Target-set Control

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Abstract

To survive in the face of uncontrollable natural variations, biological organisms have developed adaptation mechanisms that make them remarkably insensitive to variations in certain variables. Conversely, outside these ranges of admissible variation, biological function may change dramatically, usually in undesirable ways (e.g., the organisms die). As a consequence, a set-theoretic control strategy seems quite appropriate for biologically-based processes like fermentation reactors: many variables do not have to be controlled to precise setpoints, but they do have to be maintained within viable operating ranges. This paper proposes a strategy for this kind of set-theoretic control based on *zonotopes*, which are the images of n-dimensional cubes under affine transformations. This approach is well-suited to the control of linearized fundamental models or linear empirical models over a specified range of validity. In addition, the results presented here establish strong connections with classical linear control theory. Finally, these results are extended to *positive linear systems*, a class that includes many biological system models (e.g., compartmental models arising in pharmecokinetics) and that are inherently harder to control than unconstrained linear systems.

1 Introduction

This paper describes *target-set control,* a control strategy that leads to the selection of control input sequences that drive the system state vector into a specified target set. One motivation for target-set control is the control of biological processes and systems, where many variables must be kept within specified ranges, but variation within those ranges has little effect on system performance, due in part to the well-developed adaptive nature of biological organisms. For example, *homeostasis*, the term for the coordinated action by which living organisms maintain equilibrium and sustain life, has several distinguishing characteristics that are important to the engineer contemplating the control of biological systems:

1. Desired equilibrium conditions are generally in terms of ranges of acceptable values *not* precise point targets as is commonly the case in standard control engineering;
2. By design, intrinsic system robustness is achieved by ensuring that multiple combinations of input variable settings can equally well maintain the system at the desired equilibrium conditions. In other words, the problem of maintaining homeostasis has multiple equally admissible solutions. Contrast this with most standard control engineering problems for which non-uniqueness of a control solution may in fact be undesirable.
3. In their natural "settings", the states, inputs and output variables of typical biological systems are constrained to be non-negative at all times.

While one can definitely phrase the biological system control problem within the classical control theory framework (i.e. employ a model—often defined in terms of deviation variables—to compute a *unique* set of input variable values to drive system states to a unique set-point, subject to constraints), we believe that a more natural theoretical control framework for bioprocesses ought to take their distinguishing system characteristics into consideration explicitly.

2 Problem formulation

The basic problem formulation considered here is an extension of one considered previously [5], but based on a more flexible class of uncertainty sets. More specifically, this formulation assumes the existence of an approximate linear model of the general form:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{d}(k), \qquad (1)$$

where $\mathbf{x}(k)$ represents the n-dimensional state vector at time k, $\mathbf{u}(k)$ is the vector of m control inputs at time k, $\mathbf{A}$ and $\mathbf{B}$ are compatibly dimensioned matrices, and $\mathbf{d}(k)$ is an effective disturbance vector of dimension n. This model may be obtained by linearizing a fundamental model of process dynamics about some specified steady-state operating condition, via empirical model identification, or by any other means. The effective disturbance vector $\mathbf{d}(k)$ represents the combined effects on the state vector of linear model parameter uncertainty, neglected nonlinearities, discretization artifacts, and the influence of unmeasurable external disturbances. The control problem considered here is the following one:

> Given the process model (1), characterize the set of admissible sequences of control inputs $\{\mathbf{u}(k), \ldots, \mathbf{u}(k+r-1)\}$ that will drive the state vector $\mathbf{x}(k)$ into a designated target set S^* in R^n.

To solve this problem, we introduce the following sets:

- Σ_k, the set of all possible values for the state vector $\mathbf{x}(k)$ at time k,

- Δ_k, the set of all possible values for the effective disturbance vector $\mathbf{d}(k)$ at time k.

