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PARAMETER CONVERGENCE IN ADAPTIVE EXTREMUM SEEKING CONTROL

V. Adetola and M. Guay [1]

Queen's University, Kingston ON, K7L 3N6 Canada

Abstract: This paper addresses the problem of parameter convergence in adaptive extremum seeking control design. An alternate version of the popular persistence of excitation condition is proposed for a class of nonlinear systems with parametric uncertainties. The condition is translated to an asymptotic sufficient richness condition on the reference set-point. Since the desired optimal set-point is not known *a priori* in this type of problem, the proposed method includes a technique for generating perturbation signal that satisfies this condition in closed loop. This demonstrates its superiority in terms of parameter convergence. The method guarantees parameter convergence with minimal but sufficient level of perturbation. The effectiveness of the proposed method is illustrated with a simulation example.

Keywords: Extremum seeking; Persistence of excitation; Sufficient richness.

1. INTRODUCTION

Extremum seeking control (ESC) is a class of adaptive control that deals with regulation to unknown set points. This type of control has been proposed by a number of authors to handle optimization problems in nonlinear control systems and a number of applications of this method have been reported in the literature ((Krstic and Wang, 2000; Wang *et al.*, 1998; Guay and Zhang, 2003; Guay *et al.*, 2004) for example). The controller finds the operating set-points that optimize a performance or cost function. The uncertainty associated with the function makes it necessary to use some sort of adaptation and perturbation to search for the optimal operating condition.

One of the main challenges with model based or adaptive extremum-seeking control and most deterministic adaptive control approach is the ability to recover the true unknown values of the parameters. In most approaches, parameter convergence to their true values can only be ensured if the closed-loop trajectories provide sufficient excitation for the parameter estimation routine. In standard linear adaptive control approaches, this problem is tractable (Ioannou and Sun, 1996) and can be solved satisfactorily. A dither signal can be introduced momentarily in the control system to achieve the necessary excitation. For nonlinear systems, the problem of determining appropriate excitation conditions remains open. Although some limited persistence of excitation (PE) conditions have been derived, they remain difficult to apply. Such conditions appear naturally in (Guay and Zhang, 2003) for the solutions of an adaptive extremum-seeking control problem. In fact, the fulfillment of such conditions dictates the performance of the optimization routine.

This study is focused on model based extremum seeking techniques. In particular, we consider the class of adaptive ESC problems introduced in (Guay and Zhang, 2003) where the structure of the objective function is employed in the design. In contrast to non-model based approaches (see

[1] corresponding author. Email: guaym@chee.queensu.ca

(Krstic and Wang, 2000) for example), no direct measurement of the objective function is available but must be inferred through the measurements of the state variables and the estimation of model parameters. Examples of this type of problem arise when the economic function involves quantities such as costs of raw materials, operating costs and values of products aside from system's states and unknown parameters.

In the previous works in this area (for example (Guay and Zhang, 2003; Adetola and Guay, 2005; DeHaan and Guay, 2005)), convergence to the optimum is guaranteed only by assuming the satisfaction of a PE condition. Apart from the fact that it is difficult to choose a signal that satisfies such assumptions, it is necessary to select one that achieves a good compromise between the conflicting objectives of identification and control. This paper complements the previous works by translating the PE condition, which depends on the nonlinear closed loop signals, into a sufficient richness condition on the desired set-point signals. However, since the desired optimal set-point is uncertain in this type of problem, the design of a perturbation signal that satisfies this condition cannot be carried out off-line. The proposed method includes a technique for generating such signal in closed loop. The design guarantees parameter convergence with a minimum loss of regulation performance.

2. PROBLEM DESCRIPTION

Consider the following optimization problem

$$\min_{x_p} \; p(x_p, \theta) \tag{1}$$

subject to the system's dynamics

$$\begin{aligned} \dot{x}_p &= f_p(x) + \phi(x_p)\theta + G_p(x)u \\ \dot{x}_q &= f_q(x) \end{aligned} \tag{2}$$

where $x = [x_p^T \;\; x_q^T]^T \in \mathbb{R}^n$ are the systems states, $u \in \mathbb{R}^m$ is the control input. The vector $x_p \in \mathbb{R}^m$ represents the system states involved in the objective function, θ represents unknown parameter vector assumed to be uniquely identifiable and to lie in a known, convex set $\theta \in \Omega_\theta \subseteq \mathbb{R}^{n_\theta}$. The mappings $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $G_p(x) : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ are smooth. The following assumptions are made about (1) and (2).

