete impulse response is obtained. Finally, the impulse response is used to perform convolution to get the system output. Point out that fractional systems

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Fig. 1. Amplitude frequency response of a real temperature process which exhibits fractional slope on certain frequency band.



Fig. 2. Simple laboratory temperature process with an electric heater and cooler.

are linear, so all fundamental concepts of linear control theory hold true. The final simulator is implemented into an individual MATLAB/Simulink compatible s-function block. The simulator can be automatically deployed to various real-time platforms. The results are analyzed and compared to other techniques of MFPM implementation in both time and frequency domain. It is believed that the presented approach will allow to create advanced realtime simulators of fractional processes or eventually apply advanced predictive control strategies. Up to the authors' knowledge, such technique for MFPM real-time simulation was not described before.

The rest of the paper is organized as follows: The basic definitions and preliminaries are stated in Section 2. The development of multiple fractional pole simulator is described in Section 3. Practical examples of real temperature process and predictive control simulation are documented in Section 4. Conclusions and ideas for future work can be found in Section 5.

## 2. MULTIPLE FRACTIONAL POLE MODEL – MFPM

It was shown in Charef et al. (1992) that to cover the majority of physically monotone processes, one has *a priori* to consider the transfer function in the form

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$$P(s) = \frac{K}{\prod_{i=1}^{p} (\tau_i s + 1)^{n_i}},$$
(1)

where p is arbitrary integer number and  $K, \tau_i, n_i \ i = 1, 2, \ldots, p$  are positive real numbers.

Remark 1. If all  $n_i$ , i = 1, 2, ..., p are integer numbers, one obtains a classical integer-order transfer function in a Bode's form.

Remark 2. If  $p \to \infty$  then the set of all transfer functions (1) contains also processes with dead time and approximates several processes with transcendent transfer functions (like heat transfer), see e.g. Åström and Hägglund (2006).

Note, that by fractional poles one can reach arbitrary slope of asymptotic amplitude frequency response. For instance, the frequency response measured on a real temperature process depicted in Fig. 2 can be approximated by a fractional pole with order equal to 26dB/dec magnitude loss, see Fig. 1.

Also a set of ultimate process models gained from simple relay or pulse experiment is covered by MFPM in the form (1) (see Schlegel and Čech (2005); Schlegel and Čech (2014) for details). Hence the motivation for the work described below is obvious.

## 3. MULTIPLE FRACTIONAL POLE SIMULATOR

Remind that the output y(t) of any causal linear system can be defined as a convolution of the input signal u(t)and the impulse function h(t)

$$y(t) = \int_{0}^{\infty} h(\tau)u(t-\tau)\mathrm{d}\tau.$$
 (2)

The system is uniquely determined by the impulse function h(t) and the output y(t) can be computed for arbitrary input signal u(t).

Therefore, the aim in the rest of work is to compute discrete impulse response of MFPM discretized with given sampling period T.

#### 3.1 Single fractional pole impulse response

Firstly, let us consider a single fractional pole

$$P(s) = \frac{1}{(\tau s + 1)^m}.$$
 (3)

For m being integer number, it is very easy to evaluate an impulse response (using well-known Laplace transform relations) as

$$h(t) = \frac{t^{(m-1)}}{\tau^m (m-1)!} e^{-t/\tau}.$$
(4)

For non-integer m, the relation (4) can be generalized to

$$h_{\rm FP}(t) = \frac{t^{m-1}}{\tau^m \Gamma(m)} e^{-t/\tau}, \qquad (5)$$

where  $\Gamma(m)$  is a gamma function serving as a generalization of factorial function for any real m, see Fig. 3. The impulse response examples are shown in Fig. 4. Corresponding step responses are shown in Fig. 5.



Fig. 3. Gamma function – generalization of factorial.



Fig. 4. Computed impulse responses of a single fractional pole depending on its order m.

*Remark 3.* The numerical evaluation of gamma function is available in many advanced computing packages like Matlab<sup>TM</sup>, Maple<sup>TM</sup>, Mathematica<sup>TM</sup>, etc.

#### 3.2 Multiple fractional pole impulse response

In case of general MFPM in the form (1), let use denote  $h_i(t)$  the impulse response belonging to the *i*-th fractional pole. Then the impulse response of the system (1) can be computed, according to well known Laplace transform relations, as

$$h_{\rm MFP}(t) = K\left(\left(h_1(t) * h_2(t)\right) * \cdots * h_p(t)\right), \qquad (6)$$
  
where (\*) denotes convolution.

*Remark* 4. For evaluating expression (6), the numerical integration with Simpsons rule was used to improve the precision of gained impulse response.

#### 3.3 Computing MFPM step response

The step response g(t) of a multiple fractional order pole model can be computed by integrating impulse response as

$$g_{\rm MFP}(t) = \int_{0}^{t} h_{\rm MFP}(\tau) \mathrm{d}\tau.$$
(7)

An example of computed step responses of single fractional pole is shown in Fig 5.

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Fig. 5. Computed step responses of a single fractional pole depending on its order m.

*Remark 5.* The the impulse response (6) and step response (7) can be evaluated off-line with arbitrary precision available in professional RTD SW mentioned earlier. For numerical integration, the Simpsons rule was used.

Remark 6. The final time  $t_F$  for enumeration of impulse and step functions is chosen in such a way that the impulse response is sufficiently close to zero at the time  $t_F$  or equivalently the relative change of step response is sufficiently small (i.e. steady state is reached).