In what follows, given Σ_k, Δ_k, and a specific control input vector $\mathbf{u}(k)$, we first derive an expression for the set Σ_{k+1} of possible states $\mathbf{x}(k+1)$ and then extend this result to obtain an expression for the set Σ_{k+r} of possible values for $\mathbf{x}(k+r)$ for arbitrary $r \geq 1$. This multi-step result represents one important extension of our previous results [5]; another extension is the replacement of spherical uncertainty sets with more flexible *zonotopes*, described next.

3 Zonotopes

A zonotope is defined [8, p. 191] as the image of the p-cube under an affine projection map, where the p-cube is the set

$$C_p = \{\mathbf{x} \in R^p \mid |x_i| \leq 1,\ i = 1, 2, \ldots, p\}, \qquad (2)$$

and an affine projection of a set $\mathcal{A} \subset R^p$ is the set defined by

$$\mathbf{M} \bigotimes \mathcal{A} + \mathbf{b} = \{\mathbf{M}\mathbf{x} + \mathbf{b} \mid \mathbf{x} \in \mathcal{A}\}, \qquad (3)$$

where $\mathbf{M}$ is any $n \times p$ matrix and $\mathbf{b}$ is any vector in R^n. For convenience, let $Z_n(\mathbf{M}, \mathbf{b})$ denote the zonotope in R^n defined by:

$$\begin{aligned} Z_n(\mathbf{M}, \mathbf{b}) &= \mathbf{M} \bigotimes C_p + \mathbf{b} \\ &= \{\mathbf{x} \in R^n \mid \mathbf{x} = \mathbf{b} + \sum_{i=1}^{p} \lambda_i \mathbf{m}_i, \\ &\qquad |\lambda_i| \leq 1\}, \end{aligned} \qquad (4)$$

where $\mathbf{m}_i$ is the i^{th} column of the matrix $\mathbf{M}$. Note that if the matrix $\mathbf{M}$ is *diagonal*, the resulting zonotope is a *parallelepiped*, a rectangular polytope in R^p with its faces parallel to the coordinate axes.

To describe system evolution in terms of zonotopes, we need the following notion. The *Minkowski sum* of two sets $\mathcal{A}$ and $\mathcal{B}$ is defined by [8]:

$$\mathcal{A} \bigoplus \mathcal{B} = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in \mathcal{A},\ \mathbf{b} \in \mathcal{B}\}. \qquad (5)$$

Now, define Σ^0_{k+1} as the set of all possible *unforced* states $\mathbf{x}(k+1)$, obtained by setting $\mathbf{u}(k) = 0$. This set is related to Σ_k and Δ_k by the evolution equation:

$$\Sigma^0_{k+1} = [\mathbf{A} \bigotimes \Sigma_k] \bigoplus \Delta_k. \qquad (6)$$

To obtain results that are *computationally* useful from this general expression, it is necessary to specialize to a class of sets for which affine projections and Minkowski sums are easy to compute; zonotopes represent one such class.

More specifically, the zonotope $Z_n(\mathbf{M}, \mathbf{b})$ may be viewed as the Minkowski sum of p line segments in R^n, each defined by a column of the matrix $\mathbf{M}$, translated by the vector $\mathbf{b}$. That is,

$$\begin{aligned} Z_n(\mathbf{M}, \mathbf{b}) &= [\ell_1 \bigoplus \ell_2 \bigoplus \cdots \bigoplus \ell_p] + \mathbf{b} \\ \ell_i &= \{\mathbf{x} \in R^n \mid \mathbf{x} = \lambda \mathbf{m}_i,\ |\lambda| \leq 1\}. \end{aligned} \qquad (7)$$

It follows immediately from this result that the Minkowski sum of two zonotopes $Z_n(\mathbf{M}, \mathbf{b})$ and $Z_n(\mathbf{N}, \mathbf{c})$ is simply the Minkowski sum of the line segments defined by the columns of $\mathbf{M}$ and $\mathbf{N}$, offset by $\mathbf{b} + \mathbf{c}$. Defining the composite matrix $[\mathbf{M} \vdots \mathbf{N}]$ as

$$[\mathbf{M} \vdots \mathbf{N}] = [\mathbf{m}_1, \ldots, \mathbf{m}_p, \mathbf{n}_1, \ldots, \mathbf{n}_q], \tag{8}$$

the Minkowski sum of two zonotopes is given by

$$Z_n(\mathbf{M}, \mathbf{b}) \bigoplus Z_n(\mathbf{N}, \mathbf{c}) = Z_n([\mathbf{M} \vdots \mathbf{N}], \mathbf{b} + \mathbf{c}). \tag{9}$$