Assumptions

A1. The function p is C^2 in its arguments and $\partial^2 p/\partial x_p^2 \geq c_0 I > 0, \forall (x_p, \theta) \in (\mathbb{R}^m \times \Omega_\theta)$.
A2. $\exists G_p(x)^{-1} \; \forall x \in \mathbb{R}^n$.
A3. The state $x_q \in \mathbb{R}^{m-n}$ belongs to a positively invariant set for any bounded x_p.
A4. The mapping $\phi(x_p) : \mathbb{R}^n \to \mathbb{R}^{m\times n_\theta}$ is a sufficiently smooth - $C^{\beta-1}$ matrix valued function; $\beta \geq \max\{2, \text{ ceil}(\frac{n_\theta}{m})\}$, where ceil(.) rounds its argument to the nearest integer towards infinity. Moreover, $\phi(x_p)$ is assumed bounded for bounded x_p.

Assumption $A1$ state that the cost surface is strictly convex in x_p and the simplifying assumption $A2$ is only made in order to allow for a direct design of the adaptive controller.

3. EXTREMUM SEEKING SET-POINT AND CONTROLLER DESIGN

3.1 Set-point update law

Considering the fact that the cost function contains unknown parameter θ, the desired set-point measurement cannot be obtained off-line. However, if the function $p(x_p, \theta)$ is not complex, the optimal value can be determined as a function of θ by solving for x_p in $\partial p/\partial x_p = 0$. When the analytical expression of x_p is not available, the desired set-point may be obtained online using Lyapunov method.

Let $x_p^r \in \mathbb{R}^m$ denote a reference set-point for x_p and $\hat{\theta}$ denote an estimate of the unknown parameter θ. An online update law is designed such that $x_p^r(t)$ approaches the optimum value $x_p^*(\hat{\theta})$ exponentially. Let us consider an optimization Lyapunov function candidate

$$V_{sp} := \frac{1}{2}\left\| \frac{\partial p(x_p^r, \hat{\theta})}{\partial x_p^r} \right\|^2 \triangleq \frac{1}{2}\|z_r\|^2 \tag{3}$$

Taking the time derivative of V_{sp}, we have

$$\dot{V}_{sp} = \frac{\partial p}{\partial x_p^r}\left[\frac{\partial^2 p}{\partial x_p^r \partial x_p^r}\dot{x}_p^r + \frac{\partial^2 p}{\partial x_p^r \,\partial \hat{\theta}}\dot{\hat{\theta}}\right]. \tag{4}$$

Choosing the update law as

$$\dot{x}_p^r = -\left(\frac{\partial^2 p}{(\partial x_p^r)^2}\right)^{-1}\left[k_r \frac{\partial p}{\partial x_p^r}^T + \frac{\partial^2 p}{\partial x_p^r \,\partial \hat{\theta}}\dot{\hat{\theta}}\right] \tag{5}$$

with $k_r > 0$, (4) becomes

$$\dot{V}_{sp} \leq -k_r \|z_r\|^2 \tag{6}$$

Proposition 1. The optimal set-point $x_p^r(t)$ generated by (5) is feasible and converges to $x_p^*(\hat{\theta})$ exponentially.

