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Trajectory Bounds of Input-to-State Stability for Nonlinear Model Predictive Control ⋆

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Abstract: Model predictive control (MPC) is an optimization-based tool that is widely used in the chemical industry, and nonlinear MPC (NMPC) expands the technology to handle more detailed models that are accurate across a wider range of state values. Many works in the literature have studied NMPC using Input-to-State Stability (ISS). The purpose of this work is to provide a method for calculating state trajectory bounds for NMPC using ISS theory. These predictive bounds are derived in terms of parameters that may be found from a series of open loop calculations in the general nonlinear case. Results are shown for a scalar linear system and a nonlinear CSTR, and the challenges involved with higher dimensional problems are discussed.

Keywords: optimal control, nonlinear programming, model-based control, robust stability, multivariable feedback control, predictive control

1. INTRODUCTION

Model predictive control (MPC) has seen a variety of applications in chemical processes, and its advantages include a natural way of handling inequality constraints and multiple-input-multiple-output systems due to the optimization formulation of the problem. A survey of industrial uses of MPC is given in Qin and Bagdwell (2003), and a thorough treatment of MPC is given in Rawlings and Mayne (2009). Nonlinear model predictive control (NMPC) has the added advantage of being able to use a detailed first-principles dynamic model in order to provide accuracy across a wide range of states; a good introduction to NMPC is given in Grüne and Pannek (2011). Furthermore, if a sensible initialization strategy is used, an exact solution to the nonlinear programming (NLP) problem is not required, as shown in Pannocchia et al. (2011). Even so, NMPC is still not as common, partially due to the difficulty of solving large nonlinear models online. However, recent work in advanced-step NMPC allows for control regardless of model solution time, as developed in Yang and Biegler (2013) and Zavala and Biegler (2009).

The idea of input-to-state stability (ISS) is used to extend stability analysis to systems with uncertainty. The property was originally described for continuous time (CT) systems in Sontag (1989) and was extended to discrete time (DT) systems in Jiang and Wang (2001). Furthermore, ISS has been proposed as a framework for NMPC (Limon et al., 2009), and it provides a very convenient and natural way of thinking about robust stability. The goal of this work is to explicitly calculate the values of ISS bounds for NMPC.

The chief difficulty for this problem lies in finding rigorous bounds that are small enough to be useful. To aid in this, the ISS theorem is extended to allow for an uncertain term that depends on the state of the system as well as the realization of uncertainty. Also, general forms of the comparison functions are proposed to be used for the case of NMPC. Then, these comparison functions are used to formulate predictive ISS bounds through the ISS Lyapunov theorem for DT systems. A method of finding the parameters of the comparison functions is described, and computational examples include a scalar linear system and a nonlinear CSTR.

2. NMPC FORMULATION

Here we describe terminal constrained NMPC applied to the discrete time system $x_{k+1} = f(x_k, u_k, w_k)$, where the variable subscript k denotes values at discrete time t_k. The vector x_k contains the state values at time t_k, the vector u_k contains the control values at time t_k, and the vector w_k contains the disturbance values at time t_k. Assume that the desired setpoint is $x_{ss} = 0$, $u_{ss} = 0$, with $x_{ss} = f(x_{ss}, u_{ss}, 0)$. The stage cost will be $l(x_k, u_k) = x_k^T Q x_k + u_k^T R u_k$, where Q and R are positive definite matrices. This gives the following optimal control problem at time t_k:

$$min \;\; V_N = \sum_{i=0}^{N-1} l(z_i, v_i) + \phi(z_N) \tag{1a}$$

$$s.t. \;\; z_{i+1} = f(z_i, v_i, 0), z_0 = x_k, \forall\, i = 0 \dots N-1 \tag{1b}$$

$$\forall\, z_N \in \mathbb{X}_f, z_i \in \mathbb{X}, \forall\, i = 0 \dots N-1 \tag{1c}$$

where N is the control horizon, z_i is the predicted state vector at time t_{k+i}, v_k is the predicted control vector at time t_{k+i}, $\mathbb{X}$ is the feasible region of z_i which we represent

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with lower and upper bounds, $\mathbb{X}_f$ is the terminal region and $\phi(z_N)$ is the terminal cost. This problem is solved at time t_k, and then v_0 is injected to the plant, so that $u_k = v_0$. This process is repeated at each time t_k, so that the horizon shifts forward one time step for each problem solved.

Note that the above formulation contains both terminal and state constraints, and there are examples (Grimm et al., 2004) where these constraints can lead to zero robustness in the NMPC controller. This issue may be rectified by replacing the state constraints by penalty terms folded into the stage costs and replacing the terminal region with a penalty term folded into the terminal cost. Moreover, if the nominal formulation shown above is feasible, as in the examples considered later in this paper, then it is equivalent to a formulation without $\mathbb{X}_f$ and $\mathbb{X}$, but with penalty terms that have sufficiently large weights (see Pannocchia et al. (2011); Jäschke et al. (2014)).