Hence, it follows that zonotopes are closed under Minkowski addition. To see that zonotopes are also closed under affine transformations, note that

$$\begin{aligned} \mathbf{A} \bigotimes Z_n(\mathbf{M}, \mathbf{b}) + \mathbf{c} &= \mathbf{A} \bigotimes [\mathbf{M} \bigotimes C_p + \mathbf{b}] + \mathbf{c} \\ &= [\mathbf{AM}] \bigotimes C_p + [\mathbf{Ab} + \mathbf{c}] \\ &= Z_n(\mathbf{AM}, \mathbf{Ab} + \mathbf{c}). \end{aligned} \tag{10}$$

4 State evolution

To describe state evolution under this framework, suppose that the initial state homeostatic set is $\Sigma_k = Z_n(\mathbf{V}_k, \mathbf{x}_k)$ for some $n \times p$ matrix $\mathbf{V}_k$ and some nominal state vector $\mathbf{x}_k$. Next, assume the uncertainty sets Δ_{k+j} associated with the effective disturbance vectors $\mathbf{d}(k + j)$ are $\Delta_{k+j} = Z_n(\mathbf{W}_{k+j}, \mathbf{d}_{k+j})$ where $\mathbf{W}_{k+j}$ is an arbitrary $n \times q_j$ matrix describing the uncertainty in $\mathbf{d}(k+j)$ about its nominal value $\mathbf{d}_{k+j}$. It follows from Eq. (6) that the one-step unforced state evolution is described by

$$\begin{aligned} \Sigma^0_{k+1} &= [\mathbf{A} \bigotimes Z_n(\mathbf{V}_k, \mathbf{x}_k)] \bigoplus Z_n(\mathbf{W_k}, \mathbf{d}_k) \\ &= Z_n([\mathbf{AV}_k \vdots \mathbf{W}_k], \mathbf{Ax}_k + \mathbf{d}_k). \end{aligned} \tag{11}$$

The control vector $\mathbf{u}(k)$ steers the center of the evolving uncertainty set, giving us the one-step state evolution equation

$$\begin{aligned} \Sigma_{k+1} &= \Sigma^0_{k+1} + \mathbf{Bu}(k) \\ &= Z_n([\mathbf{AV}_k \vdots \mathbf{W}_k], \mathbf{Ax}_k + \mathbf{d}_k + \mathbf{Bu}(k)). \end{aligned} \tag{12}$$

Iterating this result, it follows that the r-step state uncertainty set Σ_{k+r} is the zonotope $Z_n(\mathbf{V}_{k+r}, \mathbf{x}_{k+r})$, where

$$\begin{aligned} \mathbf{V}_{k+r} &= [\mathbf{A}^r \mathbf{V}_k \vdots \mathbf{A}^{r-1} \mathbf{W}_k \vdots \cdots \vdots \mathbf{W}_{k+r-1}] \\ \mathbf{x}_{k+r} &= \mathbf{A}^r \mathbf{x}_k + \mathbf{A}^{r-1} \mathbf{d}_k + \cdots + \mathbf{d}_{k+r-1} \\ &\quad + \mathbf{A}^{r-1} \mathbf{Bu}(k) + \cdots + \mathbf{Bu}(k + r - 1). \end{aligned} \tag{13}$$