Proof. Assuming (for now, it will be shown later) that $(\hat{\theta}, \dot{\hat{\theta}})$ is bounded. This assumption coupled with assumption $A1$ ensure that (5) exist and it is finite. It follows from (6) that the origin $z_r = 0$ is exponentially stable Applying the inverse function theorem, it can be seen that the mapping z_r is a diffeomorphism. Hence it concluded that $x_p^r(t)$ converges to $\hat{\theta}$-dependent optimal set-point $x_p^*(\hat{\theta})$ exponentially fast. $\square$

3.1.1. Sufficiently rich optimal set-point Since parameter convergence is a vital issue in ESC design, we have to provide some richness condition on the set-point x_p^r to ensure that $\hat{\theta} \to \theta$ as $t \to \infty$. To achieve this, the set-point is appended with a bounded perturbation signal $d(t)$. The rich set-point is given by

$$r(t) := x_p^r(t) + d(t) \tag{7}$$

where $d(t)$ is a sufficiently smooth and uniformly bounded signal. In particular, the signal is parameterized as

$$d(t) := \sum_{k=1}^{\hbar} a_k(t) \sin(\omega_k t) = a(t)\rho(t) \tag{8}$$

where $a(t) = [a_1(t)\ a_2(t)\ \dots\ a_{\hbar}(t)]$ is the signal amplitude vector and
$\rho(t) = [\sin\omega_1 t\ \sin\omega_2 t\ \dots\ \sin\omega_{\hbar} t]$, (with $\omega_i \neq \omega_j$ for $i \neq j$), is the corresponding sinusoidal function vector. A method for generating the coefficients $a(t)$ is provided in subsection 4.1. The design ensures that $a(t) \to a^*$, the optimal value that satisfies a PE condition.

3.2 Adaptive tracking controller

Let us define the tracking and parameter estimation error vectors

$$z_c = x_p - r \quad \text{and} \quad \tilde{\theta} = \theta - \hat{\theta}. \tag{9}$$

and consider the Lyapunov function candidate

$$V_c := \frac{1}{2}\|z_c\|^2 + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \tag{10}$$

with $\Gamma = \Gamma^T > 0$. Taking the time derivative of V_c along the trajectory of (2), we have

$$\dot{V}_c = z_c^T \left(f_p(x) + \phi(x_p)\hat{\theta} + G_p(x)u - \dot{r} \right) - \dot{\hat{\theta}}^T \Gamma^{-1} \tilde{\theta} + z_c^T \phi(x_p)\tilde{\theta}$$

Considering the control law

$$u = -G_p(x)^{-1} \left(f_p(x) + \phi(x_p)\hat{\theta} - \dot{r} + k_c z_c \right), \tag{11}$$

with $k_c > 0$ and the parameter update law

$$\dot{\hat{\theta}} = \Gamma \phi(x_p)^T z_c, \tag{12}$$

it follows from (2) and (11) that

$$\dot{z}_c = \phi(x_p)\tilde{\theta} - k_c z_c, \tag{13}$$

and the time derivative of the Lyapunov function results in

$$\dot{V}_c \leq -k_c \|z_c\|^2 . \tag{14}$$

Proposition 2. Consider the closed loop system (13), adaptive control (11) and parameter update law (12), the design is such that

$$\lim_{t\to\infty} \left(z_c,\ z_c^{(k)},\ \tilde{\theta}^{(k)} \right) = 0 \tag{15}$$

with $1 \leq k \leq \beta$ and $(.)^{(k)}$ denotes $\frac{d^k}{dt^k}(.)$.

To prove this result, we need the following lemma.

Lemma 3. Barbalat's lemma (Krstic *et al.*, 1995): A signal $\zeta^{(k)} \to 0$ as $t \to \infty$ if (a) $\int_0^\infty \zeta^{(k)} dt$ exist and its finite and (b) the signal $\zeta^{(k)}$ is uniformly continuous.

Condition (a) is evident when $\zeta \in \mathcal{L}_2$ or $\zeta^{(k-1)} \to 0$ asymptotically and condition (b) can be inferred from the boundedness of $\zeta^{(k)}$ and $\zeta^{(k+1)}$.

