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A STATE SPACE APPROACH FOR BOUNDARY CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

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Abstract: Many industrial processes exhibit spatially distributed behaviour and are distributed parameter systems (DPS). Much of the current literature has focused on the control of specific types of DPS, such as those modelled by hyperbolic or parabolic partial differential equations (PDEs). State space models for these systems (termed as 2-D systems) have also been studied extensively, however additional techniques are required to deal with boundary conditions. In this paper, general boundary conditions for discretized 2-D state space models are formulated in the state space domain. A controller is developed for boundary control problems using the discretized state space method. Simulation results indicate that the resulting boundary controller can achieve desirable performance for setpoint tracking.

Keywords: Boundary Control, Distributed Parameter Systems, State Space Model, Partial Differential Equations, Linear systems, Discretized models

1. INTRODUCTION

Distributed parameter systems (DPS), in which the state variables change with space and time, are common in industry. Examples include sheet forming, fixed bed reactor, and metallurgical processes. DPS models can be obtained by application of mass and energy conservation laws and often take the form of partial differential equations (PDEs). Active research has been focused on control development for DPS based on PDE models (Christofides, 2000; Neittaanmaki and Tiba, 1994; Godasi *et al.*, 2002; Shang *et al.*, 2005). Most of these PDE-based controllers address specific classes of PDE systems (*e.g.*, hyperbolic or parabolic). Owing to lack of a general model structure and their continuous nature in both dimensions (*e.g.*, time and space), PDE models can hardly be used for model identification when first principal modelling cannot be achieved.

Research into the use of state space models for DPS started in the mid-1970s. It is noted that DPS have been termed as infinite dimensional systems for researchers using PDE model based approaches while the community using state space approaches has used the term 2-D system to represent DPS that has two independent variables. In this paper, we follow the convention of using the term 2-D systems in state space approaches. Roesser (1975) extended the state space model for lumped parameter systems (LPS) (1-D) to a 2-D discrete, linear time invariant state space model with potential applications in image processing (Roesser, 1975). Almost simultaneously, Fornasini and Marchesini also proposed a state space model for 2-D systems (Fornasini and Marchesini, 1976). It was shown that the Roesser model was more general, as the Fornasini and Marchesini model could be rewritten in the Roesser model form (Kung *et al.*, 1977). The limitations of the Roesser model are that it requires causality of the state variables in the spatial direction as

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well as in time, and it can only represent systems with unilateral boundary conditions due to the assumption of causality in space (Marszalek, 1984). In the following years, the Roesser model was studied extensively (Kaczorek, 1985; Chen and Tsai, 2002; Hernández and Arkun, 1992) and its application extended to various fields (Wellstead *et al.*, 2000; Galkowski *et al.*, 2000). More recently, a multi-dimensional state space model that does not require causality in space was presented (D'Andrea, 1998; D'Andrea and Chandra, 2002); thus, it has the potential to represent those DPS with bilateral boundary conditions. The model has been used for formulating distributed control problems for spatially interconnected systems (D'Andrea and Dullerud, 2003), vehicle formation (Fowler and D'Andrea, 2002), and sheet forming processes (Stewart, 2000). Existing state space models and control developments have mainly addressed DPS with distributed input and distributed output where the effect of boundary conditions has been simplified.

Although distributed control is becoming more practical with continuing advances in the technology of sensors and actuators (D'Andrea and Dullerud, 2003), for many systems, control actions cannot be performed at every point. An example is a boundary control system, where manipulated variables can only be implemented at the boundaries (Abu-Hamdeh, 2002). The importance of boundary control problems is well recognized and boundary controllers have been developed for systems described by PDEs or integral equation models (Chakravarti and Ray, 1999; Alvarez-Ramirez, 2001).

In this paper, a discretized 2-D state space model with general boundary conditions is formulated, and the resulting framework is used to develop a boundary controller. Simulations are performed to examine the effectiveness of the resulting boundary controller, and indicate that the developed boundary controller can generate a desirable output response to setpoint changes.