We refer to this problem as a "nominal" NMPC formulation, as opposed to a "robust" formulation, since disturbances do not appear explicitly here. Robust formulations include min-max NMPC or tube-based NMPC, which tend to suffer from conservative solutions and computational complexity, respectively, and are not considered here. Instead, we study the behavior of the nominal formulation (with sufficiently large penalties and without $\mathbb{X}_f$ and $\mathbb{X}$) upon closing the loop.

3. THE INPUT-TO-STATE STABILITY PROPERTY

Here, the ISS property is described. First, we state necessary the set and function definitions. We use $\mathbb{R}^n$ as the set of real vectors in n-dimensional space and $\mathbb{I}$ as the set of integers. A superscript + is used to add the modification that the set only contains numbers greater than or equal to zero. A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. The function α is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and $\lim_{x\to\infty} \alpha(x) = \infty$. The function $\beta : \mathbb{R}^+ \times \mathbb{I}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if $\beta(\cdot, k) \in \mathcal{K}$ for all fixed $k \in \mathbb{I}^+$, $\beta(|x_0|, \cdot)$ is decreasing for all fixed $|x_0|$, and $\lim_{k\to\infty} \beta(|x_0|, k) = 0$.

Since, for NMPC, the controls u_k are determined as some function $u_k = \kappa(x_k)$ by the optimizer, we may drop u_k as an argument of f. So, consider the discrete time nonlinear system, $x_{k+1} = f(x_k, w_k)$, $x_k \in \mathbb{X}, w_k \in \mathbb{W}$, with initial state x_0 and infinite series of inputs $\mathbf{w} = \{w_0, w_1, \dots\}$, where $\mathbb{X}$ is a closed subset of $\mathbb{R}^n$ and $\mathbb{W}$ is a compact set containing the origin. Define the following sets: $X \subset \mathbb{X}$, a closed robust positive invariant set for $x_{k+1} = f(x_k, w_k), w_k \in \mathbb{W}$, and $\mathcal{W}$, the set of infinite sequences $\mathbf{w}$ satisfying $w_k \in \mathbb{W} \ \ \forall k \in \mathbb{I}^+$. Next define the norms: $|\cdot|$, the Euclidean norm, and $||\cdot||$, the ℓ_∞ norm for sequences, $||\mathbf{w}|| := sup_{k\in\mathbb{I}^+} |w_k|$. If the system is ISS, then:

$$|x_k| \le \beta(|x_0|, k) + \gamma(||\mathbf{w}_k||) \ \ \forall k \in \mathbb{I}^+ \qquad (2)$$
$$\forall\, x_0 \in X, \ \mathbf{w}_k \in \mathcal{W}$$

where $\mathbf{w}_k = \{w_0, w_1, \dots, w_{k-1}, 0, 0, \dots\}$, $\beta(\cdot, \cdot)$ is of class $\mathcal{KL}$, and $\gamma(\cdot)$ is of class $\mathcal{K}$.

Furthermore, this property can be decomposed into two time periods: $|x_k| \le \beta(|x_0|, k) \quad \forall\ k \in \{0, \dots, k_0 - 1\}$, and $|x_k| \le \gamma(||\mathbf{w}_k||) \ \ \forall\ k \in \{k_0, k_0 + 1, \dots\}$, where k_0 is the first time that $|x_k| \le \gamma(||\mathbf{w}_k||)$. That is, the system trajectory has an asymptotic bound $\beta(|x_0|, k)$, until the first time that the trajectory crosses the boundary of the ball of radius $\gamma(||\mathbf{w}_k||)$. This ball is then positive invariant, meaning the system trajectory never leaves it, although the state value of the trajectory may take any value inside the ball.

4. THE ISS LYAPUNOV THEOREM WITH A MODIFIED UNCERTAIN TERM

Here we state a version of the Lyapunov-based ISS theorem from Jiang and Wang (2001) that is extended to allow for a uncertain term that is a function of the state as well as the realization of uncertainty, which leads to a tighter bound since one value need not hold for all states. The proof is also summarized, so that the functional forms of the bounds are apparent. Note that we only show where the proof deviates from the work of Jiang and Wang. We also rely on the extension to systems with state constraints shown in Appendix B of Rawlings and Mayne (2009). Use the system definition from the previous section, and assume that there exists a Lyapunov function $V(x)$ that admits the following comparison functions:

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \ \ \forall\, x \in X, \qquad (3a)$$
$$V(f(x, w)) - V(x) \le -\alpha_3(|x|) + \sigma(|x|, |w|) \qquad (3b)$$
$$\forall\, x \in X, \ w \in \mathbb{W}$$

where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, and $\alpha_3(\cdot) \in \mathcal{K}_\infty$, $\sigma(|x|, 0) = 0$ and $\sigma(|x|, |w|)$ is continuous and strictly increasing with respect to either argument for nonzero $|w|$. Now define the functions $\alpha_4(\cdot)$, $\rho(\cdot)$, and $\hat{\alpha}_4(\cdot)$ to have the following properties: $\alpha_4(\cdot) = \alpha_3 \circ \alpha_2^{-1}(\cdot)$, $\hat{\alpha}_4(s) \le \alpha_4(s) \quad \forall\ s$, $id - \hat{\alpha}_4(\cdot) \in \mathcal{K}_\infty$, $\rho(\cdot) \in \mathcal{K}_\infty$, and $id - \rho(\cdot) \in \mathcal{K}_\infty$, where id denotes the identity function. See lemma B.1 of Jiang and Wang (2001) for proof that $\hat{\alpha}_4(\cdot)$ exists. Now, assume that we have a solution to the following auxiliary optimization problem:

$$min\ \ b \qquad (4a)$$
$$s.t.\ \ \rho \circ \hat{\alpha}_4 \circ b \ge \sigma(|x|, ||\mathbf{w}||) \ \ \forall\, x : V(x) \le b \qquad (4b)$$
$$\rho \circ \hat{\alpha}_4 \circ V(x) \ge \sigma(|x|, ||\mathbf{w}||) \ \ \forall\, x : V(x) > b \qquad (4c)$$
$$b \ge 0 \qquad (4d)$$

where $||\mathbf{w}||$ is an upper bound on $|w_k|$. Assuming a solution to (4) is the key point that allows σ to be a function of $|x|$. In words, this problem is to determine the smallest Lyapunov function value, b, that defines a sublevel set in the state space that is positive invariant for f and a superlevel set that has an asymptotic descent property. Note that the two constraints above can be simplified to solvable forms for specific cases (see Section 8).

We can now say that the system is ISS, and we can construct β and γ by following the proof of Lemma 3.5 in Jiang and Wang (2001) . Consider the set $D = \{x : V(x) \le b\}$. If $x \in D$, then we have:

$$
\begin{aligned}
V(f(x,w)) &\le V(x) - \alpha_3(|x|) + \sigma(|x|, ||\mathbf{w}||) && \text{(5a)}\\
&\le V(x) - \alpha_4 \circ V(x) + \sigma(|x|, ||\mathbf{w}||) && \text{(5b)}\\
&\le V(x) - \hat{\alpha}_4 \circ V(x) + \sigma(|x|, ||\mathbf{w}||) && \text{(5c)}\\
&= (id - \hat{\alpha}_4) \circ V(x) + \sigma(|x|, ||\mathbf{w}||) && \text{(5d)}\\
&\le (id - \hat{\alpha}_4) \circ b + \sigma(|x|, ||\mathbf{w}||) && \text{(5e)}\\
&= (id - \hat{\alpha}_4) \circ b + \sigma(|x|, ||\mathbf{w}||)\\
&\quad + \rho \circ \hat{\alpha}_4 \circ b - \rho \circ \hat{\alpha}_4 \circ b && \text{(5f)}\\
&\le (id - \hat{\alpha}_4)(b) + \rho \circ \hat{\alpha}_4(b) && \text{(5g)}\\
&= -(id - \rho) \circ \hat{\alpha}_4(b) + b \quad \le b && \text{(5h)}\\
&\forall\, x \in D,\ x \in X,\ \mathbf{w} \in \mathcal{W}
\end{aligned}
$$

Note that step (5f) to (5g) holds due to satisfaction of (4b). Thus, the constant b is the Lyapunov function value that corresponds to the invariant ball, so set $\gamma(||\mathbf{w}_k||) = \alpha_1^{-1}(b)$. Now consider $x \notin D$:

$$
\begin{aligned}
V(f(x,w)) - V(x) &\le -\alpha_3(|x|) + \sigma(|x|, ||\mathbf{w}||) && \text{(6a)}\\
&\le -\alpha_4 \circ V(x) + \sigma(|x|, ||\mathbf{w}||) && \text{(6b)}\\
&\le -\hat{\alpha}_4 \circ V(x) + \sigma(|x|, ||\mathbf{w}||) && \text{(6c)}\\
&= -\hat{\alpha}_4 \circ V(x) + \rho \circ \hat{\alpha}_4 \circ V(x) + \sigma(|x|, ||\mathbf{w}||)\\
&\quad - \rho \circ \hat{\alpha}_4 \circ V(x) && \text{(6d)}\\
&\le -\hat{\alpha}_4 \circ V(x) + \rho \circ \hat{\alpha}_4 \circ V(x) && \text{(6e)}\\
&= -(id - \rho) \circ \hat{\alpha}_4 \circ V(x) && \text{(6f)}\\
&\forall\, x \notin D,\ x \in X,\ \mathbf{w} \in \mathcal{W}
\end{aligned}
$$