Proof of Proposition 2. It is known from (10) that V_c is a positive definite function (bounded from below by zero). Since V_c is non-increasing (14), it is concluded that $z_c(t)$ and $\hat{\theta}(t)$ are uniformly bounded. Moreover, there exist a bounded ς such that $-\infty < -\varsigma \leq V_c(\infty) - V_c(0) < 0$. This implies that $-\varsigma \leq \int_0^\infty \dot{V}_c(\tau)d\tau \Rightarrow \varsigma \geq k_c \int_0^\infty \|z_c(\tau)\|^2 d\tau \Rightarrow \|z_c\|^2_{\mathcal{L}_2} \leq \varsigma / k_c < \infty$. Since $z_c \in \mathcal{L}_2$ and $\phi(x_p)$ is bounded by assumption, it follows from (13) that $\dot{z}_c(t) \in \mathcal{L}_\infty$. Applying the above lemma, we conclude that $z_c \to 0$ as $t \to \infty$.

Also, we know that $\int_0^\infty \dot{z}_c(\sigma)d\sigma = -z_c(0)$ exists and is finite. From the fact that $\dot{z}_c$ is a function of bounded signals we deduce that $\ddot{z}_c$ is bounded, which implies that $\dot{z}_c$, is uniformly continuous and hence $\dot{z}_c \to 0$ as $t \to \infty$. Also, it follows from the adaptive law (12) that $\lim_{t\to\infty} \dot{\tilde{\theta}}(t) = 0$.

Subsequently, it will be shown that (15) holds for $1 < k \leq \beta$ by induction. Suppose $({z_c}^{(k-1)},\ \tilde{\theta}^{(k-1)}) \to 0$, then $({z_c}^{(k)},\ \tilde{\theta}^{(k)})$ satisfies condition (a). Also, condition (b) is satisfied because $({z_c}^{(k)},\ \tilde{\theta}^{(k)})$ and $({z_c}^{(k+1)},\ \tilde{\theta}^{(k+1)})$ are functions of bounded signals. Hence, $({z_c}^{(k)}, \tilde{\theta}^{(k)}) \to 0$. Since, $({z_c}^{(1)},\ \tilde{\theta}^{(1)}) \to 0$ is guaranteed, we conclude that (15) holds. □

4. PARAMETER CONVERGENCE

Consider the state error dynamic (13) and the parameter error dynamic $\dot{\tilde{\theta}} = -\Gamma\phi(x_p)^T z_c$ obtained from (12). By an argument similar to the one used in traditional adaptive control theory, a sufficient condition for parameter convergence is that the regressor $\phi(x_p)$ be persistently exciting. That is, there exists positive constants μ_0 and T such that

$$\int_t^{t+T} \phi(\tau)^T \phi(\tau) d\tau \geq \mu_0 I, \quad \forall t \geq 0$$

Though the matrix $\phi(\tau)\phi(\tau)^T$ is singular for all τ when $(n_\theta > m)$, the PE condition requires that ϕ rotates sufficiently in space that the integral of the matrix $\phi(\tau)^T\phi(\tau)$ is uniformly positive definite over any interval of some length T. However, it is difficult to check that ϕ satisfies the PE condition since the solution of the closed loop trajectories are not known *a priori*.

In the following, an alternative sufficient condition that addresses the above limitations and guarantees parameter convergence is presented. The condition requires an augmented regressor matrix to be sufficiently rich.

By differentiating (13) with respect to time, z_c^k can be written explicitly as

$$z_c^k = -\,k_c z_c^{(k-1)} + \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-j-1)!}\phi(x_p)^{(j)}\tilde{\theta}^{(k-j-1)}, \quad 1 \le k \le \beta \qquad (16)$$

Using proposition 2 and the fact that $z_c \to 0$ in the limit as $t \to \infty$, (which implies that $x_p \to r^* = x_p^*(\bar{\theta}) + a^*\rho(t)$), equation (16) results in

$$\lim_{t\to\infty} z_c^k(t) = \lim_{t\to\infty} \phi(r^*)^{(k-1)}(t)\tilde{\theta}(t) = 0, \quad 1 \le k \le \beta \qquad (17)$$

