Average Constraints in Robust Economic Model Predictive Control *

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Abstract: In this paper, we extend a previously introduced methodology for tube-based robust economic MPC to consider nonlinear average constraints, i.e., constraints on system states and inputs that need to be satisfied on average. A specifically defined integral stage cost takes the disturbance into account when considering the performance. The key idea is to use an appropriately tightened version of the average constraints by using a modified auxiliary output function in their formulation. By means of the tightened constraints, satisfaction of the original average constraints can be guaranteed despite disturbances acting on the system. For some special cases, we provide concepts to simplify the tightening (which might in general be difficult to determine). In addition, we discuss how average constraints can be used in order to enforce convergence of the closed-loop system to an invariant set. Finally, the proposed approach is illustrated with a numerical example.

1. INTRODUCTION

Within the last years, the study of economic model predictive control (MPC) schemes has gained a significant amount of attention (see e.g. Angeli et al. [2012], Amrit et al. [2011], Grüne [2013], Müller et al. [2013], Ellis et al. [2014]). In contrast to stabilizing MPC approaches, the primary control objective is not the stabilization of a desired set-point (or reference trajectory). Instead, some general performance criterion is considered, which could, e.g., be related to the economics of the considered process, such as the maximization of some product or of the profit of a plant. This means that some general cost function can be employed within the MPC algorithm, which needs not be positive definite as is typically assumed in stabilizing MPC. As a consequence, the closed-loop system is not necessarily convergent, but can exhibit some more complex, e.g., periodic, behavior (see, e.g., Angeli et al. [2012]).

The possibly non-convergent closed-loop behavior leads to the fact that besides classical pointwise-in-time constraints, which are typically considered in MPC, also constraints on averages of state and input variables become of interest in economic MPC (Angeli et al. [2012], Müller et al. [2014]). In particular, such constraints have to be dealt with online, i.e., within the optimization problem that is solved at each time step. This is in contrast to standard stabilizing MPC, where asymptotic averages of state and input variables are determined by their value at the set-point to be stabilized, and hence asymptotic average constraints can be considered offline (when determining the set-point). The concept of average constraints can be interesting for different applications such as a chemical reactor, where the average feed flow should be constrained in order to meet storage capacities.

As most of the practical applications are affected by disturbances, some effort in taking disturbances into account within economic MPC has been made. In Huang et al. [2012], based on a formulation that is related to tracking MPC, a stability result for robust economic MPC is presented. A scenario based approach is, for example, studied in Lucia et al. [2014]. In Bayer et al. [2014], the conceptual idea of tube MPC has been transferred to an economic setup by taking an appropriately modified integrated cost function into account, which can be interpreted as an averaging over all possible states induced by the disturbance.

The results in this paper are based on the latter approach. In particular, we extend the tube-based robust economic MPC scheme of Bayer et al. [2014] such that it can also handle average constraints. A first attempt in this direction was made by Bayer and Allgöwer [2014], where, however, only linear systems subject to linear average constraints could be considered. In contrast, the scheme proposed in this paper can be applied to nonlinear systems with general (possibly nonlinear) average constraints. To this end, we use an appropriately tightened version of the average constraints by using a modified auxiliary output function. This allows us to guarantee satisfaction of the average constraints by the closed-loop system despite the presence of disturbances. In addition to the above, we discuss further results which are of interest when considering average constraints, for example, how these constraints can be used to enforce convergence of the closed-loop system.

The remainder of the paper is organized as follows. In Section 2, we will recapitulate the concept of robust economic MPC. The proposed robust economic MPC scheme including average constraints as well as an analysis of the resulting closed-loop system will be provided in Section 3. Some extensions are stated in Section 4. A numerical example illustrates the proposed approach in Section 5, and the paper is concluded in Section 6.

Notation: By $\mathbb{I}_{\geq 0}$, we denote the set of non-negative integers and by $\mathbb{I}_{[a,b]}$ the set of all integers in the interval $[a,b] \in \mathbb{R}$. The relation operators are meant component-wise when applied to a vector. For example, for $a, b \in \mathbb{R}^n$, $a \leq b$ means that $a_i \leq b_i$

^{*} The authors would like to thank the German Research Foundation (DFG) for financial support of the project within the Cluster of Excellence in Simulation Technology (EXC 310/2) at the University of Stuttgart.

for all $i \in \mathbb{I}_{[1,n]}$. For the sets $X, Y \subseteq \mathbb{R}^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y : x \in X, y \in Y\}$; the Pontryagin set difference is defined by $X \oplus Y := \{z : z + y \in X, \forall y \in Y\}$. Following the definition in Angeli et al. [2012], the set of asymptotic averages of a bounded signal $v : \mathbb{I}_{>0} \to \mathbb{R}^{n_v}$ is defined by

$$\operatorname{Av}[v] := \{ \bar{v} \in \mathbb{R}^{n_v} : \exists t_n \to \infty : \lim_{n \to \infty} \frac{\sum_{k=0}^{t_n} v(k)}{t_n + 1} = \bar{v} \},$$

where t_n is an infinite subsequence of $\mathbb{I}_{\geq 0}$.