Note that step (6d) to (6e) holds due to satisfaction of (4c). Furthermore, following from (6f), we have that:

$$
\begin{aligned}
V(x_{k+1}) &\le (id - (id - \rho) \circ \hat{\alpha}_4) \circ V(x_k) && \text{(7a)}\\
V(x_{k+1}) &\le (id - (id - \rho) \circ \hat{\alpha}_4) \circ \alpha_2(|x_k|) && \text{(7b)}\\
x_{k+1} &\le \alpha_1^{-1}((id - (id - \rho) \circ \hat{\alpha}_4) \circ \alpha_2(|x_k|)) && \text{(7c)}\\
|x_k| &\le (\alpha_1^{-1}((id - (id - \rho) \circ \hat{\alpha}_4) \circ \alpha_2(|x_0|)))^k\\
&=: \beta(|x_0|, k) && \text{(7d)}\\
&\forall\, x \notin D,\ x \in X,\ \mathbf{w} \in \mathcal{W}
\end{aligned}
$$

where the superscript k denotes the function of a function, k times (the result of the expression is plugged back in, in place of $|x_0|$, k times). Note that the right-hand side of (7b) is of class $\mathcal{K}_\infty$. The final expression holds true $\forall\, k \in \{0, \ldots, k_0 - 1\}$, where k_0 is the first time such that $|x_k| \le \gamma(||\mathbf{w}_k||)$.

5. A DEGREE OF FREEDOM IN THE BOUNDS

Notice that $\rho(\cdot)$ can be any function that fulfills $\rho(\cdot) \in \mathcal{K}_\infty$, and $id - \rho(\cdot) \in \mathcal{K}_\infty$. To see the effect of $\rho(\cdot)$, inspect (4) and (7d). The choice of $\rho(\cdot)$ affects the magnitude of b and therefore affects $\gamma(||\mathbf{w}_k||)$ as it appears in (2). If we choose $\rho(s)$ to be close to s, then we are effectively choosing a smaller value b and a smaller $\gamma(||\mathbf{w}_k||)$. This means that j_0 becomes a point further forward in time, and $\gamma(||\mathbf{w}_k||)$ only has to bound x_k after some larger fraction of the initial state has decayed. This gives tighter bounding of $|x_k|$ as $k \to \infty$. On the other hand, $\beta(|x_0|, k)$ becomes larger. The opposite holds if we make $\rho(s)$ close to 0. Thus, varying $\rho(\cdot)$ will lead to different functions $\beta(\cdot, \cdot)$ and $\gamma(\cdot)$, and they will correspond to different b, $\gamma(||\mathbf{w}_k||)$, $\beta(|x_0|, k)$, and j_0.

For ease of use, we will define $\rho(s) = \epsilon_1 s$, $\epsilon_1 \in (0, 1)$, so that ϵ_1 close to 1 gives the tightest $\gamma(||\mathbf{w}_k||)$, and ϵ_1 close to 0 gives the tightest $\beta(|x_0|, k)$.

6. APPLICATION TO NMPC

The goal of this section is to describe simple but useful forms for the functions α_1, α_2, α_3, and σ that can be put to use in the context of NMPC to calculate $\gamma(||\mathbf{w}_k||)$ and $\beta(|x_0|, k)$. Previous work, for example Huang et al. (2011a,b), has shown that functions exist, but derived them in terms of Lipschitz constants and a controllability function that would be difficult to find and would provide loose bounds. So instead, we propose a general form for these functions with parameters that can be calculated.

Suppose that we let the Lyapunov function bounds have a power law form, $\alpha_i(|x|) = N_i |x|^{\mu_i}$, where N_i and μ_i are positive parameters that can be found from open loop calculations. The function σ will be addressed in detail in the next section. Note that $\alpha_1(|x|)$ and $\alpha_2(|x|)$ must provide strict lower and upper bounds on $V_N(x)$, respectively, and $N_3|x|^{\mu_3}$ must provide a strict lower bound to $l(x_0, u_0)$. Also, we require that $\mu_1 \ge \mu_2$, so that $\alpha_2(|x|) \ge \alpha_1(|x|)$ holds true near the origin.

Now we need to define $\hat{\alpha}_4(\cdot)$ so that $\hat{\alpha}_4(s) \le \alpha_4(s)\ \ \forall\, s$ and $id - \hat{\alpha}_4(\cdot) \in \mathcal{K}_\infty$ are satisfied. The function $\hat{\alpha}_4(\cdot)$ can be constructed piecewise from combinations of $\alpha_4(\cdot)$ and the identity function. First, for convenience, define $\theta = N_3 {N_2}^{-\mu_3/\mu_2}$ and $B = \mu_3/\mu_2$, so that $\alpha_4(\cdot) = \theta(\cdot)^B$. Now, we must consider three possible cases: $B < 1$, $B > 1$, or $B = 1$.