Defining

$$Z_c := \left[\, z_c^{(1)}\ z_c^{(2)}\ \ldots\ z_c^{(\Pi)} \,\right]^T \quad \text{and} \qquad (18)$$

$$\Phi := \left[\, \phi^T\ \phi^{T(1)}\ \ldots\ \phi^{T(\Pi-1)} \,\right]_{n_\theta \times (m*\Pi)} \qquad (19)$$

where $\max\{2,\ \mathrm{ceil}(\frac{n_\theta}{m})\} \le \Pi \le \beta$. Equation (17) can then be re-written in a compact form as

$$\lim_{t\to\infty} Z_c = \lim_{t\to\infty} \Phi(r^*)^T(t)\tilde{\theta}(t) = 0. \qquad (20)$$

The next step in the analysis is to decompose the time varying signal Φ into a constant matrix and a periodic part. This procedure is similar to the one presented in (Lin and Kanellakopoulos, 1999). Firstly, (19) is expressed as

$$\Phi = \left[\, \bar{\phi}_1\ \ldots\ \bar{\phi}_m\ \ \ldots\ldots\ldots\ \bar{\phi}_1^{(\Pi-1)}\ \ldots\ \bar{\phi}_m^{(\Pi-1)} \,\right] \triangleq \left[\, \psi_1\ \psi_2\ \ldots\ \psi_{m\Pi} \,\right], \quad m\Pi = m*\Pi \qquad (21)$$

where $\bar{\phi}_l^{(\cdot)}$ is the l^{th} column of matrix $\phi^{T(\cdot)}$. The trigonometric (or Fourier) series expansion for each nonlinearity vector ψ_i is computed as follows: Let $\omega_{i1}, \omega_{i2}, \cdots, \omega_{iCi}$ $(0 \le \omega_{i1} < \omega_{i2} < \cdots < \omega_{iCi})$ and $\nu_{i1}, \nu_{i2}, \cdots, \nu_{iSi}$ $(0 < \nu_{i1} < \nu_{i2} < \cdots < \nu_{iSi})$ denote the distinct frequencies appearing in the cosine terms and the sine terms of the Fourier series expansion respectively. If we let

$$\xi_i(t) = \left[\, \cos\omega_{i1}t\ \cdots\ \cos\omega_{iC_i}t\ \ \sin\nu_{i1}t \cdots\ \sin\nu_{iSi}t \,\right]^T \triangleq \left[\, \xi_{i1}(t)\ \cdots\ \xi_{iC_i}(t)\ \ \xi_{i(Ci+1)}(t) \cdots\ \xi_{Ci+Si}(t) \,\right]^T \quad i = 1, \ldots, m\Pi \qquad (22)$$

Then, each nonlinearity vector ψ_i defined in (21) can be expressed in the form

$$\psi_i = \Upsilon_i\, \xi_i(t) = \sum_{j=1}^{C_i+S_i} \Upsilon_{ij}\, \xi_{ij}(t), \quad i = 1, \ldots, m\Pi \qquad (23)$$

where Υ_i are $n_\theta \times (C_i + S_i)$ constant matrices whose elements are the real Fourier coefficients of the corresponding signals, and $\Upsilon_{ij}, j = 1, \ldots, (C_i + S_i)$ is the j^{th} column of Υ_i. This decomposition method allows one to judge the richness of the vector based on a constant matrix only. However, as pointed out in (Lin and Kanellakopoulos, 1999), the Fourier series expansion employed in the decomposition may contain an infinite number of terms, when the elements of (21) are not polynomial nonlinearities. In this case, the series expansion may be truncated. Combining (20) with equations (21) and (23), we obtain

$$\lim_{t\to\infty} \xi_{ij}(t)\, \Upsilon_{ij}^T\ \tilde{\theta}(t) = 0 \quad i = 1, \ldots, m\Pi, \quad j = 1, \ldots, C_i + S_i \qquad (24)$$

and since the scalar functions ξ_{ij} are all of the form $\cos\omega t$ or $\sin\nu t$, equation (24) is equivalent to

$$\lim_{t\to\infty} \Upsilon_{ij}^T\ \tilde{\theta}(t) = 0 \quad i = 1, \ldots, m\Pi, \quad j = 1, \ldots, C_i + S_i. \qquad (25)$$