2. ROBUST ECONOMIC MPC

In this section, we recapitulate the concept of robust economic MPC which was developed in Bayer et al. [2014]. We consider a nonlinear, discrete-time system of the form

$$x(t+1) = f(x(t), u(t), w(t)), \quad x(0) = x_0 \tag{1}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$ being continuous in (x, u, w), where $x(t) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$ is the input to the system, and $w(t) \in \mathbb{W} \subset \mathbb{R}^q$ is an unknown but bounded disturbance acting on the system at time $t \in \mathbb{I}_{\geq 0}$. The disturbance set \mathbb{W} is assumed to a compact and convex set and to contain the origin. States and inputs are required to satisfy pointwise-in-time constraints $(x(t), u(t)) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$, where \mathbb{Z} is compact.

As mentioned above, the goal in economic MPC is not necessarily stabilization of a given steady-state but rather the optimization of a more general stage cost which could be motivated by an economic performance criterion. Hence, the objective is to find a feasible input sequence such that the asymptotic average performance

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ell(x(t), u(t))$$
(2)

is minimized, where $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the stage cost. In contrast to stabilizing MPC, where usually the stage cost is assumed to be positive definite with respect to the setpoint to be stabilized, the stage cost employed here does not need to satisfy such definiteness assumptions. As mentioned above, this can lead to a closed-loop behavior which is not necessarily convergent but which can, for example, result in a cyclic behavior.

Due to the unknown disturbances, the exact system behavior cannot be predicted. However, by using the associated nominal system

$$z(t+1) = f(z(t), v(t), 0), \quad z(0) = z_0, \tag{3}$$

where z(t) is the nominal state and v(t) is the nominal input at time $t \in \mathbb{I}_{>0}$, bounds on the error

$$e(t) = x(t) - z(t) \tag{4}$$

between the real and the nominal system at each time can be provided. Using the feedback control law for the real input $u(t) = \varphi(v(t), x(t), z(t))$, we can derive the error system

$$e(t+1) = f(x(t), \varphi(v(t), x(t), z(t)), w(t)) - f(z(t), v(t), 0).$$
(5)

In order to determine bounds on the error, we make use of the following definition.

Definition 1. (Bayer et al. [2014]) A set $\Omega \subseteq \mathbb{R}^n$ is robust control invariant (RCI) for the error system (5) if there exists a feedback control law $u(t) = \varphi(v(t), x(t), z(t))$ such that for all $x(t), z(t) \in \mathbb{X}$ with $e(t) := x(t) - z(t) \in \Omega$, all $v(t) \in \mathbb{U}$, and all $w(t) \in \mathbb{W}$, it holds that $e(t+1) \in \Omega$ and $u(t) \in \mathbb{U}$.

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In the literature, approaches on determining such RCI sets can be found (see e.g. Limon et al. [2002], Bayer et al. [2013]), where the determination of the invariant sets is based on the idea of invariant level-sets of a Lyapunov function.

For linear systems, determining an RCI set boils down to determining a robust positively invariant (RPI) set. For finding RPI sets, many set theoretic approaches have been presented (see e.g. Chisci et al. [2001], Raković et al. [2005]).

In order to be able to meet the pointwise-in-time constraints for the real system despite the presence of disturbances, the nominal state and input must be restricted to a tightened set

$$\overline{\mathbb{Z}} = \{(z,v) \in \mathbb{Z} : (x,\varphi(v,x,z)) \in \mathbb{Z} \text{ for all } x \in \{z\} \oplus \Omega\}.$$

The real states and inputs are guaranteed to be in \mathbb{Z} for all possible disturbances, if the nominal states and inputs are kept within $\overline{\mathbb{Z}}$ and the feedback φ is chosen such that the error e is in the RCI set Ω .