2. 2-D STATE SPACE MODELS

DPS can commonly be classified into systems with distributed inputs and distributed outputs, and those with boundary inputs and boundary or spatially uniform outputs.

2.1 Distributed Input

The discretized state space model of a DPS with distributed input and distributed output is composed of state equations, boundary conditions and an output equation:

state equations

$$\begin{bmatrix} x^L(i+1,j) \\ x^R(i-1,j) \\ x^V(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x^L(i,j) \\ x^R(i,j) \\ x^V(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u(i,j), \quad (1)$$

boundary conditions

$$\begin{aligned} x^L(1,j) &= \alpha_{11} x^R(1,j) + \alpha_{12} x^V(1,j), \\ x^R(N,j) &= \alpha_{21} x^L(N,j) + \alpha_{22} x^V(N,j), \quad (2) \end{aligned}$$

output equation

$$y(i,j) = C_1 x^L(i,j) + C_2 x^R(i,j) + C_3 x^V(i,j), \quad (3)$$

where i indicates the spatial index and j indicates the time index, $x^L(i,j) \in R^{m_1}$ is the distributed state variable evolving horizontally (or spatially) in one direction, $x^R(i,j) \in R^{m_2}$ is the state variable evolving horizontally (or spatially) in the opposite direction, $x^V(i,j) \in R^{m_3}$ is the state variable evolving vertically (or in time), $u(i,j) \in R$ is the distributed input and $y(i,j) \in R$ is the distributed output. In the above equations, all As, Bs, Cs and αs are constant matrices with proper dimensions. Equations (1) to (3) represent a DPS with a finite range and $i \in [1,N]$.

In the existing literature, the systems addressed were assumed to be of infinite extent or periodic, meaning boundary conditions were not imposed. Recently, systems of finite extent possessing a certain symmetry structure were addressed (Langbort and D'Andrea, 2005). For PDE systems, boundary conditions play an important role in the solution technique and the process dynamics. Equation (2) provides a general formulation for boundary conditions in DPS since it can describe any linear boundary condition for PDE systems.

2.2 Boundary Input

In many DPS, the manipulated input is located on a boundary. Without loss of generality, it is assumed that the input variable appears on the boundary $x^L(1,j)$. The state space model can be described as:

state equations

$$\begin{bmatrix} x^L(i+1,j) \\ x^R(i-1,j) \\ x^V(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x^L(i,j) \\ x^R(i,j) \\ x^V(i,j) \end{bmatrix}, \quad (4)$$

boundary conditions

$$\begin{aligned} x^L(1,j) &= \alpha_{11} x^R(1,j) + \alpha_{12} x^V(1,j) \\ &\quad + \beta u(j), \\ x^R(N,j) &= \alpha_{21} x^L(N,j) + \alpha_{22} x^V(N,j), \quad (5) \end{aligned}$$

output equation

$$y(j) = \sum_{i=1}^{N} c_{1i} x^L(i,j) + \sum_{i=1}^{N} c_{2i} x^R(i,j) + \sum_{i=1}^{N} c_{3i} x^V(i,j). \quad (6)$$

Equation (6) is a general output representation including the value of the state variables on one boundary or any other spatial point, or the weighted average of the state variables. In this paper, the case when $u(j) \in R$ and $y(j) \in R$ are scalars is considered.

The state space model in Equations (4) to (6) describes the general boundary control problem when the manipulated input appears on one boundary. Physically meaningful boundary conditions, such as Dirichlet, Neumann or mixed boundary conditions for PDEs, can be written in the form of Equation (5). Most available studies on boundary control have focused on specific types of PDE systems and specific boundary conditions (*e.g.*, boundary value of the state variables is equal to the input). Equations (4) to (6) provide a general state space framework for boundary control problems which can be used in control development, model identification as well as and system studies.

3. BOUNDARY CONTROL

In this section, a boundary controller is developed for systems modelled by Equations (4) to (6).