Moreover, defining $\Upsilon_1 = \Upsilon_{11} \cdots \Upsilon_{1(C1+S1)}$, $\Upsilon_2 = \Upsilon_{21} \cdots \Upsilon_{2(C2+S2)}$ etc, and $\Upsilon^T = [\Upsilon_1\ \cdots\ \Upsilon_m]^T$, equation (25) can be written in a more compact form

$$\lim_{t\to\infty} \Upsilon^T \tilde{\theta}(t) = 0 \quad \text{or} \quad \lim_{t\to\infty} \tilde{\theta}^T(t)\, \mathcal{W}\, \tilde{\theta}(t) = 0 \qquad (26)$$

Since Υ is a constant matrix containing the set-point x_p^* and a^* in its entries (in the limit as $t \to \infty$), if the n_θ rows of Υ are linearly independent or if $\mathcal{W} = \Upsilon\Upsilon^T$ is positive definite, then $\tilde{\theta} = 0$ is guaranteed. However, it is not possible to verify this conditions *a priori* for a given dither signal because the matrix depends on unknown reference set-point (the θ-dependent solution of (1)). In the next section, we show how to generate optimal size of some pre-selected sinusoids online.

4.1 Dither signal design

It has been shown that the presence of nonlinearities in a regressor vector increase the degree of PE of a given reference signal for nonlinear systems with special structure (Lin and Kanellakopoulos, 1998; Lin and Kanellakopoulos, 1999). However, for a general nonlinear system, this may not be the case, the nonlinearities may detract or add to the excitation (Dasgupta and Shrivastava, 1991). In this work, we propose that the dither signal be chosen as a linear combination of sinusoids with at least n_θ distinct frequencies. However, since such a choice with constant arbitrary amplitude may not be optimal for nonlinear systems, a method for generating optimal coefficients of the different basis functions (sinusoids) is provided. A quadratic objective function is minimized subject to a constraint that optimizes the size of the selected frequency contents in order to ensure positive definiteness of matrix $\mathcal{W} = \Upsilon\Upsilon^T$.

The condition requires all the eigenvalues of $\mathcal{W}$ to be positive. This is true if and only if the determinant of $\mathcal{W}$ (the product of the eigenvalues) is positive since $\mathcal{W}$ is a symmetric positive semidefinite matrix.

The optimum amplitude of the dither signal is proposed as the solution of the following constrained optimization problem.

$$\begin{aligned} &\min_{a\in\mathbb{R}^{\hbar}} \quad a^TQa \\ &\text{such that} \quad \mathcal{W}_d = det(\mathcal{W}) > 0 \end{aligned} \tag{27}$$

with $Q \succ 0$. The optimization problem is tackled using an infeasible interior point technique (Vanderbei and Shanno, 1999). Firstly, a slack variable ε is added so that (27) becomes

$$\begin{aligned} &\min_{a\in\mathbb{R}^{\hbar}} \quad a^TQa \\ &\text{such that} \quad \mathcal{W}_d - \varepsilon = 0, \quad \varepsilon > 0. \end{aligned} \tag{28}$$

The constraints are then eliminated by augmenting the objective function with high costs for violating them as follows.

$$\begin{aligned} \min_{a,\varepsilon} \ P_a = & a^TQa - \frac{1}{M_1}\log(\sigma-\varepsilon) \\ & + M_2(\mathcal{W}_d-\varepsilon)^2, \quad \sigma > 0 \end{aligned} \tag{29}$$

with M_1, $M_2 > 0$. By the logarithmic barrier term, the slack variable is required to be greater than a design variable σ at all times. However, the equality constraint ($\mathcal{W}_d - \varepsilon = 0$) can be violated at any instant, its satisfaction is only achieved as the optimum solution is approached. The solution of (29) can be shown to converge to that of (27) in the limit as the positive constants M_1, $M_2 \to \infty$.