A useful rearrangement of this result is the following generalization of Eq. (12):

$$\Sigma_{k+r} = \Sigma^0_{k+r} + \Phi \mathbf{v}, \tag{14}$$

where Σ^0_{k+r} represents the r-step unforced evolution caused by the system dynamics (e.g., decay of initial conditions) and the influence of external disturbances. This term is given explicitly as

$$\begin{aligned} \Sigma^0_{k+r} &= Z_n(\mathbf{V}_{k+r}, \mathbf{y}_{k+r}), \\ \mathbf{y}_{k+r} &= \mathbf{A}^r \mathbf{x}_k + \mathbf{A}^{r-1} \mathbf{d}_k + \cdots + \mathbf{d}_{k+r-1}. \end{aligned} \tag{15}$$

The term $\Phi \mathbf{v}$ in Eq. (14) describes the influence of the control inputs applied over this time period:

$$\begin{aligned} \Phi &= [\mathbf{A}^{r-1} \mathbf{B} \vdots \cdots \vdots \mathbf{B}] \\ \mathbf{v} &= \begin{bmatrix} \mathbf{u}(k) \\ \vdots \\ \mathbf{u}(k + r - 1) \end{bmatrix}. \end{aligned} \tag{16}$$

Note that $\mathbf{v} \in R^{rm}$ describes the complete sequence of control moves made in the r steps considered here, and that Φ is the $n \times rm$ controllability matrix.

5 Target-set control

Given the results just presented for uncertain state evolution, we require one more construct to address the control problem. The *Minkowski difference* between two subsets of R^n is defined by

$$\mathcal{A} \sim \mathcal{B} = \{\mathbf{x} \in R^n \mid \mathcal{B} + \mathbf{x} \subset \mathcal{A}\}. \tag{17}$$

It is important to note that the Minkowski difference is *not* the inverse of Minkowski addition; in particular, these operations are related by [6]:

$$(\mathcal{A} \sim \mathcal{B}) \bigoplus \mathcal{B} \subset \mathcal{A}, \tag{18}$$

where the inclusion is generally proper. To see the utility of this construct, suppose the control objective is to guarantee that the state vector $\mathbf{x}(k+r)$ lies in a specified target set S^*. The set of all possible "state corrections" that are consistent with this control objective is then given by

$$\begin{aligned} \Omega_{k+r} &= \{\mathbf{z} \in R^n \mid \Sigma^0_{k+r} + \mathbf{z} \subset S^*\} \\ &= S^* \sim \Sigma^0_{k+r}. \end{aligned} \tag{19}$$

Given the sequence of r control moves specified by the vector $\mathbf{v}$ and the r-step controllability matrix Φ, the set of *feasible* control moves that are capable of meeting our control objective is given by

$$\Upsilon_{k+r} = \{\mathbf{v} \in R^{rm} \mid \Phi\mathbf{v} \in \Omega_{k+r}\}. \tag{20}$$

Note that if Φ has rank n, the usual controllability condition [4, p. 460], $\Phi\mathbf{v}$ can assume any value in R^n, implying that the r-step control problem is feasible (i.e., the set Υ_{k+r} is not empty) if and only if the set Ω_{k+r} is not empty.

To determine the set Ω_{k+r}, first note that for any two sets $\mathcal{A}, \mathcal{B} \subset R^n$ and any two n-vectors $\mathbf{a}$ and $\mathbf{b}$,

$$[\mathcal{A} + \mathbf{a}] \sim [\mathcal{B} + \mathbf{b}] = [\mathcal{A} \sim \mathcal{B}] + [\mathbf{a} - \mathbf{b}], \tag{21}$$

so the Minkowski difference of two zonotopes becomes

$$\begin{aligned} Z_n(\mathbf{M}, \mathbf{b}) \sim Z_n(\mathbf{N}, \mathbf{c}) &= [Z_n(\mathbf{M}, \mathbf{0}) \sim Z_n(\mathbf{N}, \mathbf{0})] \\ &\quad + [\mathbf{b} - \mathbf{c}]. \end{aligned} \tag{22}$$