From Equation (4), x^L can be expanded to yield:

$$\begin{aligned} x^L(i,j) &= A_{11}^{i-1} x^L(1,j) + A_{11}^{i-2} A_{12} x^R(1,j) \\ &+ \cdots + A_{12} x^R(i-1,j) + A_{11}^{i-2} A_{13} x^V(1,j) \\ &+ \cdots + A_{13} x^V(i-1,j). \end{aligned} \quad (7)$$

The state variables in an extended vector form are defined as follows:

$$\begin{aligned} \mathbf{x}^L(j) &= \left[x^L(1,j)\; x^L(2,j) \cdots x^L(N,j)\right]^T, \\ \mathbf{x}^R(j) &= \left[x^R(1,j)\; x^R(2,j) \cdots x^R(N,j)\right]^T, \\ \mathbf{x}^V(j) &= \left[x^V(1,j)\; x^V(2,j) \cdots x^V(N,j)\right]^T. \end{aligned} \quad (8)$$

Applying the boundary conditions (5), Equation (7) can be expressed as:

$$\mathbf{x}^L(j) = \mathbf{\Phi} \mathbf{x}^R(j) + \mathbf{\Gamma} \mathbf{x}^V(j) + \theta u(j) \quad (9)$$

where $\mathbf{\Phi} = [\phi_{kl}],\ \phi_{kl} \in R^{m_1 \times m_2}$, $\mathbf{\Gamma} = [\gamma_{kl}], \gamma_{kl} \in R^{m_1 \times m_3}$, $\theta = [\theta_k], \theta_k \in R^{m_1 \times 1},\ k = 1, \cdots, N,\ l = 1, \cdots, N$, and

$$\phi_{kl} = \begin{cases} A_{11}^{k-1}\alpha_{11} + A_{11}^{k-2}A_{12}, k = 1, \cdots, N, l = 1, \\ A_{11}^{k-l-1}A_{12},\ k = 3, \cdots, N,\ 1 < l < k, \\ 0,\ \text{else}, \end{cases}$$

$$\gamma_{kl} = \begin{cases} A_{11}^{k-1}\alpha_{12} + A_{11}^{k-2}A_{13},\ k = 1, \cdots, N, l = 1, \\ A_{11}^{k-l-1}A_{13},\ k = 3, \cdots, N,\ 1 < l < k, \\ 0,\ \text{else}, \end{cases}$$

$$\theta_k = A_{11}^{k-1}\beta,\ k = 1, \cdots, N. \quad (10)$$

Substituting Equation (9) and boundary conditions (5) into Equation (4), the expression for x^R in an extended matrix form is obtained:

$$\mathbf{x}^R(j) = \mathbf{\Psi} \mathbf{x}^R(j) + \mathbf{\Xi} \mathbf{x}^V(j) + \zeta u(j) \quad (11)$$

where $\mathbf{\Psi} = [\psi_{kl}],\ \psi_{kl} \in R^{m_2 \times m_2}$,$\mathbf{\Xi} = [\xi_{kl}], \xi_{kl} \in R^{m_2 \times m_3}$, $\zeta = [\zeta_k], \zeta_k \in R^{m_2 \times 1}$ and

$$\psi_{kl} = \begin{cases} A_{21}A_{11}^{k}\alpha_{11} + A_{21}A_{11}^{k-1}A_{12}, \\ \qquad k = 1, \cdots, N-1, l = 1, \\ A_{21}A_{11}^{k-l}A_{12}, \\ \qquad k = 2, \cdots, N-1, 1 < l \le k, \\ A_{22},\ k = 1, \cdots, N-1, l = k+1, \\ \alpha_{21}A_{11}^{k-1}\alpha_{11} + \alpha_{21}A_{11}^{k-2}A_{12}, k = N, l = 1 \\ \alpha_{21}A_{11}^{k-l-1}A_{12}, k = N, l = 2, \cdots, N-1, \\ 0,\ \text{else}, \end{cases}$$