If $B < 1$, then:

$$
\hat{\alpha}_4(s) := \begin{cases} \epsilon_2 s\ , & s \in \left[0, \left(\dfrac{\epsilon_2}{\theta}\right)^{\frac{1}{B-1}}\right) \\ \theta s^B\ , & s \in \left[\left(\dfrac{\epsilon_2}{\theta}\right)^{\frac{1}{B-1}}, \infty\right) \end{cases} \tag{8}
$$

If $B > 1$, then:

$$
\hat{\alpha}_4(s) := \begin{cases} \theta s^B\ , & s \in \left[0, \left(\dfrac{1}{B\theta}\right)^{\frac{1}{B-1}}\right) \\ \epsilon_2 s + \theta\left(\dfrac{1}{B\theta}\right)^{\frac{B}{B-1}} \\ -\epsilon_2\left(\dfrac{1}{B\theta}\right)^{\frac{1}{B-1}}\ , & s \in \left[\left(\dfrac{1}{B\theta}\right)^{\frac{1}{B-1}}, \infty\right) \end{cases} \tag{9}
$$

If $B = 1$, then:

$$
\hat{\alpha}_4(s) := \epsilon_2 \theta s\ ,\ \ \epsilon_2 < 1/\theta \tag{10}
$$

In all cases, ϵ_2 is user determined, $\epsilon_2 \in (0, 1)$, and a larger ϵ_2 gives a tighter bound. It is easily verifiable that these particular forms of $\hat{\alpha}_4(\cdot)$ have the necessary properties, but proof is omitted here for space reasons.

7. EXPRESSIONS FOR THE UNCERTAIN TERM

First consider the expression for the uncertain term in Huang et al. (2011a), $\sigma(\cdot) = l_V l_w(\cdot)$, where l_V is the Lipschitz constant of the Lyapunov function, and l_w is the Lipschitz constant of f with respect to w. Instead of treating l_V as a constant, let $l_V(|x|) = c|x| + t$, and define it to be an upper bound on $|dV_n(x)/d|x||$. As we will see in the examples section, this form for l_V works well for bounding real data. Note that this will be referred to as the "guaranteed" form of σ.

Now, consider a form for $\sigma(\cdot)$ that includes a further assumption. Suppose that the system $x_{k+1} = f(x_k, 0)$ exhibits nominal stability, so that there exists a Lyapunov function with the following properties:

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \quad (11a)$$
$$V(f(x,0)) - V(x) \le -\alpha_3(|x|) \quad (11b)$$
$$\forall\, x \in X$$

If we assume that $f(\cdot,\cdot)$ is Lipschitz continuous with respect to its second argument so that $|f(x,w)| \le |f(x,0)| + l_w|w|$, and that $|f(x,0)| << l_w||\mathbf{w}||$, then:

$$V(f(x,w)) - V(f(x,0))$$
$$\le \alpha_2(|f(x,w)|) - \alpha_1(|f(x,0)|) \quad (12a)$$
$$\Rightarrow V(f(x,w)) - V(x)$$
$$\le -\alpha_3(|x|) + \alpha_2(|f(x,w)|) - \alpha_1(|f(x,0)|) \quad (12b)$$
$$\le -\alpha_3(|x|) + \alpha_2(|f(x,0)| + l_w||\mathbf{w}||)$$
$$-\alpha_1(|f(x,0)|) \quad (12c)$$
$$\le -\alpha_3(|x|) + \alpha_2(|f(x,0)| + l_w||\mathbf{w}||) \quad (12d)$$
$$\approx -\alpha_3(|x|) + \alpha_2(l_w||\mathbf{w}||) \quad (12e)$$
$$\forall\, x \in X,\ \mathbf{w} \in \mathcal{W}$$

This implies that $\sigma(|w|) = \alpha_2(l_w||\mathbf{w}_k||)$, which removes dependence on $|x|$. Notice that this form may no longer be a strict bound due to the assumption mentioned above. A way of stating this assumption in practical terms is that, after control is applied, any deviation from the setpoint is much more due to the uncertainty than the control action. This assumption seems to work well when the uncertainty is only due to memoryless noise. This will be referred to as the "approximate" form of σ.

8. CONSTRAINT REFORMULATION

We now find a way to solve (4). The constraints as written are not usable, since we do not have an analytical expression for $V(x)$. However, they may be reformulated into usable constraints. For the first constraint we have:

$$\rho \circ \hat{\alpha}_4 \circ b \ge \sigma(|x|, ||\mathbf{w}_k||) \quad \forall\, x : V(x) \le b \quad (13a)$$
$$\Leftarrow \quad \rho \circ \hat{\alpha}_4 \circ b \ge \sigma(|x|, ||\mathbf{w}_k||) \quad \forall\, x : \alpha_1(|x|) \le b \quad (13b)$$
$$\Leftrightarrow \quad \rho \circ \hat{\alpha}_4 \circ b \ge \sigma(|x|, ||\mathbf{w}_k||) \quad \forall\, x : |x| \le \alpha_1^{-1}(b) \quad (13c)$$
$$\Leftarrow \quad \rho \circ \hat{\alpha}_4 \circ b \ge \sigma(\alpha_1^{-1}(b), ||\mathbf{w}_k||) \quad (13d)$$