Next, suppose $\mathcal{P}$ is a parallelepiped in R^n and $\mathcal{Z}$ is an arbitrary zonotope in R^n. By the preceeding result, there is no loss of generality in assuming that both of these sets are centered at zero. Hence, the Minkowski difference between these sets may be written as

$$\begin{aligned} \mathcal{P} \sim \mathcal{Z} &= Z_n(\mathbf{M}, \mathbf{0}) \sim Z_n(\mathbf{N}, \mathbf{0}) \\ &= \{\mathbf{z} \in R^n \mid \mathbf{z} + Z_n(\mathbf{N}, \mathbf{0}) \subset Z_n(\mathbf{M}, \mathbf{0})\} \\ &= \{\mathbf{z} \in R^n \mid -M_{ii} \leq z_i + y_i \leq M_{ii}, \\ &\qquad \mathbf{y} \in Z_n(\mathbf{N}, \mathbf{0})\}. \end{aligned} \tag{23}$$

Next, note that any vector $\mathbf{y} \in Z_n(\mathbf{N}, \mathbf{0})$ may be written as

$$\mathbf{y} = \sum_{j=1}^{p} \lambda_j \mathbf{n}_j \;\Rightarrow\; y_i = \sum_{j=1}^{p} \lambda_j N_{ij}. \tag{24}$$

Applying the triangle inequality to this result gives

$$|y_i| \leq \sum_{j=1}^{p} |\lambda_j||N_{ij}| \leq \sum_{j=1}^{p} |N_{ij}|, \tag{25}$$

since $|\lambda_j| \leq 1$ for all j. Further, the extreme values in this inequality are achievable by taking either $\lambda_j = \text{sign}\,\{N_{ij}\}$ or $\lambda_j = -\text{sign}\,\{N_{ij}\}$ for all j. Hence, the Minkowski difference result from Eq. (23) may be written more explicitly as

$$\begin{aligned} \mathcal{P} \sim \mathcal{Z} &= \{\mathbf{z} \in R^n \mid \\ &\qquad -M_{ii} + N^\diamond_{ii} \leq z_i \leq M_{ii} - N^\diamond_{ii}\}, \end{aligned} \tag{26}$$

where $N^\diamond_{ii}$ is defined as

$$N^\diamond_{ii} = \sum_{j=1}^{p} |N_{ij}|. \tag{27}$$

Defining the matrix $\mathbf{N}^\diamond$ as the $n \times n$ diagonal matrix with elements $N^\diamond_{ii}$, Eq. (26) may be written as

$$\mathcal{P} \sim \mathcal{Z} = Z_n(\mathbf{M}, \mathbf{0}) - Z_n(\mathbf{N}, \mathbf{0}) = Z_n(\mathbf{M} - \mathbf{N}^\diamond, \mathbf{0}). \tag{28}$$

Also, note that for the inequalities in Eq. (26) to be consistent—i.e., for the set $\mathcal{P} \sim \mathcal{Z}$ to be nonempty—it is necessary that $M_{ii} - N^\diamond_{ii} \geq 0$ for all i, meaning that the matrix $\mathbf{M} - \mathbf{N}^\diamond$ is positive semidefinite. Geometrically, this result means that the Minkowski difference between any parallelepiped $\mathcal{P}$ and any other zonotope $\mathcal{Z}$ is a parallelepiped.

In the context of the control problem of interest here, suppose each component of the state vector is constrained to lie in the interval $a_i \leq x_i \leq b_i$. This constraint corresponds to $\mathbf{x} \in S^* = Z_n(\mathbf{H}, \mathbf{x}^*)$ where the diagonal matrix $\mathbf{H}$ and the vector $\mathbf{x}^*$ are

$$H_{ii} = \frac{b_i - a_i}{2}, \quad x^*_i = \frac{a_i + b_i}{2}. \tag{29}$$

The set Ω_{k+r} is then given by

$$\begin{aligned} \Omega_{k+r} &= S^* \sim \Sigma^0_{k+r} \\ &= Z_n(\mathbf{H}, \mathbf{x}^*) \sim Z_n(\mathbf{V}_{k+r}, \mathbf{y}_{k+r}) \\ &= Z_n(\mathbf{H} - \mathbf{V}^\diamond_{k+r}, \mathbf{x}^* - \mathbf{y}_{k+r}). \end{aligned} \tag{30}$$