Since we assume that system (2) is fundamentally identifiable at the defining parameter values, feasibility of (27) (and hence (29)) is guaranteed by including sufficiently large number of regressor derivatives in (19). The unconstrained optimization problem (29) can be solved with gradient techniques. Let $\bar{a}^* = [a^*, \varepsilon^*]$ be the optimizer of (29), an update law that ensures $\bar{a} \to \bar{a}^*$ as $t \to \infty$ is chosen as

$$\dot{\bar{a}} = \mathrm{Proj}\left\{-k_{\bar{a}}\mathcal{D}z_{\bar{a}},\, \bar{a}\right\}, \quad \bar{a}(0) = [a_0,\ \varepsilon_0] \tag{30}$$

where Proj{.} is a standard projection algorithm (Krstic *et al.*, 1995) used to ensure that the vector $\bar{a}$ is bounded or remains in some given set. The vector $z_{\bar{a}} = \partial P_{\bar{a}}/\partial \bar{a}$ is the gradient function, $k_{\bar{a}} > 0$ is a design parameter and $\mathcal{D}$ is a positive definite matrix function. Matrix $\mathcal{D}$ can be chosen as in steepest descent method where $\mathcal{D} = I$ (identity matrix) or as in trust region where $\mathcal{D} = \left(\partial^2 P_{\bar{a}}/\partial\bar{a}^2 + (F+\kappa)I\right)^{-1}$ with F=Frobenius matrix norm of $\partial^2 P_{\bar{a}}/\partial\bar{a}^2$ and $\kappa > 0$ is a small design constant parameter. The initial conditions are to be selected such that $\varepsilon_0 > \sigma$ and some elements of a_0 equals zero to avoid excessive initial perturbation of the system.

Theorem 4. Consider the optimization problem (1) for system (2) satisfying assumptions $A1 - A4$. The controller (11), the update laws (5), (12) and (30) for sufficiently large Π ensure that the system's state $x_p(t)$ converge to a neighborhood of $x_p^*(\theta)$ - the unique minimizer of (1).

Proof. It can be deduced from proposition (2) that $\lim_{t\to\infty}\left\|x_p(t) - x_p^r(t)\right\| \le \lim_{t\to\infty}\|d(t)\|$ and it is known from proposition (1) that $\lim_{t\to\infty}\left\|x_p^r(t) - x_p^*(\hat{\theta})\right\| = 0$. Moreover, (30) ensures $\lim_{t\to\infty} a(t) = a^*$ for large enough Π. Therefore, the only solution of (26) is $\lim_{t\to\infty}\tilde{\theta}(t) = 0$, which implies $\lim_{t\to\infty}\left\|x_p^*(\hat{\theta}) - x_p^*(\theta)\right\| = 0$. Using triangle inequality, we conclude that

$$\lim_{t\to\infty}\left\|x_p(t) - x_p^*(\theta)\right\| \le \|a^*\|. \qquad \Box$$

5. SIMULATION EXAMPLE

Consider two parallel isothermal stirred-tank reactors (DeHaan and Guay, 2005) in which reagent A forms product B and waste-product C

$$A \xrightarrow{1} B$$
$$2A \xrightarrow{2} C$$

The economic steady state cost function to be optimized is given by

$$\begin{aligned} p(x_p,\ \theta) = & \sum_{i=1}^{2}[(p_{i1} + P_A - P_B)k_{i1}A_iV_i^0 \\ & + (p_{i2} + 2P_A)k_{i2}A_i^2V_i^0], \end{aligned}$$

where P_A, P_B denote component prices, p_{ij} is the net operating cost of reaction j in reactor i. The reaction kinetic constants k_{ij} are only nominally known. A_i is the concentration of reagent A in reactor i with dynamics

$$\frac{dA_i}{dt} = A^{in}\frac{F_i^{in}}{V_i} - A_i\frac{F_i^{out}}{V_i} - k_{i1}A_i - k_{i2}A_i^2$$