Then for the second constraint we have:

$$\rho \circ \hat{\alpha}_4 \circ V(x) \ge \sigma(|x|, ||\mathbf{w}_k||) \quad \forall\, x : V(x) > b \quad (14a)$$
$$\Leftarrow \rho \circ \hat{\alpha}_4 \circ \alpha_1(|x|) \ge \sigma(|x|, ||\mathbf{w}_k||) \quad \forall\, x : V(x) > b \quad (14b)$$
$$\Leftarrow \rho \circ \hat{\alpha}_4 \circ \alpha_1(|x|) \ge \sigma(|x|, ||\mathbf{w}_k||) \quad \forall\, x : \alpha_2(|x|) > b \quad (14c)$$
$$\Leftarrow \rho \circ \hat{\alpha}_4 \circ \alpha_1(|x|) \ge \sigma(|x|, ||\mathbf{w}_k||) \quad \forall\, x : |x| \ge \alpha_2^{-1}(b) \quad (14d)$$

Substitute $|x| = \alpha_1^{-1}(b)$ to see that (14d) satisfies (13d). Thus, once specific functional forms are chosen, solving this problem simplifies to solving a nonlinear equation ((14d) as an equality) and checking derivatives (that is, verifying that the derivative w.r.t. $|x|$ of the LHS of (14d) is greater than or equal to that of the RHS in the necessary range). Note that, although this reformulation leads to (4) being solvable, the resulting value of b will be larger than the optimal value of the original problem.

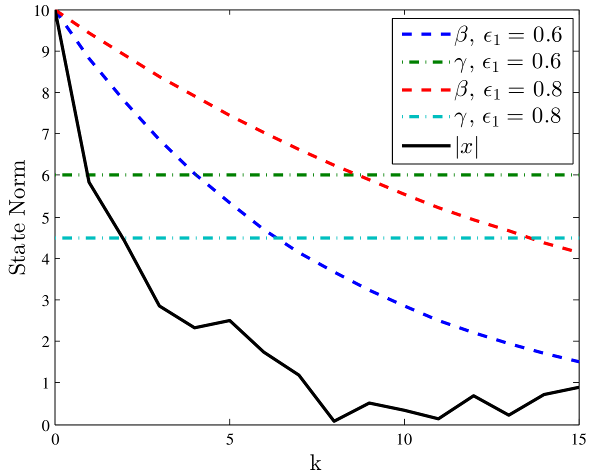

Fig. 1. Scalar system, uniform noise

9. SCALAR LQR EXAMPLE

Consider the scalar example: $f(x,u,w) = Ax + Bu + w = .75x + .25\,u + w$, $l(x,u) = x^2 + u^2$. Also, let $\mathbb{X}_f$ be the steady state. To solve this problem, we will use infinite horizon discrete-time linear quadratic regulator (LQR), and the Lyapunov function bounds may be found analytically. In this case, the Lyapunov function has the form $V_\infty(x) = \sum_{k=0}^{\infty}(x_k^T Qx + u_k^T Ru_k) = x^T Jx$ where x is the initial condition and J solves a discrete time Riccati equation. This gives $J = 2$, so set $N_1 = N_2 = 2$. We also have $u_k = Kx_k$, where $K = -(B^T JB + R)^{-1} B^T JA$. So, to find N_3, we calculate $Q + K^T RK$, which gives $N_3 = 1.11$. Also, since we have $V_\infty(x) = 2x^2 = 2|x|^2$, we can use $l_V = 4|x|$.

We will consider the case where the guaranteed form of the uncertain term is used. Consider uniform noise with $w \in [-1, 1]$ and $x_0 = 10$. Now, we must make a choice for ϵ_1, since many are possible and will provide different information. Consider $\epsilon_1 = 0.6$ and $\epsilon_1 = 0.8$, with results in Figure 1. Recall that the trajectory of the system is bounded, as shown in (2), by $\beta(|x_0, k)$ until the trajectory crosses $\gamma(||\mathbf{w}_k||)$, after which time the trajectory will always be bounded by $\gamma(||\mathbf{w}_k||)$.

Notice that this gives a rather loose bound, but it is an absolute guarantee. We can also compute $\gamma(||\mathbf{w}_k||)$ for $\epsilon_1 \approx 1$, which gives $\gamma(||\mathbf{w}_k||) = 3.6$, still a rather loose bound. However, it is a guarantee no matter the realization of the noise. Now, consider the case where $w_k = 1 \ \ \forall k$. See Figure 2, with $\epsilon_1 \approx 1$. Now, with a much "worse" case realization of the uncertainty,$\gamma(||\mathbf{w}_k||)$ gives a much tighter bound. Notice that since we chose $\epsilon_1 \approx 1$, $\beta(|x_0|, k)$ is nearly constant, so we have no guarantee of when $|x|$ will be bounded by $\gamma(||\mathbf{w}_k||)$.