The concept of robust economic MPC presented in Bayer et al. [2014] is based on the following idea: When economic stage costs are considered and when disturbances are acting on the system, it may not be the best to consider only the performance of the nominal system, as is typically done in tube-based stabilizing MPC. Instead, a cost function is used which explicitly considers the influence of the disturbance in the following way. As mentioned above, the real system state at future time instants cannot be predicted but is only known to lie in the RCI set centered at the predicted nominal states. Hence, this RCI set represents (possibly an outer approximation of) all possible states of the real system. Thus, the following integrated stage cost is used within the repeatedly solved optimization problem:

$$\ell^{\rm int}(z,v) = \int_{x \in \{z\} \oplus \Omega} \ell(x,\varphi(v,x,z)) dx.$$
(7)

This integrated cost can be interpreted as an averaging of the stage cost over all possible states of the real system. Moreover, the input φ applied to the real system is taken into consideration within the integrated cost in order to resemble also all possible inputs for the associated real states. For a more detailed discussion on these issues, we refer the reader to Bayer et al. [2014].

The optimization problem solved at each time instant t is given by

$$\min_{z(0|t),v(t)} \sum_{k=0}^{N-1} \ell^{\text{int}}(z(k|t), v(k|t))$$
(8)

s.t.
$$z(k+1|t) = f(z(k|t), v(k|t), 0),$$
 (9a)

$$(z(k|t), v(k|t)) \in \overline{\mathbb{Z}} \ \forall k \in \mathbb{I}_{[0,N-1]},$$
(9b)

$$x(t) \in \{z(0|t)\} \oplus \Omega, \tag{9c}$$

$$z(N|t) = z_s, \tag{9d}$$

where z(k|t) and v(k|t) denote the k step ahead prediction of the nominal state and input, respectively, predicted at time t. In the following, we denote by $v^*(t) := \{v^*(0|t), \dots, v^*(N - 1|t)\}$ the optimal nominal open-loop input sequence and by $z^*(t) := \{z^*(0|t), \dots, z^*(N|t)\}$ the associated nominal state trajectory starting at the optimal initial state $z^*(0|t)$.

The terminal state is constrained to the robust optimal steadystate given by

$$(z_s, v_s) = \underset{(z,v)\in\overline{\mathbb{Z}}, z=f(z,v),}{\arg\min} \int_{x\in\{z\}\oplus\Omega} \ell(x, \varphi(v, x, z)) dx, \quad (10)$$

which is needed in the MPC algorithm in order to guarantee recursive feasibility and establish average performance bounds (see Bayer et al. [2014]).

3. NONLINEAR AVERAGE CONSTRAINTS

In the following, we want to consider additional average constraints on the system as motivated in the introduction. Therefore, we introduce the auxiliary output

$$y(t) = h(x(t), u(t)),$$
 (11)

where $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is assumed to be twice continuously differentiable. With this auxiliary output at hand, the average constraints imposed on system (1) are given by

$$\operatorname{Av}[h(x,u)] \subseteq \mathbb{R}^{p}_{\leq 0}, \tag{12}$$

with $\mathbb{R}^p_{\leq 0} := \{ y \in \mathbb{R}^p | y \leq 0 \}.$

In case of *undisturbed* systems, satisfaction of the average constraint for the closed-loop system is usually guaranteed by using an additional constraint within the optimization problem, namely

$$\sum_{k=0}^{N-1} h(x(k|t), u(k|t)) \in \mathbb{Y}_t.$$

The set \mathbb{Y}_t is given through the recursive formula $\mathbb{Y}_{t+1} = \mathbb{Y}_t \oplus \mathbb{R}^p_{\leq 0} \oplus \{-h(x(t|t), u(t|t))\}$ and $\mathbb{Y}_0 = \mathbb{Y}_{00} \oplus \mathbb{R}^p_{\leq 0}$, where \mathbb{Y}_{00} is an arbitrary compact set containing the origin. This idea is, e.g., presented in Angeli et al. [2012] and Müller et al. [2014].