Further, note that this set is nonempty if and only if $\mathbf{H}_{ii} - [\mathbf{V}^\diamond_{k+r}]_{ii} \geq 0$ for $i = 1, 2, \ldots, n$. If these conditions hold, the set Υ_{k+r} of feasible controls defined in Eq. (20) corresponds to the solution set for the following collection of n simultaneous inequalities:

$$\gamma_i^- \;\leq\; [\Phi \mathbf{v}]_i \;\leq\; \gamma_i^+, \tag{31}$$

where these bounds are given by

$$\begin{aligned} \gamma_i^- &= [\mathbf{x}^* - \mathbf{y}_{k+r}]_i - [\mathbf{H} - \mathbf{V}^\diamond_{k+r}]_{ii} \\ \gamma_i^+ &= [\mathbf{x}^* - \mathbf{y}_{k+r}]_i + [\mathbf{H} - \mathbf{V}^\diamond_{k+r}]_{ii}. \end{aligned} \tag{32}$$

Recall that for a controllable system, the controllability matrix Φ has rank n for $r \geq n$, from which it follows that any state correction in the set Ω_{k+r} is achievable. For a completely controllable system, the standard (i.e., non set-theoretic) solution would be to choose $\mathbf{v}$ so that $[\Phi\mathbf{v}]_i$ falls in the center of each interval defined in Eq. (31), corresponding to a set point for the state vector of $\mathbf{x}^* - \mathbf{y}_{k+r}$ and representing a standard disturbance rejection strategy. Conversely, note that if $\gamma_i^- \leq 0 \leq \gamma_i^+$ for $i = 1, 2, \ldots, n$, it follows that one feasible solution is $\mathbf{v} = \mathbf{0}$. This solution corresponds to the classical statistical process control (SPC) strategy: no control action is necessary so long as the controlled variables lie within their target specification. One advantage of the target-set formulation considered here is that it permits us to consider a range of alternatives between these two very different control strategies. This flexibility is particularly useful in connection with positive linear systems, as the following discussion illustrates.

6 Positive linear systems

Positive linear systems are linear systems whose states, inputs, and outputs are constrained to be nonnegative at all times. The local dynamics of biological systems can often be described by positive linear systems because the variables involved are concentrations, which cannot be negative. An important special case of positive linear systems are *compartmental systems*, which may be defined as systems composed of interconnected reservoirs and which correspond to asymptotically stable positive linear systems [3, p. 147]. It is important to note, however, that this positivity constraint applies to system models written in terms of absolute state variables and *not* to those written in terms of deviation variables about some specified steady-state, since such deviations can be either positive or negative. An important practical aspect of positive linear systems is that they are inherently harder to control than unconstrained linear systems; for example, controllability conditions are much more restrictive for positive linear systems [2]. An interesting feature of the results presented in the preceeding sections of this paper is that they extend directly to the case of positive linear systems and provide some additional insights into the differences between unconstrained and positive linear systems.

To obtain this extension to positive linear systems, first define the *positive p-cube:*

$$C_p^+ = \{\mathbf{x} \in R^p \mid 0 \leq x_i \leq 1,\; i = 1, 2, \ldots, p\}, \tag{33}$$

which is a subset of the *positive orthant of R^n*, denoted R_+^n. Next, define a *positive affine projection* of a set $\mathcal{A} \subset R_+^n$ as the set $\mathbf{M} \bigotimes \mathcal{A} + \mathbf{b}$, where $\mathbf{M}$ is an $m \times n$ matrix whose elements are all nonnegative and $\mathbf{b}$ is an m-vector whose components are all nonnegative. A *positive zonotope* is the subset of R_+^n denoted $Z_n^+(\mathbf{M}, \mathbf{b})$ and defined by any positive affine projection of the positive p-cube. It is not difficult to show that the Minkowski sum of positive zonotopes is a positive zonotope, and that any positive affine projection of a positive zonotope is another positive zonotope. As a consequence, all of the state evolution results (i.e., Eqs. (13) through (16)) carry over directly to positive linear systems: if the state vector lies in the set $Z_n^+(\mathbf{V}_k, \mathbf{x}_k)$ at time k, these equations describe its subsequent evolution in response to the unforced positive system dynamics, nonnegative disturbance inputs characterized by positive zonotopes $\Delta_{k+j} = Z_n^+(\mathbf{W}_{k+j}, \mathbf{d}_{k+j})$, and nonnegative control inputs $\mathbf{u}_{k+j}$.