The inlet flows are the control inputs, while the outlet flows are governed by PI controllers which regulate reactor volume to V_i^0. Therefore,

$$\begin{aligned} \dot{x}_p = & -\underbrace{\begin{bmatrix} \dfrac{x_{p1}k_{V1}(x_{q1} - V_1^0 + x_{q3})}{x_{q1}} \\ \dfrac{x_{p2}k_{V2}(x_{q2} - V_2^0 + x_{q4})}{x_{q2}} \end{bmatrix}}_{f_p} \\ & -\underbrace{\begin{bmatrix} x_{p1} & 2x_{p1}^2 & 0 & 0 \\ 0 & 0 & x_{p2} & 2x_{p2}^2 \end{bmatrix}}_{\phi}\theta + \underbrace{\begin{bmatrix} \dfrac{A_in}{x_{q1}} & 0 \\ 0 & \dfrac{A_in}{x_{q2}} \end{bmatrix}}_{G_p}u, \end{aligned}$$

where $x_p = [A_1,\ A_2]^T$, x_{q1}, x_{q2} are the two tank volumes, x_{q3}, x_{q4} are the PI integrators, and $\theta = [k_{11}, k_{12}, k_{21}, k_{22}]^T$.

Following the design procedure, the optimizing controller, parameter estimates and the set-point signal x_p^r are generated via equations (11), (12) and (5) respectively. For the simulation, the dither signal is selected as $d_1(t) = d_2(t) = a_1(t)\sin(0.3t) + a_2(t)\sin(0.18t)$ and $\Pi = 3$ so that the augmented regressor matrix $\Phi(r^*)^T = [\phi^T\ \dot{\phi}^T\ \ddot{\phi}^T]$. The matrix Υ is obtained via the decomposition method presented in section 4. For simulation purpose, x_p^* is replaced with its estimate x_p^r at each time t and the optimal value of the dither amplitude that ensures the positive definiteness of $\mathcal{W} = \Upsilon\Upsilon^T$ is obtained via (30). Fig. 1(a) shows that the cost function converges

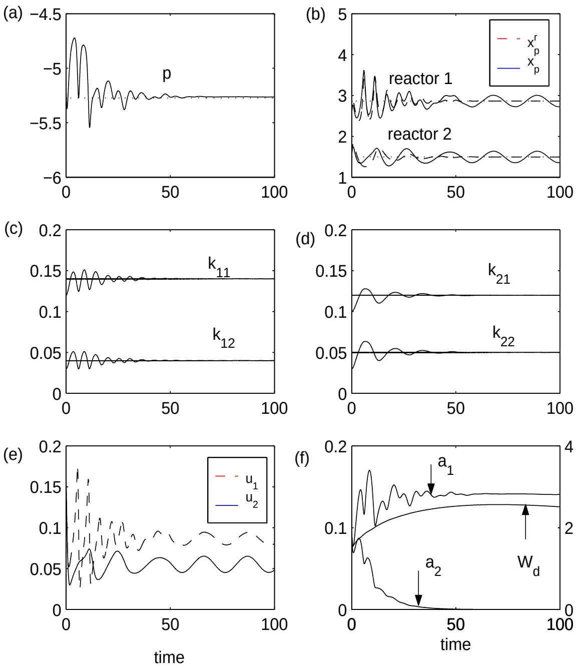

Fig. 1. Simulated system trajectories: (a) cost function, (b) reference set-point and state, (c, d) unknown parameters and estimates for reactor 1 and 2 respectively, (e) control inputs, (f) dither signal amplitude and determinant of matrix $\mathcal{W}$.

to the unknown optimal $p^*(x_p^*, \theta)$. Fig. 1(b) shows that the set-point signal converges to the optimum value $x_p^*(\theta)$ while the state x_p oscillates about the optimum. The parameter estimates converge to the true values as shown in fig. 1(c-d) and the control input, fig. 1(e), is implementable. The trajectories of the dither amplitude and the determinant are shown in fig. 1(f) for completeness. The figure showed that $a(t)$ converges to the required optimum (vertical-axis labelling on the left) and the determinant $\mathcal{W}_d$ remains positive (vertical-axis labelling on the right).

6. CONCLUSION

A persistence of excitation condition is proposed for the ESC of a class of nonlinear systems. An optimization based method is then developed for generating sufficiently rich optimum set-points that satisfies this condition online. The proposed design method guarantees parameter convergence and at the same time ensure small steady-state error in the cost function.

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