As discussed in the previous section, in case of a *disturbed* system, the real future system states and inputs cannot be predicted, but only bounds are known by means of the RCI set Ω . Hence in order to guarantee satisfaction of the average constraints (12) for the closed-loop system, one would need to guarantee that predicted states and inputs satisfy

$$\sum_{k=0}^{N-1} h(x(k|t), \varphi(v(k|t), x(k|t), z(k|t))) \in \mathbb{Y}_t,$$

$$\forall x(k|t) \in z(k|t) \oplus \Omega, \forall k \in \mathbb{I}_{[0,N-1]}.$$
(13)

However, adding this constraint to the optimization problem (8)–(9) would lead to a semi-infinite programming problem (see, e.g., Vázquez et al. [2008]), which might be intractable to solve online at each time step. Furthermore, in contrast to the pointwise-in-time constraints, it is not possible to determine offline some (constant) tightened constraint set $\overline{\mathbb{Y}}$ (similar to $\overline{\mathbb{Z}}$ in (6)) and then online in (8)–(9) require that $\sum_{k=0}^{N-1} h(z(k|t), v(k|t)) \in \overline{\mathbb{Y}}$, as the constraint set \mathbb{Y}_t is timevarying.

For linear systems with linear average constraints, this problem can be handled by using ideas from tube MPC, namely by means of set tightening via the Pontryagin set difference. In order to do so, the knowledge of the error being restricted to an RPI set can be used, see Bayer and Allgöwer [2014] and also Section 3.1B). However, this idea of using the Pontryagin set difference can only be applied if a linear output function and a linear error feedback are considered, but not for general nonlinear average constraints and dynamics. Instead, in the following, we will use ideas from *Robust Optimization*, as e.g., presented in Ben-Tal et al. [2009]. The general approach is to rewrite an optimization problem such that the disturbances are only showing up within its constraints (Robust Counterpart), and to find a feasible solution to this rewritten problem for all possible disturbances.

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Now consider again the output h. Using the feedback $u = \varphi(v, x, z)$ and x = z + e, we can rewrite h as

$$h(x, u) = h(z + e, \varphi(v, z + e, z))$$

If now an appropriate feedback control law u (see Definition 1) is applied to system (1), we know that the error e is bounded for all times within the RCI set Ω , i.e., $x(t) \in \{z(t)\} \oplus \Omega$. Hence, defining

$$\bar{h}(z,v) := \max_{e \in \Omega} h(z+e,\varphi(v,z+e,z)), \tag{14}$$

we obtain that $h(x(t), u(t)) \leq \bar{h}(z(t), v(t))$ for all $t \in \mathbb{I}_{\geq 0}$. The idea is now to design an economic MPC scheme such that the nominal closed-loop system satisfies $Av[\bar{h}(z, v)] \subseteq \mathbb{R}^p_{\leq 0}$, from where it then follows that also $Av[h(x, u)] \subseteq \mathbb{R}^p_{\leq 0}$ for all possible disturbances, as required. Before doing so in Section 3.2, in the following subsection we briefly discuss some simplifications/approximations in computing the function \bar{h} .

3.1 Simplifying the Approximations of the Average Constraint

In this section, we discuss how/in what cases the computation of \bar{h} according to (14) can be simplified.

A) Simplifications for general output functions

In order to simplify the determination of the tightened constraints, we consider the output function component-wise and introduce the following assumption.

Assumption 2. (Houska [2011]) For each $i \in \mathbb{I}_{[1,p]}$, there exists a twice continuously differentiable and non-negative function $\tilde{\lambda}_i : \mathbb{Z} \to \mathbb{R}_{>0}$ which satisfies the inequality

$$\forall \ e \in \Omega : \lambda_{\max}\left(\frac{\partial^2 h_i(z+e,\varphi(v,z+e,z))}{\partial e^2}\right) \leq 2\tilde{\lambda}_i(z,v),$$

i.e., the maximum eigenvalue of the Hessian of h_i with respect to e is for all $e \in \Omega$ bounded from above by the function $2\tilde{\lambda}_i$.

With this assumption at hand, we can follow the idea presented in Houska [2011] and use a Taylor series expansion in order to provide an approximated upper bound by

$$\begin{split} h_i(x,u) &\leq \max_{e \in \Omega} h_i(z+e,\varphi(v,z+e,z)) \\ &\leq \max_{e,s \in \Omega} \left\{ \left. h_i \right|_{e=0} + \left. \frac{\partial h_i}{\partial e} \right|_{e=0} e + \frac{1}{2} e^T \left. \frac{\partial^2 h_i}{\partial e^2} \right|_{e=s} e \right\} \\ &\leq \max_{e \in \Omega} \left. \left\{ \left. h_i \right|_{e=0} + \left. \frac{\partial h_i}{\partial e} \right|_{e=0} e + \tilde{\lambda}_i(z,v) e^T e \right\} \\ &=: \tilde{h}_i(z,v). \end{split}$$