$$\xi_{kl} = \begin{cases} A_{21}A_{11}^{k}\alpha_{12} + A_{21}A_{11}^{k-1}A_{13}, \\ \qquad k = 1, \cdots, N-1, l = 1, \\ A_{21}A_{11}^{k-l}A_{13}, \\ \qquad k = 2, \cdots, N-1, 1 < l < k, \\ A_{23}, k = 1, \cdots, N-1, l = k+1, \\ \alpha_{21}A_{11}^{k-1}\alpha_{12} + \alpha_{21}A_{11}^{k-2}A_{13}, k = N, l = 1 \\ \alpha_{21}A_{11}^{k-l-1}A_{13}, k = N, l = 2, \cdots, N-1, \\ \alpha_{22}, k = N, L = N, \\ 0,\ \text{else}, \end{cases}$$

$$\zeta_k = \begin{cases} A_{21}A_{11}^{k}\beta,\ k = 1, \cdots, N-1 \\ \alpha_{21}A_{11}^{k-1}\beta, k = N \end{cases} \quad (12)$$

From the state equation in (4), the extended matrix form of $x^V(j+1)$ can be written as:

$$\mathbf{x}^V(j+1) = \mathbf{Q}_1 \mathbf{x}^L(j) + \mathbf{Q}_2 \mathbf{x}^R(j) + \mathbf{Q}_3 \mathbf{x}^v(j) \quad (13)$$

where $\mathbf{Q}_1 = \text{diag}[A_{31}] \in R^{m_3 N \times m_1 N}$, $\mathbf{Q}_2 = \text{diag}[A_{32}] \in R^{m_3 N \times m_2 N}$ and $\mathbf{Q}_3 = \text{diag}[A_{33}] \in R^{m_3 N \times m_3 N}$.

From Equations (9) and (11), $\mathbf{x}^L$ and $\mathbf{x}^R$ can be rewritten:

$$\begin{aligned} \mathbf{x}^R(j) &= (\mathbf{I}_{m_2N \times m_2N} - \mathbf{\Psi})^{-1} \mathbf{\Xi} \mathbf{x}^V(j) \\ &+ (\mathbf{I}_{m_2N \times m_2N} - \mathbf{\Psi})^{-1} \zeta u(j), \\ \mathbf{x}^L(j) &= (\mathbf{\Phi}(\mathbf{I}_{m_2N \times m_2N} - \mathbf{\Psi})^{-1} \mathbf{\Xi} + \mathbf{\Gamma}) \mathbf{x}^V(j) \\ &+ (\mathbf{\Phi}(\mathbf{I}_{m_2N \times m_2N} - \mathbf{\Psi})^{-1} \zeta + \theta) u(j), \end{aligned} \quad (14)$$

where $\mathbf{I}_{m_2N \times m_2N}$ is an identity matrix of dimension m_2N by m_2N. Well-posedness of the systems requires that $(\mathbf{I}_{m_2N \times m_2N} - \mathbf{\Psi})$ be invertible. The output equation in (6) can be written in matrix form:

$$y(j) = \mathbf{C}_1 \mathbf{x}^L(j) + \mathbf{C}_2 \mathbf{x}^R(j) + \mathbf{C}_3 \mathbf{x}^V(j), \quad (15)$$

where $\mathbf{C}_1 = [c_{11}, c_{12}, \cdots, c_{1N}]$, $\mathbf{C}_2 = [c_{21}, c_{22}, \cdots, c_{2N}]$, and $\mathbf{C}_3 = [c_{31}, c_{32}, \cdots, c_{3N}]$.