10. APPLICATION TO A CSTR

Consider a CSTR with the consecutive competitive reactions $A + B \xrightarrow{k1} C$ and $B + C \xrightarrow{k2} D$. The CSTR has three feeds with volumetric flow rates F_A and F_B and

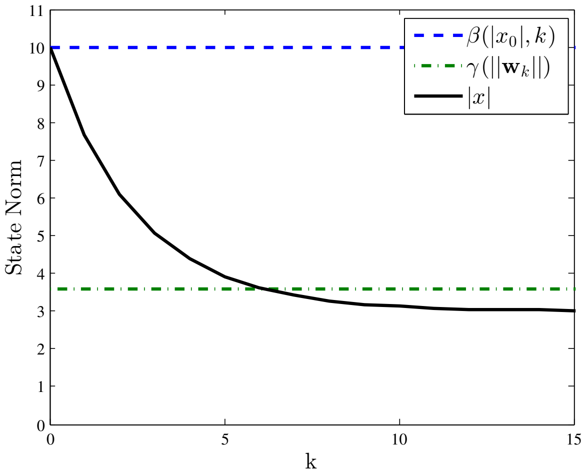

Fig. 2. Scalar system, $w_k = 1$, $\epsilon_1 = 1$

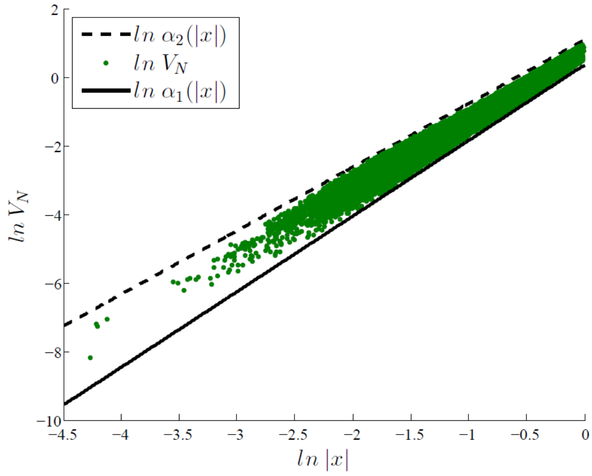

Fig. 3. CSTR Lyapunov function bounds

Table 1. Nonlinear CSTR Parameters

$k_1 = 10$	$k_2 = 4$	$C_{FA} = 5$
$C_{FB} = 5$	$V = 10$	$P_{F_A} = 1$
$P_{F_B} = 1$	$P_{F_W} = 0.5$	$P_C = 1$

Table 2. Nonlinear CSTR steady state values

$C_{A\,ss} = 1.665$	$C_{B\,ss} = 0.200$	$C_{C\,ss} = 1.044$	$C_{D\,ss} = 0.349$
$F_{A\,ss} = 14.650$	$F_{B\,ss} = 9.305$	$F_{W\,ss} = 0$	$l_{ss} = 1.044$

concentrations C_{F_A} and C_{F_B}. Each reactant feed only contains one component, and the volumetric flow rates are the control variables. Flow of pure water, F_W, is also available as a control variable. All three have an upper limit of 20. The CSTR has exiting concentrations C_A, C_B, C_C, & C_D. These are the state variables. The exiting volumetric flow rate is $F_T = F_A + F_B + F_W$. The problem will be considered with parameters shown in Table 1. The steady state objective is $min\ F_A P_{F_A} + F_B P_{F_B} + F_W P_{F_W} - F_T C_C P_C$ where P_{F_i} is the purchase price of feed i, and P_C is the sales price per mole of the product. The solution to the steady state problem is shown in Table 2.

We use a quadratic stage cost with $Q = I_4$ and $R = I_3$, as well as additive state noise. As in the last example, let $\mathbb{X}_f$ be the steady state. Now we need to find estimates for N_1, N_2, and N_3 from open loop tests, since this is not an LQR problem. To do this, we choose a sample space of the states. The bounds on the sample space are $0.5\,C_{i,ss} \leq C_i \leq 1.5\,C_{i,ss}\ \forall\, i$, where i denotes the reaction component. A uniform distribution of initial points is taken across the sample space, and 100,000 points are chosen. An open loop control problem is solved with the initial state at each one of these points. We use a step size of 0.1 and a controller horizon time of 50. Three point Radau collocation is used to discretize the differential equations, and IPOPT (Wächter and Biegler, 2006) is used to solve the optimization problems. We found that, for $\alpha_i(|x|) = N_i|x|^{\mu_i}$, $N_1 = 1.4$, $N_2 = 2.9$, $\mu_1 = 2.2$, and $\mu_2 = 1.85$ provide valid bounds within this range. The bounds are shown in Figure 3. Note that the bound parameters are chosen to give the tightest possible bounds that are true.