Obviously, it holds that $h(x, u) \leq \overline{h}(z, v) \leq \widetilde{h}(z, v), \forall e \in \Omega$. Note that \widetilde{h}_i might in general again be a non-convex function and, thus, possibly hard to find. However, in some cases further simplifications and convexifications can be performed. If, for example, $\Omega = \mathcal{B}_1(0)$ is given by the unit ball, we can find the simpler upper bound

$$\tilde{h}_i(z,v) = h_i|_{e=0} + \left\| \frac{\partial h_i}{\partial e} \right|_{e=0} \right\|_2 + \tilde{\lambda}_i(z,v).$$

Convexity of this function is of course depending on the convexity properties of the original output function h. For further details on the convexification and for further examples, the interested reader is referred to Houska [2011].

B) Linear output functions

As mentioned above, in Bayer and Allgöwer [2014] the problem of robust economic MPC including average constraints is considered for a linear system with linear output functions of the form

$$h(x,u) = h_x x + h_u u + h_c.$$

There, the Pontryagin set difference is used in order to tighten the sets of the average constraints appropriately. In fact, this idea can be generalized to all nonlinear systems for which the associated RCI set Ω can be determined with a linear error feedback of the form $\varphi(v, x, z) = v + K(x - z)$. In this case, the function \bar{h} in (14) is given by

$$\bar{h}(z,v) = h_x z + h_u v + h_c + \max_{e \in \Omega} \{h_x e + h_u K e\}.$$
 (15)

Then, in order to ensure that (12) is satisfied, one can define the tightened set $\tilde{\mathbb{Y}}_t = \mathbb{Y}_t \ominus Nh_x \Omega \ominus Nh_u \Omega$ and impose the additional constraint $\sum_{k=0}^{N-1} h(z(k|t), v(k|t)) \in \tilde{\mathbb{Y}}_t$ in problem (8)–(9), resulting in a standard finite-dimensional nonlinear program. Furthermore, computing the appropriately tightened set can as well be performed by computing the Robust Counterpart of a simple linear program (see e.g. Ben-Tal et al. [2009]), which turns out to be equivalent to the used Pontryagin set difference.

3.2 Satisfying the Average Constraint

In order to resemble the idea from nominal economic MPC, we introduce an average constraint to be satisfied by the open-loop predictions at each iteration. Namely, we require that

$$\sum_{k=0}^{N-1} \bar{h}(z(k|t), v(k|t)) \in \overline{\mathbb{Y}}_t,$$
(16)

with

$$\overline{\mathbb{Y}}_{t+1} = \overline{\mathbb{Y}}_t \oplus \mathbb{R}^p_{\leq 0} \oplus \{-\bar{h}(z(0|t), v(0|t))\},$$
(17)

for all $t \in \mathbb{I}_{\geq 0}$ and $\overline{\mathbb{Y}}_0 = \mathbb{R}^p_{\leq 0} \oplus \overline{\mathbb{Y}}_{00}$, where $\overline{\mathbb{Y}}_{00}$ is an arbitrary compact set ¹ containing the origin. Note that the set recursion in (17) is given in terms of the tightened output function \overline{h} provided in (14).

Next, we introduce the robust optimal steady-state for systems with average constraints (\bar{z}_s, \bar{v}_s) , which will be used later in the optimization as a terminal constraint in order to guarantee recursive feasibility of the MPC algorithm. This steady-state does not only need to satisfy the dynamics and the tightened pointwise-in-time constraints on the states and inputs, but it must also satisfy $\bar{h}(\bar{z}_s, \bar{v}_s) \in \mathbb{R}^p_{\leq 0}$. Taking all the necessary constraints into account and considering the integrated cost, it is given by

$$(\bar{z}_s, \bar{v}_s) = \underset{\substack{(z,v) \in \overline{\mathbb{Z}}, z = f(z,v), \\ \bar{h}(z,v) \in \mathbb{R}^p_{<0}}}{\arg\min} \int_{x \in \{z\} \oplus \Omega} \ell(x, \varphi(v, x, z)) dx.$$
(18)

Note that it is crucial for the algorithm that there exists at least one feasible steady-state which satisfies not only the tightened pointwise-in-time constraints but also the average constraint for the approximated output \bar{h} .