#### 3.4 MFPM step response discretization

Consider, that for real time simulation or predictive control one needs the simulator running with sampling time T. The final discrete impulse response  $h^d(t)$  and discrete step response  $g^d(t)$  is obtained by discretization of precomputed step response  $g_{MFP}(t)$  as

$$h_{\rm MFP}^{d}(kT) = g_{\rm MFP}(kT), k = 0$$
  
=  $g_{\rm MFP}(kT) - g_{\rm MFP}((k-1)T), k > 0$  (8)  
 $g_{\rm MFP}^{d}(kT) = g_{\rm MFP}(kT).$  (9)

*Remark 7.* The presented approach ensures the correct process static gain in the discrete simulator. This is the key advantage compared to the situation when the discrete impulse response is obtained by the direct discretization of the continuous one.

To summarize the Section, the main research result is a set of SW tools that allow to proceed with the above mentioned steps and to compute the final functions (8), (9). The SW structure is briefly outlined also in Fig. 13.

#### 4. ILLUSTRATIVE EXAMPLES

In this Section, the utilization of developed tools will be shown on four examples starting from the single fractional pole and finishing by the advanced predictive control simulation.

#### 4.1 Example 1: Single fractional pole

Consider the process which can be described by the single fractional pole

$$P(s) = \frac{1}{(5s+1)^{2.7}}.$$
(10)



Fig. 6. Example 1: Step response of single fractional pole; comparison with integer orders m = 2 and m = 3.

The discrete impulse response was obtained by applying the whole procedure described in Section 3. Results of step response simulation are depicted in Fig. 6. One can check that the resulted step response lies between two step responses obtained for integer orders m = 2 and m = 3which is intuitively correct results.

#### 4.2 Example 2: Heat transfer in the metal rod

Consider a transcendent transfer function

$$P(s) = \frac{1}{\cosh(\sqrt{s})} \tag{11}$$

describing the heat transfer in the metal rod (Åström and Hägglund (2006)). For time domain simulation or linear controller design, the transfer function (11) may be approximated on a certain frequency band by MFPM model

$$P(s) = \frac{1}{(0.017s + 1)^{4.86}(0.386s + 1)^{1.07}}.$$
 (12)

Assume that model (12) is further used for controller design. The controller will run on real plant with sampling rate T = 0.1[s]. Obviously, one needs a correct simulator of model (12) to evaluate the controller tuning settings before its installation into real plant.

*Remark 8.* Note, that the controller design for MFPM model can be carried out in frequency domain using standard linear theory (e.q. Nyquist plot shaping, sensitivity function shaping).

Using the tools developed, the discrete simulator with required sampling rate was obtained. It was tested in both time and frequency domain. In Figures 7, 8 and 9 one can check its impulse, step and frequency responses, respectively. All responses are compared to

• responses obtained by continuous filter which approximates optimally MFPM model on 4-decade frequency band. The optimality criterion is a quadratic difference between true and approximated frequency response on a respective frequency band (see Čech and Schlegel (2011) for details). The filter is finally discretized for required sampling period and is further called as 'optimal filter',



Fig. 7. Example 2: Impulse response of a discrete simulator, comparison with optimal continuous approximation filter and true impulse response



- Fig. 8. Example 2: Step response of a discrete simulator, comparison with optimal continuous approximation filter and true step response
  - true impulse and step responses computed by (6) and (7); true frequency response obtained by direct substitution  $s = j\omega$  into the MFPM (12).

#### 4.3 Example 3: Real temperature process

On real temperature process shown in Fig. 2, an identification experiment was made which resulted into the MFPM

$$P(s) = \frac{102}{(14.2s+1)(399.5s+1)^{1.1}}.$$
 (13)

Similarly to previous example, the aim is to obtain a discrete simulator with a long sampling rate T=50[s]. In Figures 10, 11 and 12 one can check the impulse, step and frequency responses, respectively.

In both examples, the proposed simulator reaches the same precision as the high order filter which approximates MFPM model. Simultaneously, one can benefit from easy and numerically more robust implementation on embedded devices.

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Fig. 9. Example 2: Frequency response of a discrete simulator, comparison with optimal continuous approximation filter and true frequency response



Fig. 10. Example 3: Impulse response of a discrete simulator, comparison with optimal continuous approximation filter and true impulse response



Fig. 11. Example 3: Step response of a discrete simulator, comparison with optimal continuous approximation filter and true step response



Fig. 12. Example 3: Frequency response of a discrete simulator, comparison with optimal continuous approximation filter and true frequency response

## 4.4 Example 4: Predictive control of fractional system

Finally, let us show how the tools may be useful for predictive control strategies. Assume that one wants to design predictive control strategy for a temperature process (13). Due to actuator constraints (saturation limits), classical PID control works not satisfactorily as it runs not in linear mode. This drawback cannot be solved by more advanced linear controller (e.g. fractional PID). Recently, a real-time function block was developed for such purposes (Schlegel and Sobota (2008)). It takes saturation limits into account. The key block parameter is a discrete step response which is used internally for computing process prediction. The vector of step response values can be gained from precise step response (7) which is obtained by integration of computed impulse response (6).

Thus the tools developed may be used for complete (even real-time) simulation of predictive control closed loop on both process model and controller side. The results of predictive control time domain simulation are shown in Fig. 14. Compared to open loop, the settling time has been significantly shortened as the controller limits are properly utilized in predictive control strategy.

## 5. CONCLUSION

The new method for creating reliable real-time simulators of multiple fractional-order pole models was presented in this paper. It is based on exact step response discretization. Comparing to other approaches, the proposed technique decreases computational burden and the resulting simulator can be deployed even to real-time embedded devices. It was shown, that the results can be used for simulation purposes or predictive control of distributed parameter processes. In the future, the results will be included into free virtual labs available at www.contlab.eu and www.pidlab.com.

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Fig. 13. Example 4: Overall SW architecture and its application in predictive control simulation



- Fig. 14. Example 4: Predictive control of fractional process – closed loop simulation; sp – setpoint, CL – closed loop response, OL – open loop response, mv – controller output
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