The most important difference between the positive system formulation and the unconstrained formulation is that the set Ω_{k+r} defined by Eq. (30) does not lie in the positive orthant. Since nonnegative input sequences can generate only state vector changes in R_+^n for a positive linear system, we must restrict consideration to the positive part of Ω_{k+r}. As in the unconstrained case, it is easy to obtain an explicit expression for this set if the target set S^* is a parallelepiped in R^n_+. Specifically, we have:

$$\begin{aligned}\Omega^+_{k+r} &= [S^* \sim \Sigma^0_{k+r}] \cap R^n_+ \\ &= [Z^+_n(\mathbf{H}, \mathbf{x}^*) \sim Z^+_n(\mathbf{V}_{k+r}, \mathbf{y}_{k+r})] \cap R^n_+ \\ &= \{x \in R_n \mid 0 \leq x_i \leq H_{ii} - \sum_{j=1}^{p} [\mathbf{V}_{k+r}]_{ij} \\ &\qquad + x^*_i - [\mathbf{y}_{k+r}]_i, i = 1, 2, \ldots, n\}. \end{aligned} \quad (34)$$

Given this result, the set Υ_{k+r} of admissible control moves consists of all nonnegative vectors $\mathbf{v}$ satisfying the following conditions for $i = 1, 2, \ldots, n$:

$$0 \leq [\Phi \mathbf{v}]_i \leq x^*_i - [\mathbf{y}_{k+r}]_i + H_{ii} - \sum_{j=1}^{p} [\mathbf{V}_{k+r}]_{ij}. \quad (35)$$

Note that the set Υ_{k+r} is non-empty if and only if the right-hand side of Eq. (35) is nonnegative for all i. *Also, note that these conditions hold if and only if the SPC solution* $\mathbf{v} = \mathbf{0}$ *is feasible.* This observation provides a very natural reference case for target-set control of positive linear systems: given a performance measure of interest and any other feasible control strategy in Υ_{k+r}, how does its performance compare with that of the SPC strategy?

Finally, note that even if Ω^+_{k+r} is non-empty and Φ^{-1} exists, it will generally *not* be possible to generate all of the positive state corrections in Ω^+_{k+r} with nonnegative control input sequences $\mathbf{v}$. In particular, note that Φ is a matrix with all nonnegative entries: even if Φ is square and Φ^{-1} exists, this inverse is necessarily an *M-matrix*, which has *non-positive* off-diagonal elements [1, ch. 6]. Consequently, unless Φ is a diagonal matrix, there will be elements of Ω^+_{k+r} that can only be reached using negative control inputs. In practice, diagonality of Φ is an extremely restrictive condition; one case where this occurs is for a diagonal $\mathbf{B}$ matrix, corresponding to the monomial matrix condition for controllability discussed by Coxson and Shapiro [2]. The contrast between this result and the unconstrained linear system result (i.e., that every state correction in Ω_{k+r} is achievable if Φ^{-1} exists) provides yet another illustration of the important practical differences between positive linear systems and unconstrained linear systems.

7 Summary

As one reviewer noted, the target-set control approach described here bears some important similarities to the geometric approach to control theory [7]. He further argued that geometric control theory is more powerful because it does not restrict consideration to zonotopes. While we agree with this argument, we also note that the restriction to zonotopes has important advantages, both computationally in the simple construction of explicit sets of admissible control values and conceptually, as in the connections between controllability and statistical process control noted in Sections 5 and 6.

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