In order to provide recursive feasibility for the average constraint, we must replace the terminal constraint (9d) by

$$z(N|t) = \bar{z}_s. \tag{19}$$

Algorithm 1 Robust Economic MPC

given: initial state
$$x(0)$$

for $t = 0, 1, 2, ...$ do
minimize (8) subject to (9a), (9b), (9c), (16), and (19)
apply $u(t) = \varphi(v^*(0|t), x(t), z^*(0|t))$ to system (1)
end for

The proposed robust economic MPC algorithm is now given in Algorithm 1.

Applying Algorithm 1 results in the optimal input for the real system $u^*(t)=\varphi(v^*(0|t),x(t),z^*(0|t))$ and the closed-loop real system

$$\begin{aligned} x(t+1) &= f(x(t), u^*(t), w(t)), \ x(0) = x_0, \\ y(t) &= h(x(t), u^*(t)), \end{aligned} \tag{20a}$$

for which the following holds.

Theorem 3. Let the optimization problem in Algorithm 1 be feasible at time t = 0. Then it is feasible for all $t \in \mathbb{I}_{>0}$ and it holds for the closed-loop system (20) that

$$(x(t), u(t)) \in \mathbb{Z}, \quad \forall t \in \mathbb{I}_{\geq 0}$$
$$\operatorname{Av}[h(x, u)] \subseteq \mathbb{R}^{p}_{< 0}.$$

Moreover, the closed-loop system in (20) has a robust asymptotic average performance which is no worse than that of the robust optimal steady-state, i.e.,

$$\ell^{\text{int}}(\bar{z}_s, \bar{v}_s) \ge \limsup_{T \to \infty} \frac{\ell^{\text{int}}(z^*(0|t), v^*(0|t))}{T}.$$

The last statement means that the average integral cost along the nominal closed-loop sequence is no worse than the integral cost at the robust optimal steady-state. This can be seen as an average performance result for the real closed loop, averaged over all possible values of the unknown real state.

Proof. Consider the standard candidate solution at time t + 1, given by $\tilde{v}(t + 1) = \{v^*(1|t), \dots, v^*(N - 1|t), \bar{v}_s\}$ and its associated state trajectory $\tilde{z}(t + 1) = \{z^*(1|t), \dots, z^*(N - 1|t), \bar{z}_s, \bar{z}_s\}$. The pointwise-in-time constraints are recursively satisfied by means of the appropriate definition of \mathbb{Z} for the nominal states and inputs, which is given such that for all possible disturbances the original constraints on the real states and inputs are satisfied. Next, we want to prove recursive feasibility for (16). Using the sequences $\tilde{v}(t+1)$ and $\tilde{z}(t+1)$, we obtain

$$\sum_{k=0}^{N-1} \bar{h}(\tilde{z}(k), \tilde{v}(k)) = \sum_{k=0}^{N-1} \bar{h}(z^*(k|t), v^*(k|t)) - \bar{h}(z^*(0|t), v^*(0|t)) + \bar{h}(\bar{z}_s, \bar{v}_s) \in \overline{\mathbb{Y}}_t \oplus \mathbb{R}^p_{\leq 0} \oplus \{-\bar{h}(z^*(0|t), v^*(0|t))\} = \overline{\mathbb{Y}}_{t+1},$$
(21)

which means that constraint (16) is recursively feasible.

In order to prove that the average constraint (12) is satisfied, we solve the recursion for $\overline{\mathbb{Y}}_t$, that is,

$$\overline{\mathbb{Y}}_t = \overline{\mathbb{Y}}_{00} \oplus (t+1)\mathbb{R}^p_{\leq 0} \oplus \left\{-\sum_{k=0}^{t-1} \bar{h}(z^*(0|k), v^*(0|k))\right\}.$$

Following the proof of [Angeli et al., 2012, Theorem 5], we can rewrite (16) as

¹ $\overline{\mathbb{Y}}_{00}$ has to be large enough such that constraint (16) is initially feasible.

$$\sum_{k=0}^{N-1} \bar{h}(z^*(k|t), v^*(k|t)) + \sum_{k=0}^{t-1} \bar{h}(z^*(0|k), v^*(0|k)) \\ \in \overline{\mathbb{Y}}_{00} \oplus (t+1) \mathbb{R}^p_{\leq 0}.$$

Note that the first sum consists of N terms only. Moreover, each of these terms can be bounded due to compactness of $\overline{\mathbb{Z}}$, continuity of h, and the definition of \overline{h} and $\overline{\mathbb{Z}}$ in (14) and (6), respectively. Considering any infinite subsequence $t_n \subseteq \mathbb{I}_{\geq 0}$ such that a limit is admitted for $s_{\overline{h}}(t_n) :=$ $\sum_{k=0}^{t_n-1} \overline{h}(z^*(0|k), v^*(0|k))/t_n$, we can see that

$$\lim_{n \to \infty} s_{\bar{h}}(t_n) \in \lim_{n \to \infty} \frac{\overline{\mathbb{Y}}_{00} \oplus (t_n + 1) \mathbb{R}_{\leq 0}^p}{t_n} = \mathbb{R}_{\leq 0}^p,$$

where the set-limit has to be understood in the sense of [Goebel et al., 2012, Definition 5.1].