Substituting Equation (14) into Equations (13) and (15) yields:

$$\begin{aligned} \mathbf{x}^V(j+1) &= \mathbf{\Omega}\mathbf{x}^V(j) + \varphi u(j), \\ y(j) &= \mathbf{F}\mathbf{x}^V(j) + Du(j), \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Omega &= \mathbf{Q}_1(\mathbf{\Phi}(\mathbf{I}_{m_2N\times m_2N} - \mathbf{\Psi})^{-1}\mathbf{\Xi} + \mathbf{\Gamma}) \\ &\quad + \mathbf{Q}_2(\mathbf{I}_{m_2N\times m_2N} - \mathbf{\Psi})^{-1}\mathbf{\Xi}) + \mathbf{Q}_3, \\ \varphi &= \mathbf{Q}_1(\mathbf{\Phi}(\mathbf{I}_{m_2N\times m_2N} - \mathbf{\Psi})^{-1}\zeta + \theta) \\ &\quad + \mathbf{Q}_2(\mathbf{I}_{m_2N\times m_2N} - \mathbf{\Psi})^{-1}\zeta \\ \mathbf{F} &= \mathbf{C}_1(\mathbf{\Phi}(\mathbf{I}_{m_2N\times m_2N} - \mathbf{\Psi})^{-1}\mathbf{\Xi} + \mathbf{\Gamma}) \\ &\quad + \mathbf{C}_2(\mathbf{I}_{m_2N\times m_2N} - \mathbf{\Psi})^{-1}\mathbf{\Xi}) + \mathbf{C}_3, \\ D &= \mathbf{C}_1(\mathbf{\Phi}(\mathbf{I}_{m_2N\times m_2N} - \mathbf{\Psi})^{-1}\zeta + \theta) \\ &\quad + \mathbf{C}_2(\mathbf{I}_{m_2N\times m_2N} - \mathbf{\Psi})^{-1}\zeta. \end{aligned} \quad (17)$$

Note that Equation (16) takes the same form as a state space model for LPS. If the setpoint profiles for the state variables $\mathbf{x}^V$ could be calculated for any output setpoint changes, the abundance of existing control theory and techniques for LPS could be applied to the control of DPS. The boundary control design presented in this paper is focused on convergence of the output to the setpoint. An alternative design problem would examine convergence of the state variables to the origin, incorporating the problem of state estimation using output measurements. This problem will be examined in future research.

Assume that $D = 0$ in Equation (17), meaning that at least one sampling time is needed for the input to affect the output. In fact, the time required for the impact of the input to reach the output can be significant for boundary control problems, especially when the input is at one boundary and the output at the opposite boundary. The delay phenomenon in boundary control problems can be systematically defined by a time delay index, k_d, such that

$$\begin{aligned} \mathbf{F}\mathbf{\Omega}^l\phi &= 0,\ l = 1, \cdots, k_d - 2, \\ \mathbf{F}\mathbf{\Omega}^{k_d-1}\phi &\neq 0. \end{aligned} \quad (18)$$

When the conditions in Equation (18) hold, the system is called to have a time delay of k_d. The future output in the next k_d steps can be written:

$$\begin{aligned} y(j+l) &= \mathbf{F}\mathbf{\Omega}^l\mathbf{x}^V(j),\ l = 0, 1, \cdots, k_d - 1, \\ y(j+k_d) &= \mathbf{F}\mathbf{\Omega}^{k_d}\mathbf{x}^V(j) + \mathbf{F}\mathbf{\Omega}^{k_d-1}\phi u(j). \end{aligned} \quad (19)$$

Then control can be developed such that

$$(y(j+k_d) - r) = \sum_{l=0}^{k_d-1} \lambda_l (y(j+l) - r), \quad (20)$$

where r is the setpoint for the output y, λ_l are tuning parameters that should satisfy $\lambda_l > 0$, $l = 0, 1, \cdots, k_d - 1$ and $\sum_{l=0}^{k_d-1} \lambda_l < 1$ for stability of the tracking error dynamics. One choice of tuning parameters is $\lambda_0 = \lambda_1 = \cdots = \lambda_{k_d-1} < \frac{1}{k_d}$. From Equation (20), the boundary controller can be formulated as:

$$u(j) = \frac{\sum_{l=0}^{k_d-1} \lambda_l(\mathbf{F}\mathbf{\Omega}^l\mathbf{x}^V(j) - r) - (\mathbf{F}\mathbf{\Omega}^{k_d}\mathbf{x}^V(j) - r)}{\mathbf{F}\mathbf{\Omega}^{k_d-1}\phi}. \quad (21)$$