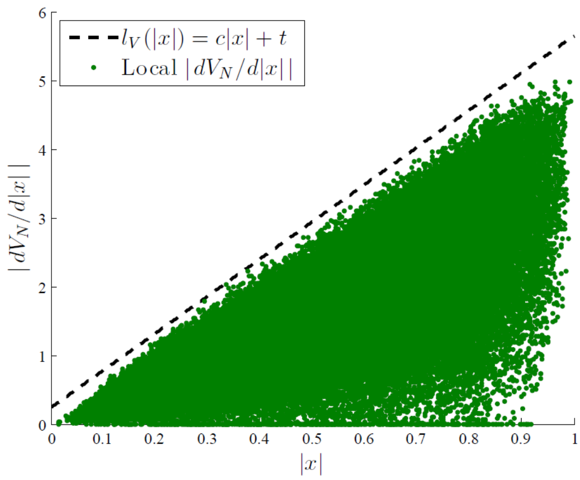

Fig. 4. Overestimation of $|\,dV_N(x)/d|x|\,|$

Also, since the stage cost is quadratic, $N_3 = 1$ and $\mu_3 = 2$ provide a valid lower bound for the first stage cost. Finally, to approximate l_V, we compute, for a given initial point x_g, $|V(x_g) - V(x_{a,i})|/|x_g - x_{a,i}|$ for the eight points $x_{a,i}$, $i = 1 \ldots 8$ closest to x_g. We then take the maximum value of these eight numbers, and use that as the local $|\,dV_N/d|x|\,|$ associated to x_g. These values are then plotted against the norm, and a linear over-estimator $l_V(|x|) = c|x| + t$ with $c = 5.4$ and $t = 0.24$ is chosen. This is shown in Figure 4.

So now everything needed is available to provide predictive trajectory bounds. Note that for a dynamic simulation, the initialization for the first NLP will be linear with time, and the subsequent NLPs will be initialized with the solution from the previous NLP moved one time step backward. This aligns with the theory in Pannocchia et al. (2011) to allow for stability even without an exact solution to a given NLP.

Consider the case with $C_{i,0} = 1.5\,C_{i,ss}$ and $w_{ki} \in [-0.1\,C_{i,ss}, 0.1\,C_{i,ss}]\ \forall\, k, i$. First use the approximate form of σ from (12e). The trajectory of the system and ISS

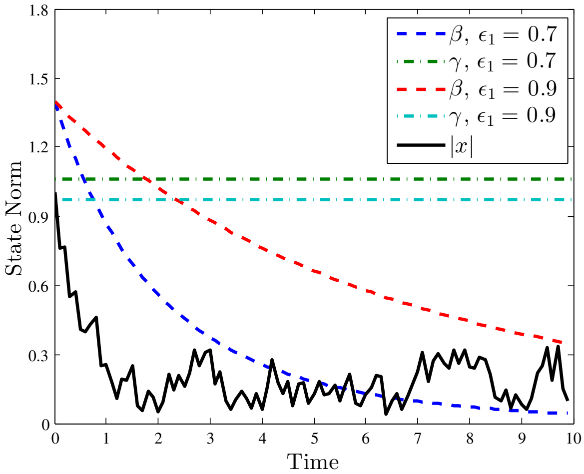

Fig. 5. CSTR with approximate γ and uniform noise

bounds for the cases that $\epsilon_1 = 0.7$ and $\epsilon_1 = 0.9$ are shown in Figure 5. Again, although the bound appears to work well here, it is not a guarantee for all realizations of the noise.

The guaranteed sensitivity expression is very conservative for this system. If we set $\epsilon_1 = 1$ and calculate the size of the invariant ball, then we see $\gamma(||\mathbf{w}_k||) = 0.937$ for the approximate expression and $\gamma(||\mathbf{w}_k||) = 6.326$ for the guaranteed expression. Although the guaranteed bound is true, it is not particularly useful. This shows the limitations with an increasing number of states.

11. CONCLUSIONS AND FUTURE WORK

This work extends the ISS results for NMPC to calculate predictive state trajectory bounds. The ISS theorem for discrete time systems is extended to allow for an uncertain term that depends on the state as well as the realization of uncertainty. Functional forms for the Lyapunov function bounds are proposed, and a method for calculating their parameters is shown. Example calculations are shown for a linear scalar system and a nonlinear CSTR.

Though guaranteed bounds can be calculated, they may be very conservative for systems with many states. Furthermore, there are two main issues that arise upon applying this method to systems with many states. First, sampling the state space to find bounds on the Lyapunov function becomes computationally intensive, and it is unclear if a particular sampling method is more advantageous than others. Second, this method functions by condensing all of the information held in the states into one number, the Lyapunov funciton value, and this leads to a larger loss of information in the case of many states. Future work to improve the quality of these could involve a new form for $\sigma(\cdot)$ or new assumptions that provide good performance in practice. Also, a method of solving (4) that does not lead to an overly-conservative solution would provide significant benefits.

This method directly extends to the economic case, though the variance in the objective function will lead to additional difficulties. Furthermore, this can be applied to systems in which the uncertainty takes a specific form, such as in a feed composition, but more effort will be required to find an expression for l_w. Finally, an on-line procedure for updating the bounds would be useful. This would involve a strategy for updating $\rho(\cdot)$ over time, or some way of combining information from the bounds for multiple choices of $\rho(\cdot)$.

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