Up to now, we have merely shown that $\operatorname{Av}[\bar{h}(z,v)] \subseteq \mathbb{R}_{\leq 0}^p$. In order to show satisfaction of the average constraint (12) for the real closed-loop system, we have to recall that each element of $\operatorname{Av}[\bar{h}(z,v)]$ is a limit point of the sequence $s_{\bar{h}}(t) = \sum_{k=0}^{t-1} \bar{h}(z^*(0|k), v^*(0|k))/t$ (when taking the limit along a specific subsequence t_n). Due to the fact that $\operatorname{Av}[\bar{h}(z,v)] \subseteq \mathbb{R}_{\leq 0}^p$, if follows that $\limsup_{t\to\infty} [s_{\bar{h}}(t)]_i \leq 0$, for all $i \in \mathbb{I}_{[1,p]}$. From the definition of \bar{h} in (14) and the definition of the input u(t) in Algorithm 1, it follows that the closed-loop system (20) satisfies $h(x(t), u(t)) \leq \bar{h}(z^*(0|t), v^*(0|t))$ for all $t \in \mathbb{I}_{\geq 0}$. Hence we have $s_h(t) \leq s_{\bar{h}}(t)$ for all $t \in \mathbb{I}_{\geq 0}$, where $s_h(t) := \sum_{k=0}^{t-1} h(x(k), u(k))/t$, which in particular implies that $\limsup_{t\to\infty} [s_h(t)]_i \leq \limsup_{t\to\infty} [s_{\bar{h}}(t)]_i \leq 0$, for all $i \in \mathbb{I}_{[1,p]}$. But this means that

$$\operatorname{Av}[h(x,u)] \subseteq \mathbb{R}^p_{<0},$$

i.e., the closed-loop system satisfies the average constraints as required.

The last statement – the bound on the asymptotic performance – follows directly from [Bayer et al., 2014, Theorem 4]. \Box

4. FURTHER EXTENSIONS

In the following, we briefly discuss some further results and extensions, which were previously obtained in nominal economic MPC (or robust economic MPC without average constraints), and which can be transferred to the setting of this paper with little effort.

A) Terminal Set Constraint and Terminal Cost

For ease of presentation within constraint (9d) and (19), we have restricted the predicted terminal nominal state of the system to lie at the robust optimal steady-state. However, this terminal constraint can be relaxed to a terminal set constraint. To this end, in [Mayne et al., 2000, Amrit et al., 2011], appropriate conditions on the terminal set $\overline{\mathbb{X}}_f \subseteq \mathbb{X}$ and the terminal cost $V_f: \overline{\mathbb{X}}_f \to \mathbb{R}$ are given. These conditions include finding an appropriate terminal controller $\kappa_f: \mathbb{X} \to \mathbb{U}$. However, even when these conditions are satisfied, recursive feasibility of (16) can in general not be guaranteed. Namely, we see that we must guarantee that

$$\bar{h}(z,\kappa_{\mathrm{f}}(z)) \in \mathbb{R}^{p}_{<0}, \forall z \in \overline{\mathbb{X}}_{\mathrm{f}}$$

If this condition is satisfied, we can directly apply the result presented above with the appropriate terminal set constraint and the terminal cost.

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On the other hand, if this condition is *not* satisfied, we have to define a suitably contracting terminal region and modify the definition of $\overline{\mathbb{Y}}_t$ such that recursive feasibility and satisfaction of the average constraints can still be guaranteed (see [Müller et al., 2014, Section 3] for more details).

B) Enforcing Convergence by Average Constraints

As discussed above, the closed-loop system resulting from application of an economic MPC algorithm does not necessarily converge to a steady-state, but can exhibit some more complex behavior. However, in certain applications, closed-loop convergence is an important requirement which need be satisfied and cannot be traded off against an improved performance. In this case, appropriately chosen average constraints can be used to enforce convergence. Namely, for nominal economic MPC, several methods were presented in Müller et al. [2014] how the output function h can be defined such that (asymptotic) convergence of the closed-loop system can be guaranteed. These methods can directly be transferred to the setting of this paper, i.e., for robust economic MPC, in order to determine suitable functions \bar{h} . In this way, asymptotic convergence of the nominal system to some desired feasible steady-state \tilde{z}_s can be ensured, and the real closed-loop system then converges to the set $\{\tilde{z}_s\} \oplus \Omega$.