The controller described in Equation (21) can drive the output to the setpoint, with its behaviour depending on the choice of the tuning parameters λ_l. The time delay index is usually $k_d \geq 1$. When the input is on one boundary and the output on the opposite boundary, the time delay index is greatest and could be close to or greater than the residence time. In this case, the boundary control input in Equation (21) is a function of the state variables at all spatial points. The performance of the controlled output can be improved because the controller takes into account the response of the current output and future outputs (within the delay horizon). An additional advantage of this control method is that the resulting control is robust to sensor failures at some spatial points.

4. SIMULATIONS

The boundary controller developed above is evaluated using two examples: a second order parabolic PDE system and a hyperbolic countercurrent heat exchanger.

4.1 Parabolic PDE

Consider a system described by

$$\frac{\partial s(z,t)}{\partial t} = a\frac{\partial s(z,t)}{\partial z} + b\frac{\partial^2 s(z,t)}{\partial z^2} - cs(z,t) \quad (22)$$

with boundary conditions

$$\begin{aligned} s(0,t) &= u(t), \\ \frac{\partial s(z,t)}{\partial z}\Big|_{z=L} &= 0, \end{aligned} \quad (23)$$

where t indicates time, z indicates the spatial coordinate, $s(z,t)$ is the distributed state variable, and a=0.005, b=0.1, c=0.05. The manipulated input $u(t)$ is the inlet value of the state variable $s(z,t)$. The controlled output is the outlet value of the state variable $s(z,t)$, *i.e.*, $y(t) = s(L,t)$. The parabolic system described here may represent an isothermal tubular reactor with diffusion and convection.

A state space model for the system described by (22) and (23) can be derived using different approaches (*e.g.*, numerical discretization, state space identification, analytical solutions). For the sake of simplicity, the finite difference method is applied to the PDE system and yield the state space model:

$$\begin{bmatrix} x^L(i+1,j) \\ x^R(i-1,j) \\ x^V(i,j+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \eta_1 & \eta_2 & \eta_3 \end{bmatrix} \begin{bmatrix} x^L(i,j) \\ x^R(i,j) \\ x^V(i,j) \end{bmatrix} \quad (24)$$

$$\begin{aligned} x^L(1,j) &= u(j), \\ x^R(N,j) &= x^V(N,j), \\ y(j) &= x^V(N,j). \end{aligned} \quad (25)$$

Using the controller in (21), it is found that the system has a time delay of $k_d = N = 6$. The tuning parameters used are $\lambda = [\,0\ 0.01\ 0.15\ 0.12\ 0.21\ 0.5\,]$. Figure (1) shows the closed-loop response to setpoint changes. It is observed that the output reaches the setpoint quickly with smooth control action.

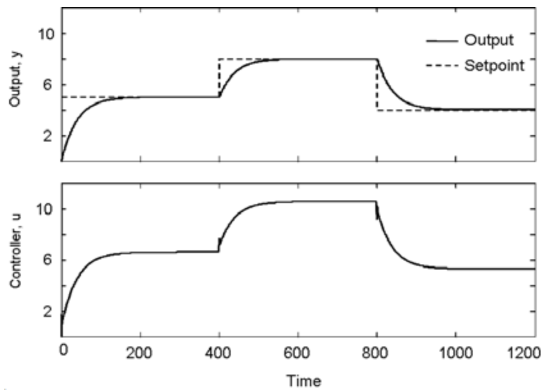

Fig. 1. Response of the parabolic PDE with boundary control to setpoint changes

4.2 *Countercurrent Heat Exchanger*

This example shows the application of the developed boundary controller to a countercurrent heat exchanger. Figure (2) shows the diagram of the system. The controller is designed to regulate the outlet temperature of the cold water by manipulating the inlet temperature of the hot water. Assuming that heat transfer to the environment is negligible, the system can be modelled as follows (Abdelghani-Idrissi *et al.*, 2001):

$$\begin{aligned} \frac{\partial T_h}{\partial t} &= -V_h \frac{\partial T_h}{\partial z} + V_h N_h (T_w - T_h), \\ \frac{\partial T_c}{\partial t} &= V_c \frac{\partial T_c}{\partial z} + V_c N_c (T_w - T_c), \\ \frac{\partial T_w}{\partial t} &= V_h C_h N_h (T_h - T_w) \\ &\quad + V_c C_c N_c (T_w - T_c), \end{aligned} \quad (26)$$

with boundary conditions

$$\begin{aligned} T_h(0,t) &= u(t), \\ T_c(L,t) &= T_{co} \end{aligned} \quad (27)$$

where subscripts h, c, and w refer to hot stream, cold stream, and wall temperatures respectively.

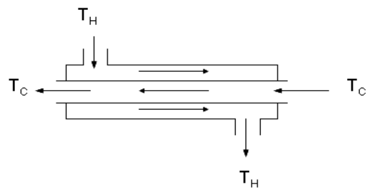

Fig. 2. Diagram of the countercurrent heat exchanger

The state space model of the system takes the form of:

$$\begin{bmatrix} x_h^L(i+1,j) \\ x_c^R(i-1,j) \\ x_w^V(i,j+1) \\ x_h^V(i,j+1) \\ x_c^V(i,j+1) \end{bmatrix} = \left[\begin{array}{c|c|ccc} 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & 0 & \alpha_5 & \alpha_6 & 0 \\ 0 & \alpha_7 & \alpha_8 & 0 & \alpha_9 \end{array}\right] \begin{bmatrix} x_h^L(i,j) \\ x_c^R(i,j) \\ x_w^V(i,j) \\ x_h^V(i,j) \\ x_c^V(i,j) \end{bmatrix}$$

$$y(j) = K_1 x^V(j) \quad (28)$$

with boundary conditions:

$$\begin{aligned} x_h^L(1,j) &= u(j) \\ x_c^R(N,j) &= T_{co} \end{aligned} \quad (29)$$

The boundary control performance was evaluated by examining the closed-loop output response to changes in the setpoint. Using the developed control method, the delay time was calculated to be 3 time units. Figure (3) shows that the outlet cold-water temperature displays a stable response and quick convergence to the setpoint under the boundary controller with $\lambda = [\,0\ 0.3\ 0.6\,]$.

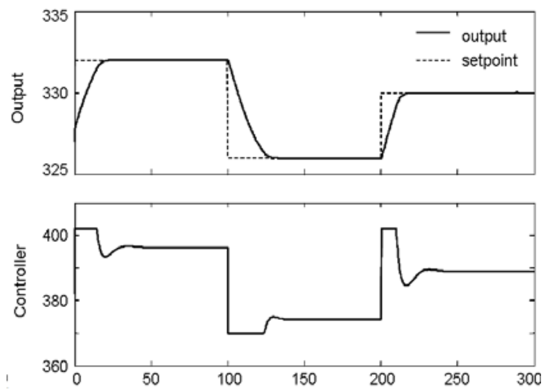

Fig. 3. Response of the heat exchanger with boundary control to changes in setpoint

5. CONCLUSIONS

This work has demonstrated how boundary conditions can be incorporated and used for controller design in discretized 2-D state space models for distributed parameter systems. State space models with general boundary conditions are formulated for systems with either distributed or boundary inputs. The controller design approach is presented for systems with boundary input. Time delay can arise because of the spatially distributed nature of the process, and is defined in terms of the state space model parameters. The resulting controller demonstrates good performance for systems with a significant time delay because the controller determines the control input taking into account the projected behaviour of the output over the time delay horizon.The developed boundary controller was evaluated by simulation using both a parabolic system and a countercurrent hyperbolic system. The results show that, under the boundary controller, the output converges to the setpoint in a quick and smooth way without aggressive control action.

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