5. NUMERICAL EXAMPLE

In the following, we apply the approach presented in this paper to a nonlinear example from the literature. The system considered is presented in Limon et al. [2002] and Bayer et al. [2013] and given by

$$\begin{aligned} x_1^+ &= 0.55x_1 + 0.12x_2 + (-0.6x_1 + x_2 + 0.01)u + w_1 \\ x_2^+ &= 0.67x_2 + (x_1 - 0.8x_2 + 0.15)u + w_2. \end{aligned}$$

For the constraints, we assume $\mathbb{X} = \{x \in \mathbb{R}^2 : ||x||_{\infty} \leq 1\}$ and $\mathbb{U} = \{u \in \mathbb{R} : |u| \leq 0.1\}$. The disturbances are bounded in $\mathbb{W} = \{w \in \mathbb{R}^2 : ||w||_{\infty} \leq 0.005\}$. Using the approach in Bayer et al. [2013], we can derive the RCI set to be given by $\Omega = \{e \in \mathbb{R}^2 : ||e||_{\infty} \leq 1/30\}$. Within the economic setup, we want to minimize the value of x_2 , thus the cost is given by $\ell(x, u) = x_2$. Our second goal, besides minimization of x_2 , is the satisfaction of two average constraints expressed by the output functions

$$h_1(x) = x_1^2 - 100 x_2^2$$
 and $h_2(u) = -u$.

The first constraint restricts the states on average to a sector, while the second should keep the input positive on average. For the initial state we set $x = [0.1, 0.1]^T$ and we choose $\overline{\mathbb{Y}}_{00} = \{y \in \mathbb{R}^2 : |y_i| \leq 0.6, i = \{1, 2\}\}$. The prediction horizon is given by N = 20.

We can see that $(\bar{z}_s, \bar{v}_s) = (0.0180, 0.0385, 0.0925)$ is the robust optimal steady-state including the average constraints as defined in (18). On the other hand, if we compute the robust optimal steady-state without average constraints as defined in (10), we end up at $(z_s, v_s) = (-0.0055, -0.0578, -0.1)$.

As can be seen from considering the simulation results, the closed-loop behavior can be separated into three phases. In the first phase (iteration 0 to 10), the system converges to the the robust optimal steady-state (z_s, v_s) without average constraints. Note that (z_s, v_s) would be feasible including the average constraint expressed by h_1 , however, not for the one expressed by h_2 . By choosing $\overline{\mathbb{Y}}_{00}$ rather large, it is possible



Fig. 1. Closed-loop behavior of the real system. x_1 is given in blue, x_2 in green, \bar{z}_s in magenta.



Fig. 2. Closed-loop input to the real system. u is given in blue, \bar{v}_s in magenta.

to keep the system for some iterations at the economically best steady-state without violating the average constraint (16). After some iterations (around iteration 25), this additional freedom is no longer available. The length of this transient phase can be tuned by the size of $\overline{\mathbb{Y}}_{00}$. Once the additional freedom induced by $\overline{\mathbb{Y}}_{00}$ is no longer available and in order to meet both average constraints, the input must become positive for some iterations. Yet, the constraints only need to be satisfied on average, and thus, the input can again become negative after a few iterations, see Fig. 2. This leads to a cyclic closed-loop behavior, see Fig. 1 and Fig. 2. By means of this cyclic behavior, a closed-loop performance can be achieved, which is better than staying at (\bar{z}_s, \bar{v}_s) , while still satisfying the average constraints. Note that a steady-state near the origin, which would lead to a closedloop performance comparable to the one of the cyclic solution and which would meet the average constraint expressed by h_2 , is infeasible due to the average constraint expressed by h_1 .

6. CONCLUSION

In this paper, an idea for handling average constraints within a robust economic MPC approach was presented. Based on an appropriate reformulation of the auxiliary output, recursive feasibility as well as satisfaction of the average constraint for the disturbed closed-loop system was provided. Furthermore, some extensions where briefly discussed regarding terminal set constraints and the possibility to use average constraints for enforcing convergence.

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